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CONVERSES OF JESSEN'S INEQUALITY ON TIME SCALES II

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Abstract. We obtain new refinements of converse Jessen's inequality with respect to the multiple Lebesgue delta integral. The applicability of our results is illustrated in refinements of converse inequalities regarding monotonicity properties of generalized means, power means and some refinements of converse Hölder's inequality, which are all proved in the time scale setting.

1. Introduction

1.1. On time scale calculus

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [13] in 1988 as a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences, extending to cases "in between" and offering a formalism for studying hybrid discrete-continuous dynamic systems. It has applications in any field that requires simultaneous modelling of discrete and continuous time. Now, we briefly introduce the time scales calculus and refer to [1, 14, 15] and the books [6, 19] for further details.

By a time scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, the convention is $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). If $\sigma(t) > t$, then we say that t is *right-scattered*, and if $\rho(t) < t$, then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $\sigma(t) = t$, then t is said to be *right-dense*, and if $\rho(t) = t$, then t is said to be *left-dense*.

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Points that are simultaneously right-dense and left-dense are called *dense*. The mapping $\mu:\mathbb{T}\to[0,\infty)$ defined by

$$\mu(t)=\sigma(t)-t$$

is called the *graininess function*. If $\mathbb{T}$ has a left-scattered maximum M, then we denote $\mathbb{T}^\kappa=\mathbb{T}\setminus\{M\}$; otherwise $\mathbb{T}^\kappa=\mathbb{T}$. If $f:\mathbb{T}\to\mathbb{R}$ is a function, then we define the function $f^\sigma:\mathbb{T}\to\mathbb{R}$ by

$$f^\sigma(t)=f(\sigma(t))\quad\text{for all}\quad t\in\mathbb{T}.$$

In the following considerations, $\mathbb{T}$ will denote a time scale, $I_{\mathbb{T}}=I\cap\mathbb{T}$ will denote a time scale interval (for any open or closed interval I in $\mathbb{R}$), and $[0,\infty)_{\mathbb{T}}$ will be used for the time scale interval $[0,\infty)\cap\mathbb{T}$.

DEFINITION 1. Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighborhood U of t such that

$$\left|f(\sigma(t))-f(s)-f^\Delta(t)(\sigma(t)-s)\right|\leqslant\varepsilon\,|\sigma(t)-s|\quad\text{for all}\quad s\in U_{\mathbb{T}}.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t\in\mathbb{T}^\kappa$.

For all $t\in\mathbb{T}^\kappa$, we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^\Delta(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t iff the limit $\lim\limits_{s\to t}\frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\Delta(t)=\lim\limits_{s\to t}\frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t))=f(t)+\mu(t)f^\Delta(t)$.

DEFINITION 2. A function $f:\mathbb{T}\to\mathbb{R}$ is called *rd-continuous* if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f\in\mathrm{C}^1_{\mathrm{rd}}$) if $f^\Delta(t)$ exists for all $t\in\mathbb{T}^\kappa$ and $f^\Delta\in\mathrm{C}_{\mathrm{rd}}$.

DEFINITION 3. A function $F:\mathbb{T}\to\mathbb{R}$ is called a *delta antiderivative* of $f:\mathbb{T}\to\mathbb{R}$ if $F^\Delta(t)=f(t)$ for all $t\in\mathbb{T}^\kappa$. Then we define the *delta integral* by

$$\int_a^t f(s)\Delta s=F(t)-F(a).$$

The importance of rd-continuous function is revealed by the following result.

THEOREM 1. *Every rd-continuous function has a delta antiderivative.*

Now we give some properties of the delta integral.

THEOREM 2. *If* $a,b,c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ *and* $f,g \in \mathrm{C}_{\mathrm{rd}}$, *then*

(i) $\int_a^b (f(t)+g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^a f(t)\Delta t = 0$;

(vi) if $f(t) \geqslant 0$ *for all* t, *then* $\int_a^b f(t)\Delta t \geqslant 0$.

1.2. On positive linear functionals and time scale integrals

First we recall the following definition from [22].

DEFINITION 4. Let E be a nonempty set and L be a linear class of real-valued functions $f: E \to \mathbb{R}$ having the following properties.

(L_1) If $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$, then $(\alpha f + \beta g) \in L$.

(L_2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

A positive linear functional is a functional $A: L \to \mathbb{R}$ having the following properties.

(A_1) If $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$, then $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$.

(A_2) If $f \in L$ and $f(t) \geqslant 0$ for all $t \in E$, then $A(f) \geqslant 0$.

In [2, 3, 4, 10], the authors presented a series of inequalities for the time scale integral and showed that it is not necessary to prove that kind of inequalities "from scratch" in the time scale setting as they can be obtained easily from well-known inequalities for positive linear functionals since the time scale integral is in fact a positive linear functional. Consequently, the results on classical inequalities, whose generalizations for the positive linear functionals are given in the monograph [22], could be used for obtaining new inequalities for the time scale integral.

Now we quote three theorems from [2] that we need in our research.

THEOREM 3. *Let $\mathbb{T}$ be a time scale. For $a,b \in \mathbb{T}$ with $a < b$, let*

$$E = [a,b) \cap \mathbb{T} \quad \text{and} \quad L = \mathrm{C_{rd}}(E,\mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, delta integral

$$\int_a^b f(t)\Delta t,$$

is a positive linear functional which satisfies conditions (A_1) and (A_2).

Corresponding versions of Theorem 3 for nabla and α-diamond integrals are also given in [2].

Multiple Riemann integration and multiple Lebesgue integration on time scale was introduced in [8] and [9], respectively, and both integrals are also a positive linear functionals.

THEOREM 4. *Let $\mathbb{T}_1,\ldots,\mathbb{T}_n$ be time scales. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leqslant i \leqslant n$, let*

$$\mathscr{E} \subset ([a_1,b_1) \cap \mathbb{T}_1) \times \cdots \times ([a_n,b_n) \cap \mathbb{T}_n)$$

be Lebesgue Δ-measurable and let L be the set of all Δ-measurable functions from $\mathscr{E}$ to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, multiple Lebesgue delta integral on time scales

$$\int_{\mathscr{E}} f(t)\Delta t,$$

is positive linear functional and satisfies conditions (A_1) and (A_2).

THEOREM 5. *Under the assumptions of Theorem 4, delta integral*

$$\frac{\int_{\mathscr{E}} |h(t)| f(t)\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t},$$

where $h : \mathscr{E} \to \mathbb{R}$ is Δ-integrable and $\int_{\mathscr{E}} |h(t)|\Delta t > 0$, is also a positive linear functional satisfying (A_1), (A_2) and $A(1) = 1$.

1.3. On Jessen and Lah–Ribarič inequalities

Using the known Jessen inequality for positive linear functionals ([22, Theorem 2.4]) and Theorem 5, M. Anwar, R. Bibi, M. Bohner and J. Pečarić proved in [2] the following generalization of Jessen's inequality on time scales.

THEOREM 6. *Assume $\phi \in C(I,\mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval. Let $\mathscr{E} \subset \mathbb{R}^n$ be as in Theorem 4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}} |h(t)|\Delta t > 0$. Then*

$$\phi\left(\frac{\int_{\mathscr{E}} |h(t)|f(t)\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t}\right) \leqslant \frac{\int_{\mathscr{E}} |h(t)|\phi(f(t))\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t}.$$

Lah and Ribarič proved in [20] the converse of Jensen's inequality for convex functions (see also [21]). Beesack and Pečarić gave in [7] the generalization of Lah–Ribarič's inequality for positive linear functionals. Applying the fact that the multiple Lebesgue delta time scale integral is a positive linear functional (Theorem 5) to Beesack–Pečarić's result from [7], the following theorem is proved in [2].

THEOREM 7. *Assume $\phi \in C(I,\mathbb{R})$ is convex, where $I = [m,M] \subset \mathbb{R}$, with $m < M$. Let $\mathscr{E} \subset \mathbb{R}^n$ be as in Theorem 4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}} |h(t)|\Delta t > 0$. Then*

$$\frac{\int_{\mathscr{E}} |h(t)|\phi(f(t))\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t} \leqslant \frac{M - \frac{\int_{\mathscr{E}} |h(t)|f(t)\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t}}{M-m}\phi(m) + \frac{\frac{\int_{\mathscr{E}} |h(t)|f(t)\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t} - m}{M-m}\phi(M). \tag{1}$$

Recently, in their paper [16], R. Jakšić and J. Pečarić presented new converses of the Jessen and Lah-Ribarič inequalities. Now, we quote their main result.

THEOREM 8. *Let ϕ be a continuous convex function on an interval of real numbers I and $m,M \in \mathbb{R}$, $m < M$, with $[m,M] \subset Int(I)$, where $Int(I)$ is the interior of I. Let L satisfy conditions (L_1), (L_2) on $\mathscr{E}$ and let A be any positive linear functional on L with $A(1) = 1$. If $f \in L$ satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \qquad \text{for every} \qquad t \in \mathscr{E}$$

and $\phi \circ f \in L$, then

$$\begin{aligned} 0 &\leqslant A(\phi(f)) - \phi(A(f)) \\ &\leqslant (M - A(f))(A(f) - m) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t;m,M) \\ &\leqslant (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \end{aligned} \tag{2}$$

where $\phi'_-(M) = \lim_{x \to M^-} \frac{\phi(x)-\phi(M)}{x-M}$ is a left hand derivate of ϕ at M, and $\phi'_+(M) = \lim_{x \to M^+} \frac{\phi(x)-\phi(M)}{x-M}$ is a right hand derivate of ϕ at M, $x \in I$. We also have the inequalities

$$0 \leqslant A(\phi(f)) - \phi(A(f)) \leqslant \frac{1}{4}(M-m)^2 \Psi_\phi(A(f);m,M)$$
$$\leqslant \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \tag{3}$$

where $\Psi_\Phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big).$$

If ϕ *is concave on* I*, then all inequalities in* (2) *and* (3) *are reversed.*

2. New results

In this section, we prove converses of Jessen's inequality on time scales which refine the results given in [5] for multiple Lebesgue delta integral. For simplicity, we introduce the following notations

$$L_\Delta(f) = \int_{\mathscr{E}} f(t)\Delta t \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathscr{E}} f(t)|h(t)|\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t},$$

where $f,h : \mathscr{E} \to \mathbb{R}$ are Δ-integrable and $\int_{\mathscr{E}} |h(t)|\Delta t > 0$.

THEOREM 9. *Let* $\phi \in C(I,\mathbb{R})$ *be convex, where* $I = [m,M] \subset \mathbb{R}$*, with* $m < M$*. Assume* $\mathscr{E} \subset \mathbb{R}^n$ *is as in Theorem 4 and suppose* f *is* Δ*-integrable on* $\mathscr{E}$ *such that* $f(\mathscr{E}) = I$*. Moreover, let* $h : \mathscr{E} \to \mathbb{R}$ *be* Δ*-integrable such that* $\int_{\mathscr{E}} |h(t)|\Delta t > 0$*. Then*

$$\begin{aligned}
0 &\leqslant \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\
&\leqslant \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\
&\leqslant \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \cdot \frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\
&\leqslant \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)),
\end{aligned} \tag{4}$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big).$$

If ϕ *is concave on* I*, then all inequalities in* (4) *are reversed.*

Proof. Since ϕ is a convex function, first inequality in (4) follows from Theorem 6. From Theorem 7, we have

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\
&\leqslant \frac{M - \overline{L}_\Delta(f,h)}{M-m}\phi(m) + \frac{\overline{L}_\Delta(f,h) - m}{M-m}\phi(M) - \phi\left(\overline{L}_\Delta(f,h)\right)
\end{aligned} \tag{5}$$

$$
\begin{aligned}
&= \frac{1}{M-m}\left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\\
&\quad\times\left(\frac{\phi(M)-\phi\left(\overline{L}_\Delta(f,h)\right)}{M-\overline{L}_\Delta(f,h)}-\frac{\phi\left(\overline{L}_\Delta(f,h)\right)-\phi(m)}{\overline{L}_\Delta(f,h)-m}\right)\\
&= \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\Psi_\phi\left(\overline{L}_\Delta(f,h);m,M\right)\\
&\leqslant \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M),
\end{aligned}
$$

which is the second inequality in (4), provided that $\overline{L}_\Delta(f,h)\neq m,M$. When $\overline{L}_\Delta(f,h)$ is equal to m or M then inequality (4) is obvious.

Since,

$$
\begin{aligned}
&\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)=\frac{1}{M-m}\sup_{t\in\langle m,M\rangle}\left\{\frac{\phi(M)-\phi(t)}{M-t}-\frac{\phi(t)-\phi(m)}{t-m}\right\}\\
&\leqslant \frac{1}{M-m}\Big(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t}+\sup_{t\in\langle m,M\rangle}\frac{-(\phi(t)-\phi(m))}{t-m}\Big)\\
&= \frac{1}{M-m}\Big(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t}-\inf_{t\in\langle m,M\rangle}\frac{\phi(t)-\phi(m)}{t-m}\Big)=\frac{\phi'_-(M)-\phi'_+(m)}{M-m},
\end{aligned}
$$

the third inequality in (4) is true. The last inequality in (4) follows from the elementary estimate $\frac{(M-x)(x-m)}{M-m}\leqslant\frac{1}{4}(M-m)$, for every $x\in\mathbb{R}$. If the function ϕ is concave, then $-\phi$ is convex, so applying (4) to $-\phi$ gives the reversed inequalities in (4). This complete the proof. □

REMARK 1. According to (5), with the same assumptions as in Theorem 9, following inequalities are also true

$$
\begin{aligned}
0 &\leqslant \overline{L}_\Delta(\phi(f),h)-\phi\left(\overline{L}_\Delta(f,h)\right)\\
&\leqslant \frac{1}{4}(M-m)^2\Psi_\phi\left(\overline{L}_\Delta(f,h);m,M\right)\\
&\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned}
$$

3. Applications

In this section we use the results from Theorem 9 to get new converse inequalities for generalized means, power means and the Hölder inequality in the time scale setting.

3.1. Generalized means

Applying classical results to the monotonicity properties of generalized means with respect to the functional A, found in [12, p. 75, Theorem 92] and [22, p. 108, Theorem 4.3], R. Jakšić and J. Pečarić proved in [16] the following converse.

THEOREM 10. *Let L satisfy (L_1), (L_2) and A satisfy (A_1), (A_2) and $A(1)=1$. Suppose $\psi,\chi: I\to\mathbb{R}$ are continuous and strictly monotone and $\phi=\chi\circ\psi^{-1}$ is convex, where $I\supset[m,M]$, $-\infty<m<M<\infty$. Then, for every $f\in L$ such that $m\leqslant f(t)\leqslant M$, $t\in[m,M]$ and $\psi(f),\chi(f)\in L$, we have*

$$\begin{aligned}
0 &\leqslant \chi(M_\chi(f,A))-\chi(M_\psi(f,A))\\
&\leqslant (M_\psi-A(\psi(f)))(A(\psi(f))-m_\psi)\sup_{t\in\langle m,M\rangle}\Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi)\\
&\leqslant (M_\psi-A(\psi(f)))(A(\psi(f))-m_\psi)\frac{[\chi\circ\psi^{-1}]'_-(M_\psi)-[\chi\circ\psi^{-1}]'_+(m_\psi)}{M_\psi-m_\psi}\\
&\leqslant \frac{1}{4}(M_\psi-m_\psi)([\chi\circ\psi^{-1}]'_-(M_\psi)-[\chi\circ\psi^{-1}]'_+(m_\psi)),
\end{aligned}\tag{6}$$

where $[m_\psi,M_\psi]=\psi([m,M])$ and $M_\psi(f,A)=\psi^{-1}(A(\psi(f)))$ is a generalized mean with respect to the operator A and function ψ. If ϕ is concave, then all inequalities in (6) are reversed.

Using the generalized mean on time scales, with respect to the multiple Lebesgue delta integral ([5]), from Theorem 9 and Theorem 10 we deduce the following result.

THEOREM 11. *Suppose $I=[m,M]$, $-\infty<m<M<\infty$, $\psi,\chi: I\to\mathbb{R}$ are continuous and strictly monotone and $\phi=\chi\circ\psi^{-1}$ is convex. Assume $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 4 and $f,h:\mathscr{E}\to\mathbb{R}$ are Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=I$ and $\int_{\mathscr{E}}|h(t)|\Delta t>0$. Then,*

$$\begin{aligned}
0 &\leqslant \chi\left(M_\chi\left(f,\overline{L}_\Delta\right)\right)-\chi\left(M_\psi\left(f,\overline{L}_\Delta\right)\right)\\
&\leqslant \left(M_\psi-\overline{L}_\Delta(\psi(f),h)\right)\left(\overline{L}_\Delta(\psi(f),h)-m_\psi\right)\sup_{t\in\langle m,M\rangle}\Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi)\\
&\leqslant \left(M_\psi-\overline{L}_\Delta(\psi(f),h)\right)\left(\overline{L}_\Delta(\psi(f),h)-m_\psi\right)\\
&\quad\times\frac{(\chi\circ\psi^{-1})'_-(M_\psi)-(\chi\circ\psi^{-1})'_+(m_\psi)}{M_\psi-m_\psi}\\
&\leqslant \frac{1}{4}\left(M_\psi-m_\psi\right)\left((\chi\circ\psi^{-1})'_-(M_\psi)-(\chi\circ\psi^{-1})'_+(m_\psi)\right),
\end{aligned}\tag{7}$$

where $[m_\psi,M_\psi]=\psi([m,M])$. If ϕ is concave, then all inequalities in (7) are reversed.

Proof. The claim follows from Theorem 4, Theorem 9 and Theorem 10. □

3.2. Power means

The following result on power means with respect to a positive linear functional is proved in [16].

THEOREM 12. *Let L satisfy (L_1), (L_2) and A satisfy (A_1), (A_2) with $A(1)=1$. Let $0<m\leqslant f(t)\leqslant M<\infty$ for $t\in E$, $f^r, f^s, \log f\in L$ for $r,s\in\mathbb{R}$, $r<s$ and let*

$$\phi(t)=\begin{cases} t^{s/r} & : r\neq 0, s\neq 0,\\ \frac{1}{r}\log t & : r\neq 0, s=0,\\ e^{st} & : r=0, s\neq 0.\end{cases} \tag{8}$$

If $0<r<s$ or $r<0<s$, then

$$\begin{aligned} 0 &\leqslant (M^{[s]}(f,A))^s-(M^{[r]}(f,A))^s\\ &\leqslant (M^r-A(f^r))(A(f^r)-m^r)\sup_{t\in\langle m,M\rangle}\Psi_\Phi(t^r;m^r,M^r)\\ &\leqslant \frac{s}{r}(M^r-A(f^r))(A(f^r)-m^r)\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\\ &\leqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}). \end{aligned} \tag{9}$$

If $r<s<0$, then

$$\begin{aligned} 0 &\geqslant (M^{[s]}(f,A))^s-(M^{[r]}(f,A))^s\\ &\geqslant (M^r-A(f^r))(A(f^r)-m^r)\sup_{t\in\langle m,M\rangle}\Psi_\Phi(t^r;m^r,M^r)\\ &\geqslant \frac{s}{r}(M^r-A(f^r))(A(f^r)-m^r)\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\\ &\geqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}). \end{aligned} \tag{10}$$

If $s=0$ and $r<0$, then

$$\begin{aligned} 0 &\leqslant \log(M^{[0]}(f,A))-\log(M^{[r]}(f,A))\\ &\leqslant (M^r-A(f^r))(A(f^r)-m^r)\sup_{t\in\langle m,M\rangle}\Psi_\Phi(t^r;M^r,m^r)\\ &\leqslant -\frac{1}{r}\cdot\frac{(M^r-A(f^r))(A(f^r)-m^r)}{M^r m^r}\\ &\leqslant \frac{1}{4r}(m^r-M^r)\left(\frac{1}{m^r}-\frac{1}{M^r}\right). \end{aligned} \tag{11}$$

If $r=0$ and $s>0$, then

$$\begin{aligned} 0 &\leqslant (M^{[s]}(f,A))^s-(M^{[0]}(f,A))^s\\ &\leqslant (\log M-A(\log f))(A(\log f)-\log m)\sup_{t\in\langle m,M\rangle}\Psi_\Phi(\log t;\log m,\log M)\\ &\leqslant s(\log M-A(\log f))(A(\log f)-\log m)\frac{M^s-m^s}{\log M-\log m}\\ &\leqslant s(M^s-m^s)\log\frac{M}{m}. \end{aligned} \tag{12}$$

According to definition of power mean on time scales with respect of the multiple Lebesgue delta integral ([5]), from the Theorem 12 we derive the following result.

THEOREM 13. *Suppose* $\mathscr{E} \subset \mathbb{R}^n$ *is as in Theorem 4,* f *is* Δ*-integrable on* $\mathscr{E}$ *such that* $f(\mathscr{E}) = I$ *and* $0 < m \leqslant f(t) \leqslant M < \infty$*, for* $t \in \mathscr{E}$*,* $m, M \in \mathbb{R}$*. Let* $h : \mathscr{E} \to \mathbb{R}$ *be* Δ*-integrable such that* $\int_{\mathscr{E}} |h(t)|\Delta t > 0$*. For* $r, s \in \mathbb{R}$ *suppose* f^r*,* f^s*,* $(\log f)$ *are* Δ*-integrable on* $\mathscr{E}$*.*

(i) If $0 < r < s$ *or* $r < 0 < s$*, then*

$$\begin{aligned} 0 &\leqslant \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right)^s \\ &\leqslant \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r; m^r, M^r) \\ &\leqslant \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\leqslant \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right). \end{aligned} \tag{13}$$

(ii) If $r < s < 0$*, then*

$$\begin{aligned} 0 &\geqslant \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right)^s \\ &\geqslant \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r; m^r, M^r) \\ &\geqslant \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\geqslant \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right). \end{aligned} \tag{14}$$

(iii) If $s = 0$ *and* $r < 0$*, then*

$$\begin{aligned} 0 &\leqslant \log\left(M^{[0]}\left(f,\overline{L}_\Delta\right)\right) - \log\left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right) \\ &\leqslant \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r; M^r, m^r) \\ &\leqslant -\frac{1}{r} \cdot \frac{\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)}{M^r m^r} \\ &\leqslant -\frac{1}{4r} \cdot \frac{(M^r - m^r)^2}{M^r m^r}. \end{aligned} \tag{15}$$

(iv) If $r = 0$ *and* $s > 0$*, then*

$$\begin{aligned} 0 &\leqslant \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[0]}\left(f,\overline{L}_\Delta\right)\right)^s \\ &\leqslant \left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right) \\ &\quad \times \sup_{t\in\langle m,M\rangle} \Psi_\phi(\log t; \log m, \log M) \end{aligned}$$

$$\leqslant \left(\log M - \overline{L}_\Delta(\log f, h)\right)\left(\overline{L}_\Delta(\log f, h) - \log m\right) \cdot \frac{s(M^s - m^s)}{\log M - \log m} \tag{16}$$
$$\leqslant s\,(M^s - m^s)\log\frac{M}{m}.$$

Proof. The claim follows from Theorem 4, Theorem 9 and Theorem 12. □

3.3. Hölder's inequality

The following theorem gives Hölder's inequality for delta time scale integrals proved in [2].

THEOREM 14. *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 4 and $|w||f|^p$, $|w||g|^q$, $|wfg|$ are Δ-integrable on $\mathscr{E}$, where $p > 1$ and $q = p/(p-1)$. Then,*

$$\int_{\mathscr{E}} |w(t)f(t)g(t)|\Delta t \leqslant \left(\int_{\mathscr{E}} |w(t)||f(t)|^p\Delta t\right)^{\frac{1}{p}}\left(\int_{\mathscr{E}} |w(t)||g(t)|^q\Delta t\right)^{\frac{1}{q}}.$$

This inequality is reversed if $0 < p < 1$ and $\int_{\mathscr{E}} |w(t)||g(t)|^q\Delta t > 0$, and it is also reversed if $p < 0$ and $\int_{\mathscr{E}} |w(t)||f(t)|^p\Delta t > 0$.

Next, we refine the converse Hölder's inequalities on time scales, proved in [5].

THEOREM 15. *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 4 and w, f, g are real functions on $\mathscr{E}$ such that $w, f, g \geqslant 0$. For $m, M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ let $m \leqslant f(t)g^q(t) \leqslant M$, $t \in \mathscr{E}$. If wf^p, wg^q, wfg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} w(t)g^q(t)\Delta t > 0$, where $p > 1$ and $q = p/(p-1)$, then*

$$0 \leqslant \int_{\mathscr{E}} w(t)f^p(t)\Delta t \cdot \left(\int_{\mathscr{E}} w(t)g^q(t)\Delta t\right)^{\frac{p}{q}} - \left(\int_{\mathscr{E}} w(t)f(t)g(t)\Delta t\right)^p$$
$$\leqslant \left(M\int_{\mathscr{E}} w(t)g^q(t)\Delta t - \int_{\mathscr{E}} w(t)f(t)g(t)\Delta t\right)\left(\int_{\mathscr{E}} w(t)f(t)g(t)\Delta t - m\int_{\mathscr{E}} w(t)g^q(t)\Delta t\right)$$
$$\times \sup_{t\in\langle m,M\rangle} \Psi_\phi(t; m, M) \cdot \left(\int_{\mathscr{E}} w(t)g^q(t)\Delta t\right)^{p-2} \tag{17}$$
$$\leqslant \left(M\int_{\mathscr{E}} w(t)g^q(t)\Delta t - \int_{\mathscr{E}} w(t)f(t)g(t)\Delta t\right)\left(\int_{\mathscr{E}} w(t)f(t)g(t)\Delta t - m\int_{\mathscr{E}} w(t)g^q(t)\Delta t\right)$$
$$\times \frac{p(M^{p-1} - m^{p-1})}{M - m} \cdot \left(\int_{\mathscr{E}} w(t)g^q(t)\Delta t\right)^{p-2}$$

$$\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\left(\int_{\mathscr{E}} w(t)g^q(t)\Delta t\right)^p.$$

For $p<0$, inequalities (17) hold if $\int_{\mathscr{E}} w(t)f(t)g(t)\Delta t>0$, $t\in\mathscr{E}$. In case $0<p<1$, all inequalities in (17) are reversed.

Proof. Inequalities (17) follow directly from Theroem 9 by taking the function ϕ to be of the form $\phi(t)=t^p$ and replacing h by wg^q and f by $fg^{-\frac{q}{p}}$. For $p<0$ and $p>1$, the function t^p is convex and inequalities (17) follow from inequalities (4). For $0<p<1$, the function t^p is concave and, according to Theorem 9, all inequalities in (17) will be reversed. □

THEOREM 16. *Assume $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 4 and $f,g\geqslant 0$ such that f^p, g^q, fg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} g^q(t)\Delta t>0$, where $0<p<1$ and $q=p/(p-1)$. For $m,M\in\mathbb{R}$ such that $-\infty<m<M<\infty$, let $m\leqslant f(t)g^{-q}(t)\leqslant M$, $t\in\mathscr{E}$. Then,*

$$\begin{aligned}
0&\leqslant \int_{\mathscr{E}} f(t)g(t)\Delta t-\left(\int_{\mathscr{E}} f^p(t)\Delta t\right)^{\frac{1}{p}}\left(\int_{\mathscr{E}} g^q(t)\Delta t\right)^{\frac{1}{q}}\\
&\leqslant \frac{1}{\int_{\mathscr{E}} g^q(t)\Delta t}\left(M\int_{\mathscr{E}} g^q(t)\Delta t-\int_{\mathscr{E}} f^p(t)\Delta t\right)\cdot\left(\int_{\mathscr{E}} f^p(t)\Delta t-m\int_{\mathscr{E}} g^q(t)\Delta t\right)\\
&\quad\times\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \qquad (18)\\
&\leqslant \frac{1}{p}\cdot\frac{M^{-\frac{1}{q}}-m^{-\frac{1}{q}}}{M-m}\cdot\frac{1}{\int_{\mathscr{E}} g^q(t)\Delta t}\left(M\int_{\mathscr{E}} g^q(t)\Delta t-\int_{\mathscr{E}} f^p(t)\Delta t\right)\\
&\quad\times\left(\int_{\mathscr{E}} f^p(t)\Delta t-m\int_{\mathscr{E}} g^q(t)\Delta t\right)\\
&\leqslant \frac{1}{4p}(M-m)\left(M^{-\frac{1}{q}}-m^{-\frac{1}{q}}\right)\int_{\mathscr{E}} g^q(t)\Delta t.
\end{aligned}$$

For $p<0$, inequalities (18) hold if $\int_{\mathscr{E}} f^p(t)\Delta t>0$, $t\in\mathscr{E}$. In case $p>1$, all inequalities in (18) are reversed.

Proof. Inequalities (18) follow directly from Theroem 9 by taking the function ϕ to be of the form $\phi(t)=t^{\frac{1}{p}}$ and replacing h by g^q and f by $\frac{f^p}{g^q}$. Namely, when $p<1$, the function $t^{\frac{1}{p}}$ is convex and inequalities (18) follow from inequalities (4). For $p>1$, the function t^p is concave and, according to Theorem 9, all inequalities in (18) will be reversed. □

THEOREM 17. *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 4 and $f,g \geqslant 0$ such that g^q, fg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} g^q(t)\Delta t > 0$, where $p<0$ or $p>1$ and $q = p/(p-1)$. Let $m,M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ and $m \leqslant f(t)g^{1-q}(t) \leqslant M$, $t \in \mathscr{E}$. Then,*

$$
\begin{aligned}
0 &\leqslant \int_{\mathscr{E}} f^p(t)\Delta t \cdot \left(\int_{\mathscr{E}} g^q(t)\Delta t\right)^{\frac{p}{q}} - \left(\int_{\mathscr{E}} f(t)g(t)\Delta t\right)^p \\
&\leqslant \left(M\int_{\mathscr{E}} g^q(t)\Delta t - \int_{\mathscr{E}} f(t)g(t)\Delta t\right)\left(\int_{\mathscr{E}} f(t)g(t)\Delta t - m\int_{\mathscr{E}} g^q(t)\Delta t\right) \\
&\quad \times \left(\int_{\mathscr{E}} g^q(t)\Delta t\right)^{p-2} \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \qquad (19) \\
&\leqslant \left(M\int_{\mathscr{E}} g^q(t)\Delta t - \int_{\mathscr{E}} f(t)g(t)\Delta t\right)\left(\int_{\mathscr{E}} f(t)g(t)\Delta t - m\int_{\mathscr{E}} g^q(t)\Delta t\right) \\
&\quad \times \frac{p(M^{p-1}-m^{p-1})}{M-m} \cdot \left(\int_{\mathscr{E}} g^q(t)\Delta t\right)^{p-2} \\
&\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\left(\int_{\mathscr{E}} g^q(t)\Delta t\right)^p .
\end{aligned}
$$

In case $0<p<1$, all inequalities in (19) are reversed.

Proof. Inequalities (19) follow directly from Theroem 9 by taking the function ϕ to be of the form $\phi(t) = t^p$ and replacing h by g^q and f by fg^{1-q}. Namely, for $p<0$ or $p>1$, the function t^p is convex and inequalities (19) follow from inequalities (4). For $0<p<1$, the function t^p is concave and, according to Theorem 9, all inequalities in (19) will be reversed. □

3.4. Additional improvements

Recently, R. Jakšić and J. E. Pečarić proved in [17] new refinement of the converse Jensen inequality for normalized positive linear functionals, given in Theorem 8. Using that result, we derive the following theorem which refines inequality (4) from Theorem 9.

THEOREM 18. *Let $\phi \in C(I,\mathbb{R})$ be convex, where $I = [m,M] \subset \mathbb{R}$, with $m<M$. Assume $\mathscr{E} \subset \mathbb{R}^n$ and L are as in Theorem 4 with additional property that for every $f,g \in L$ we have that $\min\{f,g\} \in L$ and $\max\{f,g\} \in L$. Let f be Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}} |h(t)|\Delta t > 0$.*

Then,

$$
\begin{aligned}
0 &\leqslant \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\
&\leqslant \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - \overline{L}_\Delta(\tilde{f},h)\delta_\phi \\
&\leqslant \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \cdot \frac{\phi_-'(M) - \phi_+'(m)}{M-m} - \overline{L}_\Delta(\tilde{f},h)\delta_\phi \qquad (20)\\
&\leqslant \frac{1}{4}(M-m)(\phi_-'(M) - \phi_+'(m)) - \overline{L}_\Delta(\tilde{f},h)\delta_\phi,
\end{aligned}
$$

where

$$\tilde{f} = \tfrac{1}{2} - \tfrac{|f - \frac{m+M}{2}|}{M-m}, \quad \delta_\phi = \phi(m) + \phi(M) - 2\phi\left(\tfrac{m+M}{2}\right)$$

and $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \tfrac{1}{M-m}\left(\tfrac{\phi(M)-\phi(t)}{M-t} - \tfrac{\phi(t)-\phi(m)}{t-m}\right).$$

If ϕ is concave on I, then the above inequalities are reversed.

Proof. Inequality (20) follows directly from the main result of [17] and the fact that multiple Lebesgue delta integral is a positive linear functional. □

REMARK 2. Using the resoning shown in Theorem 18, the refinements of all inequalities proved in this paper in Theorem 10, Theorem 11, Theorem 12 and Theorem 13 can be obtained.

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