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ON WEIGHTED INTEGRAL EQUATIONS WITH NEGATIVE EXPONENTS

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Abstract. This paper is concerned with the integral equation

$$u(x)=\int_{R^n}|x-y|^p k(y)u^q(y)dy,$$

where $n\geqslant 1$, $p\neq 0$, $q<0$, and the weighted function $k(x)$ is smooth. This equation comes from the prescribing curvature problems. In addition, it is related to the best functions of the reversed Hardy-Littlewood-Sobolev inequality. We consider the existence and the estimates of increasing rates of the positive entire solutions in the case that $k(x)$ is bounded and unbounded respectively.

1. Introduction

In 2004, Li [9] introduced an integral equation

$$u(x)=\int_{R^n}|x-y|^p u^q(y)dy \tag{1.1}$$

with $q<0$ which comes from the conformal geometry, and posed whether or not does (1.1) admit any positive (regular) solutions for all $n\geqslant 1$, $p>0$ and $q<-1-2n/p$. In 2007, Xu [13] studied this problem and obtained the positive answers: Eq. (1.1) has a C^1 positive solution if and only if $q=-(p+2n)/p$. Now, u is given by

$$u(x)=a(b^2+|x-x_0|^2)^{-\lambda/2} \tag{1.2}$$

with $a,b>0$ and $x_0\in R^n$. In addition, if $-n<p<0$ and $q<0$, then (1.1) has no C^1 positive solution.

Also Xu introduced another integral equation

$$u(x)=\int_{R^n}|x-y|^p k(y)u^{-(1+2n/p)}(y)dy \tag{1.3}$$

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for any given smooth function $k(x)$ on R^n. It is associated with the study of the Kazdan-Warner conditions in the study of the prescribing curvature problems (cf. [6]). He showed that if $u>0$ is a smooth solution of the integral equation, then the identity

$$\int_{R^n}[x\cdot\nabla k(x)]u^{-2n/p}(x)dx=0 \tag{1.4}$$

holds as long as $k(x)\geqslant a>0$ is a bounded smooth function such that $|\nabla k(x)|\leqslant c|x|^{p-1}$ for some positive constant c and $|x|$ sufficiently large. In 2015, to investigate the best functions of the reversed Hardy-Littlewood-Sobolev inequality (cf. [5]), Lei studied more general integral system of the Euler-Lagrange equations with variational coefficients in [8]

$$\begin{cases} u(x)=c_1(x)\displaystyle\int_{R^n}\frac{|x-y|^\lambda dy}{v^q(y)}, & u>0 \text{ in } R^n, \\ v(x)=c_2(x)\displaystyle\int_{R^n}\frac{|x-y|^\lambda dy}{u^p(y)}, & v>0 \text{ in } R^n, \end{cases} \tag{1.5}$$

where $p,q,\lambda>0$, and $c_1(x)$, $c_2(x)$ are double bounded functions. A function $k(x)$ is called double bounded, if there is $C>1$ such that $C^{-1}\leqslant k(x)\leqslant C$ for all $x\in R^n$.

If we replaced the critical exponent $-(1+2n/p)$ by a general real number q, then (1.3) becomes

$$u(x)=\int_{R^n}|x-y|^p k(y)u^q(y)dy. \tag{1.6}$$

In this paper, we investigate the existence/nonexistence and the asymptotic behavior of the positive solutions of (1.6).

THEOREM 1. *(1) If $-n<p<0$ and $q<0$, $k(x)$ is a double bounded function. Then (1.6) has no continuous positive solution.*

(2) If $p>0$, and $q<-(1+\frac{n}{p})$, then there exists a radial solution of (1.6) for some double bounded function $k(x)$.

REMARK 1.1. For system (1.5), Lei also pointed out in [8] that when $\lambda>0$ and the Serrin type condition $\min\{p,q\}>(n+\lambda)/\lambda$ holds, there exists a radial solution for some double bounded functions $c_1(x),c_2(x)$. For single equations, the exponent $-(1+\frac{n}{p})$ is a Serrin's exponent. Different from the integral equations with positive exponents, the Serrin's exponents here are difficult to be verified that they are critical.

THEOREM 2. *Let $p>0$, $q<0$, and $k\in L^\infty(R^n)\cap C^1(R^n)$ be a positive function. If u is a continuous positive solution of (1.6), then*

(1) $u(x)\simeq|x|^p$ when $|x|\to\infty$. Here $u(x)\simeq|x|^\theta$ means that there exists $C\geqslant 1$ such that $C^{-1}|x|^\theta\leqslant u(x)\leqslant C|x|^\theta$ when $|x|\to\infty$.

(2) The exponent q is critical (i.e. $q=-(1+2n/p)$) if and only if

$$\int_{R^n}[x\cdot\nabla k(x)]u^{1+q}(x)dx=0. \tag{1.7}$$

REMARK 1.2. (1) Lei also studied system (1.5) and proved in [8] that if $\lambda > 0$ and $max\{p,q\} > (n+\lambda)/\lambda$, then u,v are increasing with the fast rate λ: $u(x), v(x) \simeq |x|^{\lambda}$ as $|x| \to \infty$. When $p = q$ and $u \equiv v$, those rates are same as the result of (1) in Theorem 2. In this paper we employ different idea from the paper [8]. In addition, if $p < 0$ and $q > 0$ in (1.6), or $p,q,\lambda < 0$ in (1.5), then these increasing rates become the corresponding fast decay rates (cf. [12] and the references therein).

(2) Eq. (1.4) is the same as (1.7) in the critical case. In addition, if (1.7) is true, then (1.6) has continuous positive solution if and only if q is the critical exponent $-(1+2n/p)$.

Next we consider the unbounded weight $k(x)$. As a representative example, $k(x)$ can be taken a radial power function. Let $k(x) = |x|^l$ in (1.6), then equation (1.6) turns to

$$u(x) = \int_{R^n} |x-y|^p |y|^l u^q(y) dy. \tag{1.8}$$

When $p < 0$, $q > 0$ and $l < 0$, this equation is related to the best constant of the Hardy-Sobolev inequality (cf. [7] and [11]). In addition, when $n \geqslant 3$ and $p = 2-n$, it appears in the study of differential geometry (cf. [10] and the reference therein). When $l > 0$, it is associated with the study of the fractional order Henon equations (cf. [1] and [2] with $n \geqslant 3$ and $p = 2-n$ particularly).

THEOREM 3. *Eq. (1.8) has no continuous positive solution when either*
(1) $p > 0$, $q > 0$ *and* $l \geqslant 0$*; or*
(2) $-n-l < p < 0$ *and* $q < 0$.

THEOREM 4. *Assume that* u *is a continuous positive solution of (1.8). Then* $u(x) \simeq |x|^p$ *when* $|x| \to \infty$ *if either*
(1) $-n < l \leqslant 0$, $p > 0$, $q < -\frac{n+p}{p}$*; or*
(2) $p > 0$, $q < -\frac{n+p+l}{p}$ *and* $l > 0$.

2. Results on (1.6)

Proof of Theorem 1. (1) First, by Lemma 3.11.3 in [14], we have that for all $r > 0$,

$$\begin{aligned} \frac{1}{w_n r^n} \int_{B_r(0)} u(x) dx &= \int_{R^n} \left\{ \frac{1}{w_n r^n} \int_{B_r(0)} |x-y|^p dx \right\} k(y) u^q(y) dy \\ &\leqslant C \int_{R^n} |y|^p k(y) u^q(y) dy = Cu(0), \end{aligned} \tag{2.1}$$

where C is a constant depending only on p and n and independent of r.

Using the Hölder inequality, we obtain

$$\begin{aligned} 1 &= \frac{1}{w_n r^n} \int_{B_r(0)} u^{-\alpha}(x) u^{\alpha}(x) dx \\ &\leqslant \left\{ \frac{1}{w_n r^n} \int_{B_r(0)} [u^{-\alpha}(x)]^{\beta} dx \right\}^{\frac{1}{\beta}} \left\{ \frac{1}{w_n r^n} \int_{B_r(0)} u^{\alpha\delta}(x) dx \right\}^{\frac{1}{\delta}}. \end{aligned} \tag{2.2}$$

Let α,β,δ satisfy $\alpha\beta=-q$, $\frac{1}{\beta}+\frac{1}{\delta}=1$ and $\alpha\delta=1$. We get $\alpha=\frac{q}{q-1}$, $\beta=1-q$, $\delta=\frac{q-1}{q}$. Thus we have

$$C^q u^q(0)\leqslant \frac{1}{w_n r^n}\int_{B_r(0)} u^q(x)dx. \tag{2.3}$$

Since $p<0$, if $|x|<r$, then $r^p<|x|^p$. Therefore by multiplying both sides of (2.3) by $w_n r^{n+p}$, we get

$$\begin{aligned} C^q w_n r^{n+p} u^q(0) &\leqslant r^p\int_{B_r(0)} u^q(x)dx \\ &\leqslant \int_{B_r(0)} |x|^p u^q(x)dx \leqslant c\int_{B_r(0)} |x|^p k(x)u^q(x)dx = cu(0). \end{aligned} \tag{2.4}$$

Noticing that $n+p>0$, and letting $r\to\infty$, we reach a contradiction.

(2) Set

$$u(x)=(1+|x|^2)^{p/2}. \tag{2.5}$$

Clearly, when $|x|\leqslant 2R$ for $R>0$, we can find $c>1$ such that

$$c^{-1}\leqslant u(x)\leqslant c. \tag{2.6}$$

Write $I(x):=\int_{R^n}|x-y|^p u^q(y)dy$. By (2.6) it follows that

$$I(x)\geqslant \int_{B_{2R}(x)\setminus B_R(x)} |x-y|^p u^q(y)dy \geqslant c^{-1}.$$

On the other hand,

$$\int_{B_R(x)} |x-y|^p u^q(y)dy\leqslant c.$$

Noting $q<-(1+n/p)$, we see that

$$\int_{B_R^c(x)} |x-y|^p u^q(y)dy\leqslant c.$$

Thus $c^{-1}\leqslant I(x)\leqslant c$. Combining with (2.6) yields $c^{-1}\leqslant \frac{u(x)}{I(x)}\leqslant c$. We set $R(x)=\frac{u(x)}{I(x)}$, then $R(x)$ is double bounded, and

$$u(x)=R(x)\int_{R^n}|x-y|^p u^q(y)dy.$$

Write $w(x)=\frac{u(x)}{R(x)}$ and $k(x)=R^q(x)$, then $k(x)$ is a double bounded function, and $w(x)$ is a solution of (1.6).

Next, we consider the case of $|x|>2R$.

By the condition of Theorem 1, we see

$$q<-\frac{n+p}{p}. \tag{2.7}$$

Clearly,

$$\int_{R^n}|x-y|^p u^q(y)dy = \int_{B_1(0)}|x-y|^p u^q(y)dy + \int_{B_{2|x|}(0)\setminus B_1(0)}|x-y|^p u^q(y)dy$$
$$+\int_{R^n\setminus B_{2|x|}(0)}|x-y|^p u^q(y)dy := I_1+I_2+I_3.$$

First, there exists $C>0$ such that

$$\frac{|x|^p}{C}\leqslant I_1\leqslant C|x|^p.$$

Next, from $|x-y|\leqslant 3|x|$ as $|y|\leqslant 2|x|$, we get by (2.5) that

$$I_2\leqslant C|x|^p\int_1^{2|x|} r^{n+pq}\frac{dr}{r}\leqslant C|x|^p.$$

Finally, from $|y|/2\leqslant |x-y|\leqslant 3|y|/2$ as $|y|\geqslant 2|x|$, we get by $q<-\frac{n+p}{p}$ that

$$I_3\leqslant C\int_{2|x|}^{\infty} r^{n+p+pq}\frac{dr}{r}\leqslant C.$$

Combining the estimates of I_1,I_2,I_3, we see that

$$\frac{|x|^p}{C}\leqslant \int_{R^n}|x-y|^p u^q(y)dy\leqslant C|x|^p \tag{2.8}$$

for some $C>1$. By (2.5) and (2.8), we obtain

$$\frac{1}{C}\int_{R^n}|x-y|^p u^q(y)dy\leqslant u(x)\leqslant C\int_{R^n}|x-y|^p u^q(y)dy.$$

Write $R(x)=u(x)[\int_{R^n}|x-y|^p k(y)u^q(y)dy]^{-1}$. Then, $R(x)$ is double bounded and

$$u(x)=R(x)\int_{R^n}|x-y|^p u^q(y)dy.$$

Set $w(x)=\frac{u(x)}{R(x)}$ and $k(x)=R^q(x)$, then $k(x)$ is a double bounded function, and $w(x)$ is also a solution of (1.6). $\square$

Proof of Theorem 2. (1) Assume that u is a positive solution of (1.6). We claim $u(x)\simeq |x|^p$ when $|x|\to\infty$.

To see this, first we observe

$$\int_{R^n\setminus B_1(0)} u^q(y)k(y)dy\leqslant \int_{R^n\setminus B_1(0)} u^q(y)|y|^p k(y)dy\leqslant u(0)<\infty.$$

Therefore $ku^q\in L^1(R^n)$ since $ku^q\in L^1(B_1(0))$ is clear.

From (1.6) we have

$$
\begin{aligned}
|x|^p u\left(\frac{x}{|x|^2}\right) &= \int_{R^n} |x|^p \Big|\frac{x}{|x|^2} - y\Big|^p u^q(y)k(y)dy \\
&= \int_{R^n} |y|^p \Big|x - \frac{y}{|y|^2}\Big|^p u^q(y)k(y)dy.
\end{aligned}
\tag{2.9}
$$

Letting $|x| \to 0$, we obtain

$$
\lim_{|x|\to 0}\left[|x|^p u\left(\frac{x}{|x|^2}\right)\right] = \int_{R^n} u^q(y)k(y)dy < \infty. \tag{2.10}
$$

We should point out that here we can take the limit by using the dominated convergence theorem. In fact, when $p > 0$ and $|x| \leqslant 1$,

$$
\Big|x - \frac{y}{|y|^2}\Big|^p \leqslant (|x| + 1/|y|)^p \leqslant (1 + 1/|y|)^p
$$

and hence $|y|^p(1+1/|y|)^p u^q(y)k(y) \in L^1(R^n)$.

By doing variable change in (2.10), we can see that there are constants $R > 0$ large and $C > 0$ such that

$$
0 < C^{-1}|x|^p \leqslant u(x) \leqslant C|x|^p \tag{2.11}
$$

for all $|x| \geqslant R$.

(2) Assume that u is a positive solution of (1.6).

By (2.11), we have

$$
\begin{aligned}
\int_{R^n\setminus B_R(0)} u^{q+1}(x)k(x)dx &= \int_{R^n\setminus B_R(0)} u^q(x)u(x)k(x)dx \\
&\leqslant C\int_{R^n\setminus B_R(0)} u^q(x)|x|^p k(x)dx \leqslant Cu(0).
\end{aligned}
$$

This implies $ku^{q+1} \in L^1(R^n)$.

Denote

$$
\nabla u(x) = \int_{R^n} p|x-y|^{p-2}(x-y)u^q(y)k(y)dy.
$$

In order to see this is well defined, we have to show that

$$
v(x) := \int_{R^n} |x-y|^{p-1}u^q(y)k(y)dy < \infty.
$$

In fact, let us fix a point $x \in R^n$.

If $p \geqslant 1$, then $|x-y| \leqslant (|x|+|y|)$, hence $|x-y|^{p-1} \leqslant (|x|+|y|)^{p-1}$.

When $|x| \leqslant 1$, we have $|x-y|^{p-1} \leqslant (1+|y|)^p$ and $(1+|y|)^p u^q(y)k(y) \in L^1(R^n)$ as we have already seen.

When $|x| \geqslant 1$, observing that $|x-y|^{p-1} \leqslant 2^p(|x|^p+|y|^p)$, we conclude that $v(x)$ is finite for any given $x \in R^n$, since ku^q, $|y|^p u^q(y)k(y) \in L^1(R^n)$.

If $0<p<1$, we can obtain that

$$\begin{aligned}
v(x) &= \int_{|x-y|\leqslant 1} |x-y|^{p-1}u^q(y)k(y)dy + \int_{|x-y|\geqslant 1} |x-y|^{p-1}u^q(y)k(y)dy \\
&\leqslant C\int_0^1 r^{p+n-2}dr + \int_{R^n} u^q(y)k(y)dy < \infty.
\end{aligned}$$

So ∇u is well defined.

We claim that if (1.6) has a continuous positive solution for $p>0$, then (1.7) holds if and only if $q=-(1+2n/p)$.

For $\lambda \neq 0$, there holds

$$\begin{aligned}
u(\lambda x) &= \int_{R^n} |\lambda x - y|^p k(y)u^q(y)dy = \int_{R^n} \lambda^p |x-z|^p k(\lambda z)u^q(\lambda z)\lambda^n dz \\
&= \lambda^{p+n}\int_{R^n} |x-z|^p k(\lambda z)u^q(\lambda z)dz.
\end{aligned}$$

Differentiating with respect to λ yields

$$\begin{aligned}
x\cdot\nabla u(\lambda x) &= (p+n)\lambda^{p+n-1}\int_{R^n} |x-z|^p k(\lambda z)u^q(\lambda z)dz \\
&\quad +\lambda^{p+n}\int_{R^n} |x-z|^p[z\cdot\nabla k(\lambda z)u^q(\lambda z) + qk(\lambda z)u^{q-1}(\lambda z)z\cdot\nabla u(\lambda z)]dz.
\end{aligned} \tag{2.12}$$

Let $\lambda = 1$, then (2.12) turns to

$$x\cdot\nabla u(x) = (p+n)u(x) + \int_{R^n} |x-z|^p[z\cdot\nabla k(z)u^q(z) + k(z)z\cdot\nabla u^q(z)]dz. \tag{2.13}$$

Multiplying by $k(x)u^q(x)$ on both sides of (2.13) and integrating over R^n, we have

$$\begin{aligned}
&\frac{1}{q+1}\int_{R^n} k(x)x\cdot\nabla u^{q+1}(x)dx - (p+n)\int_{R^n} u^{q+1}(x)k(x)dx \\
&= \int_{R^n}\int_{R^n} |x-z|^p u^q(x)k(x)[z\cdot\nabla(u^q(z)k(z))]dzdx \\
&= \int_{R^n} z\cdot\nabla(u^q(z)k(z))\int_{R^n} |x-z|^p u^q(x)k(x)dxdz \\
&= \int_{R^n} z\cdot\nabla(u^q(z)k(z))u(z)dz \\
&= \int_{R^n} k(z)z\cdot\nabla u^q(z)u(z)dz + \int_{R^n} z\cdot\nabla k(z)u^{q+1}(z)dz.
\end{aligned} \tag{2.14}$$

If (1.7) holds, (2.14) turns to

$$\begin{aligned}
&\frac{1}{q+1}\int_{R^n} k(x)x\cdot\nabla u^{q+1}(x)dx - (p+n)\int_{R^n} u^{q+1}(x)k(x)dx \\
&= \int_{R^n} k(z)z\cdot\nabla u^q(z)u(z)dz = \frac{q}{q+1}\int_{R^n} k(z)z\cdot\nabla u^{q+1}(z)dz.
\end{aligned} \tag{2.15}$$

Clearly, (2.15) is equivalent to

$$\frac{q-1}{q+1}\lim_{R\to\infty}\int_{B_R} k(x)x\cdot\nabla u^{q+1}(x)dx+(p+n)\lim_{R\to\infty}\int_{B_R} u^{q+1}(x)k(x)dx=0.$$

Integrating by parts, we get

$$\begin{aligned}&\frac{q-1}{q+1}\lim_{R\to\infty}\int_{\partial B_R} Rk(x)u^{q+1}(x)ds-\frac{q-1}{q+1}\lim_{R\to\infty}\int_{B_R}(x\cdot\nabla k(x))u^{q+1}(x)dx\\&-\frac{n(q-1)}{q+1}\lim_{R\to\infty}\int_{B_R} k(x)u^{q+1}(x)dx+(p+n)\lim_{R\to\infty}\int_{B_R} u^{q+1}(x)k(x)dx=0.\end{aligned}\tag{2.16}$$

Due to the fact that $ku^{q+1}\in L^1(R^n)$, we know

$$\lim_{R_j\to\infty} R_j\int_{\partial B_{R_j}} ku^{q+1}ds=0.$$

By (1.7), (2.16) turns to

$$[(n+p)-\frac{n(q-1)}{q+1}]\lim_{R\to\infty}\int_{B_R} k(x)u^{q+1}(x)dx=0.$$

Therefore we get $q=-(1+2n/p)$.

If $q=-(1+2n/p)$ holds, we can also deduce (1.7) by the same calculation above. □

3. Results on (1.8)

Proof of Theorem 3. (1) When $|x|\gg 1$, there holds

$$\begin{aligned}u(x)&\geqslant\int_{B_1(0)\setminus B_{\frac{1}{2}}(0)}|x-y|^p|y|^l u^q(y)dy\\&\geqslant c(\min_{B_1(0)\setminus B_{\frac{1}{2}}(0)}u^q)\int_{B_1(0)\setminus B_{\frac{1}{2}}(0)}|x-y|^p|y|^l dy\geqslant c|x|^p.\end{aligned}\tag{3.1}$$

So we get

$$\begin{aligned}u(x)&\geqslant\int_{R^n\setminus B_{2|x|}(0)}|x-y|^p|y|^l u^q(y)dy\\&\geqslant C|x|^p\int_{R^n\setminus B_{2|x|}(0)}|y|^{l+pq}dy\geqslant C|x|^p\int_{2|x|}^{\infty} r^{n+l+pq}\frac{dr}{r}.\end{aligned}\tag{3.2}$$

Noting $n+l+pq>0$, we see that u blows up.

(2) The proof of (2) in Theorem 3 is the same as the proof of (1) in Theorem 1. The different one is multiplying both sides of (2.3) by $w_n r^{n+p+l}$ instead of $w_n r^{n+p}$. □

Proof of Theorem 4. (1) First we have the following lower bound estimate for large $|x|$

$$\begin{aligned} u(x) &\geqslant \int_{B_1(0)\setminus B_{\frac{1}{2}}(0)} |x-y|^p|y|^l u^q(y)dy \\ &\geqslant c(\min_{B_1(0)\setminus B_{\frac{1}{2}}(0)} u^q)\int_{B_1(0)\setminus B_{\frac{1}{2}}(0)} |x-y|^p|y|^l dy \geqslant c|x|^p. \end{aligned} \tag{3.3}$$

(2) Next, consider the case of $-n < l \leqslant 0$. When $|x| \gg 1$, there holds

$$\begin{aligned} u_1(x) &:= \int_{B_1(0)} |x-y|^p|y|^l u^q(y)dy \\ &\leqslant C(\max_{B_1(0)} u^q)\int_{B_1(0)} |x-y|^p|y|^l dy \\ &\leqslant C|x|^p \int_0^1 r^{n+l-1}dr \\ &\leqslant C|x|^p. \end{aligned} \tag{3.4}$$

When $y \in B_{2|x|}(0)$, $|x-y| \leqslant |x|+|y| \leqslant 3|x|$. Therefore, by (3.3) and $q < -\frac{n+p}{p}$ we get

$$\begin{aligned} u_2(x) &:= \int_{B_{2|x|}(0)\setminus B_1(0)} |x-y|^p|y|^l u^q(y)dy \\ &\leqslant C|x|^p \int_{B_{2|x|}(0)\setminus B_1(0)} |y|^l|y|^{pq}dy \\ &\leqslant C|x|^p \int_1^{2|x|} r^{n+l+pq}\frac{dr}{r} \\ &\leqslant C|x|^p. \end{aligned} \tag{3.5}$$

When $y \in R^n \setminus B_{2|x|}(0)$, $|x-y| \leqslant |x|+|y| \leqslant 3|y|/2$. Therefore, by (3.3) and $q < -\frac{n+p}{p}$ we get

$$\begin{aligned} u_3(x) &:= \int_{R^n\setminus B_{2|x|}(0)} |x-y|^p|y|^l u^q(y)dy \\ &\leqslant C\int_{R^n\setminus B_{2|x|}(0)} |y|^{p+l+pq}dy \\ &\leqslant C|x|^{n+p+l+pq}. \end{aligned} \tag{3.6}$$

Noting $n+p+l+pq<0$ which is implied by $q < -\frac{n+p}{p}$, from the estimates of (3.4)–(3.6), it follows

$$u(x) \leqslant C|x|^p. \tag{3.7}$$

Combining this result with (3.3) we have

$$u(x) \simeq |x|^p$$

when $|x| \to \infty$.

(3) Finally consider the case of $l > 0$. When $|x| \gg 1$, there holds

$$
\begin{aligned}
u_1(x) &:= \int_{B_R(0)} |x-y|^p |y|^l u^q(y) dy \\
&\leqslant C(\max_{B_R(0)} u^q) \int_{B_R(0)} |x-y|^p |y|^l dy \\
&\leqslant C|x|^p.
\end{aligned}
\tag{3.8}
$$

When $y \in B_{2|x|}(0)$, $|x-y| \leqslant |x| + |y| \leqslant 3|x|$. Therefore, by (3.3) and $q < -\frac{n+p+l}{p}$ we get

$$
\begin{aligned}
u_2(x) &:= \int_{B_{2|x|}(0)\setminus B_R(0)} |x-y|^p |y|^l u^q(y) dy \\
&\leqslant C|x|^p \int_R^{2|x|} r^{n+l+pq} \frac{dr}{r} \leqslant C|x|^p.
\end{aligned}
\tag{3.9}
$$

When $y \in R^n \setminus B_{2|x|}(0)$, $|x-y| \leqslant |x| + |y| \leqslant 3|y|/2$. Therefore, by (3.3) and $q < -\frac{n+p+l}{p}$ we get

$$
\begin{aligned}
u_3(x) &:= \int_{R^n\setminus B_{2|x|}(0)} |x-y|^p |y|^l u^q(y) dy \\
&\leqslant C \int_{R^n\setminus B_{2|x|}(0)} |y|^{p+l+pq} dy \leqslant C|x|^{n+p+l+pq}.
\end{aligned}
\tag{3.10}
$$

Noting $n+p+l+pq < 0$ which is implied by $q < -\frac{n+p+l}{p}$, from the estimates of (3.8)–(3.10), it follows

$$
u(x) \leqslant C|x|^p. \tag{3.11}
$$

Combining this result with (3.3) we have

$$
u(x) \simeq |x|^p
$$

when $|x| \to \infty$. □

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