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GENERALIZATION OF MAJORIZATION THEOREM VIA TAYLOR'S FORMULA

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Abstract. We give generalization of majorization theorem for the class of n-convex functions by using Taylor's formula. We use inequalities for the Čebyšev functional to obtain bounds for the identities related to generalizations of majorization inequalities. We present mean value theorems and $n-$exponential convexity for the functional obtained from the generalized majorization inequalities. At the end we discuss the results for particular families of function and give means.

1. Introduction

Majorization gives us the precise answer about the location of the components of the vector $\mathbf{x}$ respected to that of vector $\mathbf{y}$. The well known Majorization theorem given by Marshall and Olkin [11] (see also [15], p. 320):

THEOREM 1. *Let I be an interval in $\mathbb{R}$ and let $\mathbf{x},\mathbf{y}$ be two n-tuples such that $x_i, y_i \in I$ $(i = 1,\ldots,n)$. Then*

$$\sum_{i=1}^{n} \phi(y_i) \leqslant \sum_{i=1}^{n} \phi(x_i)$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ iff

$$\sum_{i=1}^{m} y_{[i]} \leqslant \sum_{i=1}^{m} x_{[i]}$$

holds for $m = 1,2,\ldots,n-1$ and

$$\sum_{i=1}^{n} y_{[i]} = \sum_{i=1}^{n} x_{[i]}.$$

The generalization of Theorem 1 was given by Fuchs in [6] as weighted Majorization Theorem (see also [15], p. 323):

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THEOREM 2. *Let* $\mathbf{x}, \mathbf{y}$ *be two decreasing* n*-tuples from an interval* I*, let* $\mathbf{w} = (w_1, \ldots, w_n)$ *be a real* n*-tuple such that*

$$\sum_{i=1}^{k} w_i y_i \leqslant \sum_{i=1}^{k} w_i x_i, \text{ for } k = 1, \ldots, n-1; \tag{1}$$

and

$$\sum_{i=1}^{n} w_i y_i = \sum_{i=1}^{n} w_i x_i. \tag{2}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$*, we have*

$$\sum_{i=1}^{n} w_i \phi(y_i) \leqslant \sum_{i=1}^{n} w_i \phi(x_i). \tag{3}$$

The following integral version of Theorem 2 is a simple consequence of Theorem A in [13] (see also [15], p. 328):

THEOREM 3. *Let* $x, y : [a,b] \to [\alpha, \beta]$ *such that* $[\alpha, \beta] \subset I$ *be decreasing and* $w : [a,b] \to \mathbb{R}$ *be continuous functions. If*

$$\int_a^v w(t)y(t)dt \leqslant \int_a^v w(t)x(t)dt \text{ for every } v \in [a,b] \tag{4}$$

and

$$\int_a^b w(t)y(t)dt = \int_a^b w(t)x(t)dt \tag{5}$$

hold, then for every continuous convex function $\phi : I \to \mathbb{R}$*, we have*

$$\int_a^b w(t)\phi((y(t))dt \leqslant \int_a^b w(t)\phi(x(t))dt. \tag{6}$$

For other integral version and generalization of majorization theorem see ([11], p. 583) (see also [12], [1], [10]). The classical Taylor's formula with integral remainder can be stated as:

THEOREM 4. *Let* n *be a positive integer and* $\phi : [a,b] \to \mathbb{R}$ *be such that* $\phi^{(n-1)}$ *is absolutely continuous, then for all* $x \in [a,b]$ *the Taylor's formula at the point* $c \in [a,b]$ *is*

$$\phi(x) = T_{n-1}(\phi; c, x) + R_{n-1}(\phi; c, x),$$

where $T_{n-1}(\phi; c, x)$ *is a Taylor's polynomial of degree* $n-1$*, i.e.*

$$T_{n-1}(\phi; c, x) = \sum_{k=0}^{n-1} \frac{\phi^{(k)}(c)}{k!}(x-c)^k$$

and the remainder is given by

$$R_{n-1}(\phi; c, x) = \frac{1}{(n-1)!} \int_c^x \phi^{(n)}(t)(x-t)^{n-1} dt.$$

In rest of the paper, we need the following real valued function of our interest defined as:

$$(x-t)_+ = \begin{cases} x-t, & t \leqslant x, \\ 0, & t > x. \end{cases}$$

For two Lebesgue integrable functions $f,h : [\alpha,\beta] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

In [5] the authors proved the following theorems:

THEOREM 5. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha,\beta] \to \mathbb{R}$ be an absolutely continuous function with $(.-\alpha)(\beta-.)[h']^2 \in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \tag{7}$$

The constant $\frac{1}{\sqrt{2}}$ in (7) is the best possible.

THEOREM 6. *Assume that $h : [\alpha,\beta] \to \mathbb{R}$ is monotonic nondecreasing on $[\alpha,\beta]$ and $f : [\alpha,\beta] \to \mathbb{R}$ be an absolutely continuous with $f' \in L_\infty[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{2(\beta-\alpha)} ||f'||_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dh(x). \tag{8}$$

The constant $\frac{1}{2}$ in (8) is the best possible.

2. Main results

We start the section with the proof of identities obtained by using Taylor's formula.

THEOREM 7. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 1$ and let $\mathbf{w} = (w_1,\ldots,w_m)$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$. Then*

$$\begin{aligned} &\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) \\ &= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!} \left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k \right) \\ &\quad + \frac{1}{(n-1)!}\int_\alpha^\beta \left[\sum_{i=1}^m w_i((x_i-t)_+)^{n-1} - \sum_{i=1}^m w_i((y_i-t)_+)^{n-1} \right] \phi^{(n)}(t)dt, \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
&\sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) \\
&= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^{m} w_i(\beta - x_i)^k - \sum_{i=1}^{m} w_i(\beta - y_i)^k\right)(-1)^k \\
&\quad - \frac{1}{(n-1)!}\int_{\alpha}^{\beta}(-1)^{n-1}\left[\sum_{i=1}^{m} w_i((t-x_i)_+)^{n-1} - \sum_{i=1}^{m} w_i((t-y_i)_+)^{n-1}\right]\phi^{(n)}(t)dt. \quad (10)
\end{aligned}
$$

Proof. Using Taylor's formula at point α in $\sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i)$, we have

$$
\begin{aligned}
&\sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) \\
&= \sum_{i=1}^{m} w_i\left(\sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(x_i-\alpha)^k + \frac{1}{(n-1)!}\int_{\alpha}^{x_i}\phi^{(n)}(t)(x_i-t)^{n-1}dt\right) \\
&\quad - \sum_{i=1}^{m} w_i\left(\sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(y_i-\alpha)^k + \frac{1}{(n-1)!}\int_{\alpha}^{y_i}\phi^{(n)}(t)(y_i-t)^{n-1}dt\right) \\
&= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^{m} w_i(x_i-\alpha)^k - \sum_{i=1}^{m} w_i(y_i-\alpha)^k\right) \\
&\quad + \frac{1}{(n-1)!}\int_{\alpha}^{x_i}\sum_{i=1}^{m} w_i(x_i-t)^{n-1}\phi^{(n)}(t)dt - \frac{1}{(n-1)!}\int_{\alpha}^{y_i}\sum_{i=1}^{m} w_i(y_i-t)^{n-1}\phi^{(n)}(t)dt \\
&= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^{m} w_i(x_i-\alpha)^k - \sum_{i=1}^{m} w_i(y_i-\alpha)^k\right) \\
&\quad + \frac{1}{(n-1)!}\int_{\alpha}^{\beta}\sum_{i=1}^{m} w_i((x_i-t)_+)^{n-1}\phi^{(n)}(t)dt - \frac{1}{(n-1)!}\int_{\alpha}^{\beta}\sum_{i=1}^{m} w_i((y_i-t)_+)^{n-1}\phi^{(n)}(t)dt
\end{aligned}
$$

where

$$
\int_{\alpha}^{\beta}\sum_{i=1}^{m} w_i((x_i-t)_+)^{n-1}\phi^{(n)}(t)dt = \int_{\alpha}^{x_i}\sum_{i=1}^{m} w_i(x_i-t)^{n-1}\phi^{(n)}(t)dt + \int_{x_i}^{\beta} 0
$$

and

$$
\int_{\alpha}^{\beta}\sum_{i=1}^{m} w_i((y_i-t)_+)^{n-1}\phi^{(n)}(t)dt = \int_{\alpha}^{y_i}\sum_{i=1}^{m} w_i(y_i-t)^{n-1}\phi^{(n)}(t)dt + \int_{y_i}^{\beta} 0
$$

So by using above result we will get (9).

Similarly using Taylor's formula at point β in $\sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i)$, we get (10). □

Integral version of the above theorem can be stated as:

THEOREM 8. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 1$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions. Then*

$$\begin{aligned}
&\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\
&= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\int_a^b w(\tau)\left[(x(\tau)-\alpha)^k - (y(\tau)-\alpha)^k\right]d\tau\right) \\
&\quad + \frac{1}{(n-1)!}\int_\alpha^\beta \left(\int_a^b w(\tau)\left[((x(\tau)-t)_+)^{n-1} - ((y(\tau)-t)_+)^{n-1}\right]d\tau\right)\phi^{(n)}(t)dt,
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau \\
&= \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\int_a^b w(\tau)\left[(\beta-x(\tau))^k - (\beta-y(\tau))^k\right]d\tau\right)(-1)^k \\
&\quad - \frac{1}{(n-1)!}\int_\alpha^\beta (-1)^{n-1}\left(\int_a^b w(\tau)\left[((t-x(\tau))_+)^{n-1} - ((t-y(\tau))_+)^{n-1}\right]d\tau\right)\phi^{(n)}(t)dt.
\end{aligned} \tag{12}$$

In the following theorem we obtain generalizations of majorization inequality for n-convex functions.

THEOREM 9. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 1$ and let $\mathbf{w} = (w_1,\dots,w_m)$, $\mathbf{x} = (x_1,\dots,x_m)$ and $\mathbf{y} = (y_1,\dots,y_m)$ be m-tuples such that $x_i, y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\dots,m)$. Then*

(i) if ϕ is n-convex function and

$$\sum_{i=1}^m w_i((x_i-t)_+)^{n-1} - \sum_{i=1}^m w_i((y_i-t)_+)^{n-1} \geqslant 0, \quad t \in [\alpha,\beta], \tag{13}$$

then

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) \geqslant \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k - \sum_{i=1}^m w_i(y_i-\alpha)^k\right). \tag{14}$$

(ii) If ϕ is n-convex function and

$$(-1)^{n-1}\left(\sum_{i=1}^m w_i((t-x_i)_+)^{n-1} - \sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\right) \leqslant 0, \quad t \in [\alpha,\beta], \tag{15}$$

then

$$\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) \geqslant \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k - \sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k. \tag{16}$$

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)} \geqslant 0$ (see [15], p. 16). Hence we can apply Theorem 7 to obtain (14) and (16) respectively. □

In the following Corollary, we give generalization of Fuch's majorization theorem.

COROLLARY 1. *Let all the assumptions of Theorem 7 be satisfied,* $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ *be decreasing* m*-tuples and* $\mathbf{w} = (w_1, \ldots, w_m)$ *be any* m*-tuple such that* $x_i, y_i \in [\alpha, \beta]$, $w_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ *which satisfies* (1), (2) *and* $\phi : [\alpha, \beta] \to \mathbb{R}$ *is* n*-convex function. Then*

(i) for $n \geqslant 1$, (14) *holds. Moreover, let the inequality* (14) *be satisfied. If the function*

$$F_1(x) := \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(x-\alpha)^k. \tag{17}$$

is convex, the R.H.S. of (14) *is non negative, that is* (3) *holds.*

(ii) If n is even, then (16) *holds. Moreover, let the inequality* (16) *be satisfied. If the function*

$$F_2(x) := \sum_{k=1}^{n-1} \frac{(-1)^k \phi^{(k)}(\beta)}{k!}(\beta - x)^k. \tag{18}$$

is convex, the R.H.S. of (16) *is non negative, that is* (3) *holds.*

Proof. (i) On account of given m-tuples satisfying (1), (2) and the function $x \mapsto ((x-t)_+)^{n-1}$ being convex for given n, (13) holds by virtue of Theorem 2. Therefore by following Theorem 9 we can obtain (14). Moreover, we can rewrite the R.H.S. of (14) in the form of the L.H.S. with $\phi = F_1$, where F_1 is defined in (17) and will be obtained after reorganization of this side. Since F_1 is assumed to be convex, therefore using the given conditions on m-tuples and by following Theorem 2 the non negativity of R.H.S. of (14) is immediate, that is (3) holds.

Similarly we can prove the part (ii). □

3. New upper bounds via Čebyšev functional

In the sequel, we consider Theorems 5 and 6 to derive generalizations of the results proved in the previous section. Let $\mathbf{w} = (w_1, \ldots, w_m)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be m-tuples such that $x_i, y_i \in [\alpha, \beta]$, $w_i \in \mathbb{R}$ $(i = 1, \ldots, m)$, denote

$$\mathfrak{R}(t) = \sum_{i=1}^{m} w_i((x_i - t)_+)^{n-1} - \sum_{i=1}^{m} w_i((y_i - t)_+)^{n-1}, \quad t \in [\alpha, \beta], \tag{19}$$

$$\mathfrak{B}(t) = (-1)^{n-1}\left(\sum_{i=1}^{m} w_i((t - x_i)_+)^{n-1} - \sum_{i=1}^{m} w_i((t - y_i)_+)^{n-1}\right), \quad t \in [\alpha, \beta]. \tag{20}$$

THEOREM 10. *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous for some $n\geqslant 1$ with $(.-\alpha)(\beta-.)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$ and let $\mathbf{w}=(w_1,\dots,w_m)$, $\mathbf{x}=(x_1,\dots,x_m)$ and $\mathbf{y}=(y_1,\dots,y_m)$ be m-tuples such that $x_i,y_i\in[\alpha,\beta]$, $w_i\in\mathbb{R}$ $(i=1,\dots,m)$ and let the functions $\mathfrak{R}$, $\mathfrak{B}$ be defined by (19), (20) respectively . Then*

(i) the remainder $\mathfrak{K}_n^1(\alpha,\beta;\phi)$ given in the following identity

$$\sum_{i=1}^m w_i\phi(x_i)-\sum_{i=1}^m w_i\phi(y_i)$$
$$=\sum_{k=1}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k-\sum_{i=1}^m w_i(y_i-\alpha)^k\right)$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-1)!}\int_\alpha^\beta \mathfrak{R}(t)dt+\mathfrak{K}_n^1(\alpha,\beta;\phi), \tag{21}$$

satisfies the estimation

$$|\mathfrak{K}_n^1(\alpha,\beta;\phi)|\leqslant\frac{1}{(n-1)!}[\Delta(\mathfrak{R},\mathfrak{R})]^{\frac12}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac12}.$$

(ii) The remainder $\mathfrak{K}_n^2(\alpha,\beta;\phi)$ given in the following identity

$$\sum_{i=1}^m w_i\phi(x_i)-\sum_{i=1}^m w_i\phi(y_i)$$
$$=\sum_{k=1}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k-\sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-1)!}\int_\alpha^\beta \mathfrak{B}(t)dt-\mathfrak{K}_n^2(\alpha,\beta;\phi), \tag{22}$$

satisfies the estimation

$$|\mathfrak{K}_n^2(\alpha,\beta;\phi)|\leqslant\frac{1}{(n-1)!}[\Delta(\mathfrak{B},\mathfrak{B})]^{\frac12}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right|^{\frac12}.$$

Proof. Applying Theorem 5 for $f\mapsto\mathfrak{R}$ and $h\mapsto\phi^{(n)}$ and employ similar method as in Theorem 16 [9]. □

The following Grüss type inequalities can be obtained by using Theorem 6

THEOREM 11. *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n)}$ $(n\geqslant 1)$ is absolutely continuous function and $\phi^{(n+1)}\geqslant 0$ on $[\alpha,\beta]$ and let the functions $\mathfrak{R}$, $\mathfrak{B}$ be defined by (19), (20) respectively. Then, we have*

(i) the representation (21) and the remainder $\mathfrak{K}_n^1(\alpha,\beta;\phi)$ satisfies the bound

$$|\mathfrak{K}_n^1(\alpha,\beta;\phi)|\leqslant\frac{1}{(n-1)!}||\mathfrak{R}'||_\infty\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2}-\frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right].$$

(*ii*) *The representation* (22) *and the remainder* $\mathfrak{K}_n^2(\alpha,\beta;\phi)$ *satisfies the bound*

$$|\mathfrak{K}_n^2(\alpha,\beta;\phi)| \leqslant \frac{1}{(n-1)!}||\mathfrak{B}'||_\infty\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2}-\frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right].$$

Proof. Applying Theorem 6 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(n)}$ and employ similar method as in Theorem 17 [9]. □

Now we intend to give the Ostrowski type inequalities related to generalizations of majorization's inequality.

THEOREM 12. *Assume that all the assumptions of Theorem 7 hold. Moreover, assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $1/p+1/q=1$. Let $|\phi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be a R-integrable function for some $n \geqslant 1$. Then, we have:*

(*i*)

$$\begin{aligned}&\left|\sum_{i=1}^m w_i\phi(x_i)-\sum_{i=1}^m w_i\phi(y_i)\right.\\ &\left.-\sum_{k=1}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^m w_i(x_i-\alpha)^k-\sum_{i=1}^m w_i(y_i-\alpha)^k\right)\right|\\ &\leqslant \frac{1}{(n-1)!}||\phi^{(n)}||_p\left(\int_\alpha^\beta\left|\sum_{i=1}^m w_i((x_i-t)_+)^{n-1}-\sum_{i=1}^m w_i((y_i-t)_+)^{n-1}\right|^q dt\right)^{1/q}.\end{aligned} \tag{23}$$

(*ii*)

$$\begin{aligned}&\left|\sum_{i=1}^m w_i\phi(x_i)-\sum_{i=1}^m w_i\phi(y_i)\right.\\ &\left.-\sum_{k=1}^{n-1}\frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^m w_i(\beta-x_i)^k-\sum_{i=1}^m w_i(\beta-y_i)^k\right)(-1)^k\right|\\ &\leqslant \frac{1}{(n-1)!}||\phi^{(n)}||_p\left(\int_\alpha^\beta\left|(-1)^{n-1}\left[\sum_{i=1}^m w_i((t-x_i)_+)^{n-1}-\sum_{i=1}^m w_i((t-y_i)_+)^{n-1}\right]\right|^q dt\right)^{1/q}.\end{aligned} \tag{24}$$

The constant on the R.H.S. of (23) *and* (24) *are sharp for* $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. To prove above results, we employ similar method adopted in Theorem 19 [9]. □

4. Associated linear functionals and exponential convexity

In the present section we will construct some linear functionals as differences of the L. H. S and R. H. S. of some of the inequalities derived earlier. The obtained linear functionals will be used in the construction of new families of exponentially convex functions and some related results will be derived.

Some definitions and basic results regarding exponentially convex functions can be seen from [2], [7] and [14] which are used in sequel.

REMARK 1. By the virtue of Theorem 9, we define the positive linear functionals with respect to n-convex function ϕ as follows

$$\Omega_1(\phi) := \sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left(\sum_{i=1}^{m} w_i(x_i-\alpha)^k - \sum_{i=1}^{m} w_i(y_i-\alpha)^k\right) \geqslant 0, \tag{25}$$

$$\Omega_2(\phi) := \sum_{i=1}^{m} w_i\phi(x_i) - \sum_{i=1}^{m} w_i\phi(y_i) - \sum_{k=1}^{n-1} \frac{\phi^{(k)}(\beta)}{k!}\left(\sum_{i=1}^{m} w_i(\beta-x_i)^k - \sum_{i=1}^{m} w_i(\beta-y_i)^k\right)(-1)^k \geqslant 0. \tag{26}$$

The Lagrange and Cauchy type mean value theorems related to defined functionals are in the following theorems.

THEOREM 13. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha,\beta]$. If the inequalities in (14) and (16) are valid, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\Omega_i(\phi) = \phi^{(n)}(\xi_i)\Omega_i(\varphi); \quad i = 1,2,$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Omega_i(\cdot)$ are defined in Remark 1.

Proof. Similar to the proof of Theorem 4.1 in [8] (see also [3]). □

THEOREM 14. *Let $\phi,\lambda : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi,\lambda \in C^n[\alpha,\beta]$. If the inequalities in (14) and (16) are valid, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\frac{\Omega_i(\phi)}{\Omega_i(\lambda)} = \frac{\phi^{(n)}(\xi_i)}{\lambda^{(n)}(\xi_i)}; \quad i = 1,2,$$

provided that the denominators are non-zero and $\Omega_i(\cdot)$ are defined in Remark 1.

Proof. Similar to the proof of Corollary 4.2 in [8] (see also [3]). □

Theorem 14 enables us to define Cauchy means, because if

$$\xi_i = \left(\frac{\phi^{(n)}}{\lambda^{(n)}}\right)^{-1}\left(\frac{\Omega_i(\phi)}{\Omega_i(\lambda)}\right),$$

which show that ξ_i $(i=1,2)$ are means of α, β for given functions ϕ and λ.

Next we construct the non trivial examples of $n-$exponentially and exponentially convex functions from positive linear functionals $\Omega_i(\cdot)$ $(i=1,2)$. We use the idea given in [14].

THEOREM 15. *Let $\Theta = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is $n-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Then for the linear functionals $\Omega_i(\cdot)$ $(i=1,2)$ as defined in Remark 1, the following statements are valid for each $i=1,2$:*

(i) The function $t \to \Omega_i(\phi_t)$ is $n-$exponentially convex in the Jensen sense on J and the matrix $[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leqslant n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geqslant 0$$

for all $m \in \mathbb{N}$, $m = 1,2,...,n$.

(ii) If the function $t \to \Omega_i(\phi_t)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. Similar to the proof of Theorem 23 [9]. □

The following corollary is an immediate consequence of the above theorem.

COROLLARY 2. *Let $\Theta = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Then for the linear functional $\Omega_i(\cdot)$ $(i=1,2)$, the following statements hold:*

(i) The function $t \to \Omega_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}, m \leqslant n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\Omega_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geqslant 0$$

for all $m \in \mathbb{N}$, $m = 1,2,...,n$.

(ii) If the function $t \to \Omega_i(\phi_t)$ is continuous on J, then it is exponentially convex on J.

COROLLARY 3. *Let $\Theta = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0, \ldots, x_n; \phi_t]$ is $2-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \ldots, x_n \in I$. Let $\Omega_i(\cdot)$ $(i = 1, 2)$ be linear functionals. Then the following statements hold:*

(i) If the function $t \mapsto \Omega_i(\phi_t)$ is continuous on J, then it is $2-$exponentially convex function on J. If $t \mapsto \Omega_i(\phi_t)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\Omega_i(\phi_s)]^{t-r} \leqslant [\Omega_i(\phi_r)]^{t-s} [\Omega_i(\phi_t)]^{s-r},$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $t \mapsto \Omega_i(\phi_t)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leqslant u$ and $q \leqslant v$, we have

$$\mu_{p,q}(\Omega_i, \Theta) \leqslant \mu_{u,v}(\Omega_i, \Theta), \tag{27}$$

where

$$\mu_{p,q}(\Omega_i, \Theta) = \begin{cases} \left(\frac{\Omega_i(\phi_p)}{\Omega_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\Omega_i(\phi_p)}{\Omega_i(\phi_p)}\right), & p = q, \end{cases} \tag{28}$$

for $\phi_p, \phi_q \in \Theta$.

Proof. Similar to the proof of Corollary 2 [9]. □

5. Applications to Cauchy means

In the running section, we use a family of functions which fulfil the conditions of Theorem 15, Corollary 2 and Corollary 3.

EXAMPLE 1. Let us consider a family of functions

$$\Theta = \{\phi_t : [0, \infty) \to \mathbb{R} : t > 0\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{1, \ldots, n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{1, \ldots, n-1\}, \end{cases}$$

with $0 \log 0 = 0$. Since $\frac{d^n \phi_t}{dx^n}(x) = x^{t-n} > 0$, the function ϕ_t is n-convex for $x > 0$ and $t \mapsto \frac{d^n \phi_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 15 we also have that $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 2 we conclude that $t \mapsto \Omega_i(\phi_t)$ $(i=1,2)$ are exponentially convex in the Jensen sense. Hence, for this family of functions, it is easy to give explicitly $\mu_{t,q}(\Omega_i,\Theta)$ $(i=1)$ from (28),

$$\mu_{t,q}(\Omega_1,\Theta)=\begin{cases}\left(\dfrac{\sum\limits_{i=1}^{m} w_i x_i^t-\sum\limits_{i=1}^{m} w_i y_i^t}{\sum\limits_{i=1}^{m} w_i x_i^q-\sum\limits_{i=1}^{m} w_i y_i^q}\times\dfrac{t(t-1)\cdots(t-n+1)}{q(q-1)\cdots(q-n+1)}\right)^{\frac{1}{t-q}}, & t\neq q,\\ \exp\left(\dfrac{\sum\limits_{i=1}^{m} w_i x_i^t\log x_i-\sum\limits_{i=1}^{m} w_i y_i^t\log y_i}{\sum\limits_{i=1}^{m} w_i x_i^t-\sum\limits_{i=1}^{m} w_i y_i^t}+\sum\limits_{k=0}^{n-1}\dfrac{1}{k-t}\right), & t=q\notin\{1,\ldots,n-1\},\\ \exp\left(\dfrac{\sum\limits_{i=1}^{m} w_i x_i^t\log^2 x_i-\sum\limits_{i=1}^{m} w_i y_i^t\log^2 y_i}{2\left(\sum\limits_{i=1}^{m} w_i x_i^t\log x_i-\sum\limits_{i=1}^{m} w_i y_i^t\log y_i\right)}+\sum\limits_{\substack{k=0\\k\neq t}}^{n-1}\dfrac{1}{k-t}\right), & t=q\in\{1,\ldots,n-1\}.\end{cases}$$

Similarly, one can give $\mu_{t,q}(\Omega_i,\Theta)$ $(i=2)$ from (28). Now using Theorem 14 we conclude that

$$\alpha\leqslant\left(\frac{\Omega_i(\phi_t)}{\Omega_i(\phi_q)}\right)^{\frac{1}{t-q}}\leqslant\beta,\quad i=1,2.$$

Hence $\mu_{t,q}(\Omega_i,\Theta)$ $(i=1,2)$ are means and their monotonicity is followed by (27).

We conclude the paper with the following remarks:

REMARK 2. One can also consider families of functions given in the last section of [4] to construct a large families of functions which are exponentially convex and new monotonic means.

REMARK 3. All the results given above can also be given in integral versions using Theorem 8.

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