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EIGENVALUE INEQUALITIES RELATED TO THE ANDO–HIAI INEQUALITY

MOHAMMAD BAGHER GHAEMI AND VENUS KALEIBARY

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Abstract. In this paper, we show that if f is a doubly concave function on $[0,\infty)$ and $0 < sA \leqslant B \leqslant tA$ for some scalars $0 < s \leqslant t$ with $w = t/s$, then for every $k = 1,2,\cdots,n$,

$$\lambda_k(f(A)\sharp f(B)) \leqslant \frac{w^{\frac{1}{4}} + w^{-\frac{1}{4}}}{2}\lambda_k(f(A\sharp B)),$$

where $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ is the symmetric geometric mean. As an application, we give some reverses of Ando-Hiai and Golden-Thompson type inequalities. These new reverse inequalities, improve some known results.

1. Introduction

In what follows, capital letters A,B,H and K means bounded linear operators on an n-dimentional complex Hilbert space $(\mathscr{H},\langle\cdot\rangle)$. An operator A is called positive if $\langle Ax,x\rangle \geqslant 0$ for every $x \in \mathscr{H}$ and then we write $A \geqslant 0$. For a pair A,B of Hermitian operators, we say $A \leqslant B$ if $B - A \geqslant 0$. Let f be a continuous real function on $(0,\infty)$. Then f is said to be operator monotone if $A \leqslant B$ implies $f(A) \leqslant f(B)$ for positive definite operators A,B. Also, f is said to be operator concave if $f(\alpha A + (1-\alpha)B) \geqslant \alpha f(A) + (1-\alpha)f(B)$ for all positive definite operators A,B and $\alpha \in [0,1]$.

In our preceding paper [8], we presented some results on operator concave functions involving operator means. In the main theorem we showed that if $f:[0,\infty) \longrightarrow [0,\infty)$ is an operator concave function and A,B are positive operators such that $0 < sA \leqslant B \leqslant tA$, then for all $\alpha \in [0,1]$

$$f(A)\sharp_\alpha f(B) \leqslant \max\{S(s),S(t)\}f(A\sharp_\alpha B), \tag{1}$$

where $S(t) = \dfrac{t^{\frac{1}{t-1}}}{e\log(t^{\frac{1}{t-1}})}$ is the so called Specht's ratio, and $\sharp_\alpha$ is α-geometric mean.

In this paper we study an analogous of inequality (1), with the generalized Kantorovich constant $K(w,\alpha)$. For this purpose, we need the assumption of doubly concavity of $f(t)$. A non-negative continuous function $f(t)$ defined on a positive interval $I \subset [0,\infty)$, is said doubly concave if:

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1. $f(t)$ is concave in the usual sense;

2. $f(t)$ is geometrically concave, i.e., $f(x^{\alpha}y^{1-\alpha}) \geqslant f(x)^{\alpha}f(y)^{1-\alpha}$ for all $x, y \in I$, and $\alpha \in [0,1]$.

If $f(t)$ and $g(t)$ are doubly concave on I, then so is their geometric mean $f(t)^{\alpha}g(t)^{1-\alpha}$ for $\alpha \in [0,1]$ and their minimum $\min\{f(t), g(t)\}$. These properties say that there are a lot of doubly concave functions. We state some examples of such functions in the following and refer to [4], for more examples.

The most important examples of doubly concave functions on $I = [0,\infty)$ are $t \mapsto t^p$ with exponent $p \in [0,1]$. Other simple examples are $t \mapsto t/(t+1)$, $t \mapsto t/\sqrt{t+1}$ and $t \mapsto 1-e^{-t}$. On $I = [1,\infty)$, the function $\log t$ and on $I = [0,1]$, the function $-t\log t$ are also doubly concave.

In Section 2 we present the main result of paper on an eigenvalue inequality involving doubly concave functions and geometric means, as mentioned in abstract.

Section 3 and 4 are devoted to state some elegant application of the main theorem. Let A and B be positive operators. By the weakly log-majorization $A \prec_{w\log} B$ we mean that

$$\prod_{j=1}^{k}\lambda_j(A) \leqslant \prod_{j=1}^{k}\lambda_j(B), \qquad k = 1, 2, \cdots, n,$$

where $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$ are the eigenvalues of A listed in decreasing order. If equality holds when $k = n$, we have the log-majorization $A \prec_{\log} B$. It is known that the weakly log-majorization $A \prec_{w\log} B$ implies $\|A\|_u \leqslant \|B\|_u$ for any unitarily invariant norm $\|\cdot\|_u$, i.e. $\|UAV\|_u = \|A\|_u$ for all A and all unitaries U, V. See [2] for theory of majorization.

To study the Golden-Thompson inequality, Ando-Hiai in [1] developed the following log-majorization inequalities:

$$A^r \sharp_{\alpha} B^r \prec_{\log} (A \sharp_{\alpha} B)^r, \qquad r \geqslant 1, \tag{2}$$

or equivalently

$$(A^p \sharp_{\alpha} B^p)^{\frac{1}{p}} \prec_{\log} (A^q \sharp_{\alpha} B^q)^{\frac{1}{q}}, \qquad 0 < q \leqslant p. \tag{3}$$

There are some literatures [10, 11] on the converse of these inequalities in terms of operator norm $\|\cdot\|$.

In section 3 we consider another converse of Ando-Hiai inequalities (2) and (3), in terms of eigenvalue inequalities, which generalizes some results of [10] to all unitarily invariant norms.

Section 4 is devoted to get results on the reverse Golden-Thompson inequalities. The Golden-Thompson trace inequality, which is of importance in statistical mechanics and in the theory of random matrices, states that $Tre^{H+K} \leqslant Tre^{H}e^{K}$ for arbitrary Hermitian operators H and K.

This inequality has been complemented in several ways [1, 9]. Ando and Hiai in [1] proved that for every unitarily invariant norm $\|\cdot\|_u$ and $p>0$

$$\|(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}\|_u \leqslant \|e^{(1-\alpha)H+\alpha K}\|_u. \tag{4}$$

Seo in [12] showed converse of the Golden-Thompson type inequality (4), and in section 4 we establish another reverse of this inequality. Our results are eigenvalue inequalities parallel to the reverse inequalities obtained in [3] and [5] for the classical Araki and Golden-Thompson inequalities.

2. Doubly concave functions and geometric means

We start this section with our main result. It provides a new reverse for Ando-Hiai inequality and thereupon a new reverse for Golden-Thompson type inequality, as we see in the sequel. The following lemmas will be used to prove this main result.

LEMMA 1. [2, p. 58] (The Minimax Principle) *Let A be a Hermitian operator on $\mathscr{H}$. Then*

$$\lambda_k(A) = \max_{\dim \mathscr{F}=k} \min\{\langle Ah,h\rangle;\, h\in\mathscr{F},\ \|h\|=1\},$$

where $\mathscr{F}$ is a subspace of $\mathscr{H}$.

LEMMA 2. *Let h be a norm one vector, A be a positive definte operator and $f(t)$ be any concave function defined on $[0,\infty)$. Then*

$$\langle f(A)h,h\rangle \leqslant f(\langle Ah,h\rangle).$$

This lemma is derived from the standard Jensen's inequality.

The constant $K(w,\alpha)$ occurs in the following lemma, playing a key role in the proof of Theorem 1 below.

LEMMA 3. [6, Lemma 8] *Let $A,B>0$ with $0<sA\leqslant B\leqslant tA$ for some scalars $0<s\leqslant t$ with $w=t/s$. Then, for all vectors h and all $\alpha\in[0,1]$*

$$\langle A\sharp_\alpha Bh,h\rangle \leqslant \langle Ah,h\rangle^{1-\alpha}\langle Bh,h\rangle^{\alpha} \leqslant K(w,\alpha)^{-1}\langle A\sharp_\alpha Bh,h\rangle,$$

where $K(w,\alpha)$ is the generalized Kantorovich constant defined for $w>0$ by:

$$K(w,\alpha) := \frac{w^\alpha - w}{(\alpha-1)(w-1)}\left(\frac{\alpha-1}{\alpha}\,\frac{w^\alpha-1}{w^\alpha-w}\right)^{\alpha}. \tag{5}$$

It is known that $K(w,\alpha)\in(0,1]$ for $\alpha\in[0,1]$. See [7] *for some important properties of $K(w,\alpha)$.*

THEOREM 1. *Let f be a doubly concave function on $[0,\infty)$ and $0 < sA \leqslant B \leqslant tA$ for some scalars $0 < s \leqslant t$ with $w = t/s$. Then for all $\alpha \in [0,1]$ and $k = 1,2,\cdots,n$,*

$$\lambda_k(f(A)\sharp_\alpha f(B)) \leqslant K(w,\alpha)^{-1}\lambda_k(f(A\sharp_\alpha B)),$$

where $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ is the α-geometric mean and $K(w,\alpha)$ is the generalized Kantorovich constant defined as (5).

Proof. By Lemma 2 for every vector h with $\|h\| = 1$, we can write

$$\langle f(A)h,h\rangle \leqslant f(\langle Ah,h\rangle). \tag{6}$$

Also, for any integer k less than or equal to the dimension of the space, we have a subspace $\mathscr{F}$ of dimension k such that

$$\begin{aligned}
\lambda_k(f(A)\sharp_\alpha f(B)) &= \min_{h\in\mathscr{F}:\|h\|=1} \langle f(A)\sharp_\alpha f(B)h,h\rangle \\
&\leqslant \min_{h\in\mathscr{F}:\|h\|=1} \langle f(A)h,h\rangle^{1-\alpha}\langle f(B)h,h\rangle^\alpha \\
&\leqslant \min_{h\in\mathscr{F}:\|h\|=1} \big(f(\langle Ah,h\rangle)\big)^{1-\alpha}\big(f(\langle Bh,h\rangle)\big)^\alpha \\
&\leqslant \min_{h\in\mathscr{F}:\|h\|=1} f\big(\langle Ah,h\rangle^{1-\alpha}\cdot\langle Bh,h\rangle^\alpha\big) && (7)\\
&\leqslant \min_{h\in\mathscr{F}:\|h\|=1} f\big(K(w,\alpha)^{-1}\langle A\sharp_\alpha Bh,h\rangle\big) && (8)\\
&\leqslant K(w,\alpha)^{-1}\min_{h\in\mathscr{F}:\|h\|=1} f(\langle A\sharp_\alpha Bh,h\rangle) && (9)\\
&= K(w,\alpha)^{-1}\min_{h\in\mathscr{F}:\|h\|=1} \langle f(A\sharp_\alpha B)h,h\rangle \\
&\leqslant K(w,\alpha)^{-1}\lambda_k(f(A\sharp_\alpha B)).
\end{aligned}$$

We have used the minmax principle, Lemma 3 and inequality (6) respectively. Also (7) follows from the geometrically concavity of f, (8) follows from Lemma 3 and monotony of f, and (9) follows from this fact that for every nonnegative concave function f and every $z \geqslant 1$, $f(zx) \leqslant zf(x)$. Note that for $\alpha \in [0,1]$, $K(w,\alpha)^{-1} \geqslant 1$. In two last statements we have used the monotony of f and the minmax principle again. □

Note that the above statement is equivalent to the existence of a unitary operator U satisfying in the following inequality:

$$f(A)\sharp_\alpha f(B) \leqslant K(w,\alpha)^{-1}Uf(A\sharp_\alpha B)U^*.$$

3. Reverse of the Ando-Hiai inequality

By the following theorem we present a new reverse of Ando-Hiai inequalities (2) and (3), as an application of Theorem 1.

THEOREM 2. *Let $0 < sA \leqslant B \leqslant tA$ for some scalars $0 < s \leqslant t$ with $w = t/s$. Then for all $\alpha \in [0,1]$ and $k = 1, 2, \cdots, n$,*

$$\lambda_k(A^r \sharp_\alpha B^r) \leqslant K(w,\alpha)^{-r} \lambda_k(A \sharp_\alpha B)^r, \qquad 0 < r \leqslant 1, \tag{10}$$

or equivalently

$$\lambda_k(A \sharp_\alpha B)^r \leqslant K(w^r,\alpha)^{-1} \lambda_k(A^r \sharp_\alpha B^r), \qquad r \geqslant 1, \tag{11}$$

$$\lambda_k(A^q \sharp_\alpha B^q)^{\frac{1}{q}} \leqslant K(w^p,\alpha)^{-\frac{1}{p}} \lambda_k(A^p \sharp_\alpha B^p)^{\frac{1}{p}}, \qquad 0 < q \leqslant p, \tag{12}$$

where $K(w,\alpha)$ is the generalized Kantorovich constant defined as (5).

Proof. From inequality (8) in the proof of Theorem 1, we have

$$\begin{aligned}
\lambda_k(f(A) \sharp_\alpha f(B)) &\leqslant \min_{h \in \mathscr{F}: \|h\|=1} f\left(\langle K(w,\alpha)^{-1}(A \sharp_\alpha B)h, h\rangle\right) \\
&= \min_{h \in \mathscr{F}: \|h\|=1} \langle f(K(w,\alpha)^{-1}(A \sharp_\alpha B))h, h\rangle \\
&\leqslant \lambda_k f(K(w,\alpha)^{-1}(A \sharp_\alpha B)),
\end{aligned}$$

where monotony of f and the minmax principle are used in two last statements, respectively. Now by letting $f(t) = t^r$, $0 < r \leqslant 1$, inequality (10) is obtained. For $r \geqslant 1$, we have $0 < \frac{1}{r} \leqslant 1$ and by (10)

$$\lambda_k(A^{\frac{1}{r}} \sharp_\alpha B^{\frac{1}{r}}) \leqslant K(w,\alpha)^{-\frac{1}{r}} \lambda_k(A \sharp_\alpha B)^{\frac{1}{r}}.$$

Replacing A and B by A^r and B^r respectively

$$\lambda_k(A \sharp_\alpha B) \leqslant K(w^r,\alpha)^{-\frac{1}{r}} \lambda_k(A^r \sharp_\alpha B^r)^{\frac{1}{r}}.$$

Note that the generalized condition number A^r and B^r is w^r. By taking r-th power on both sides we get the desired inequality (11). Similarly, the equivalence of (10)–(12) is proved. $\square$

Note that eigenvalue inequlaities immedialtly imply log-majorizations and unitarily invariant norm inequalities.

REMARK 1. Let $0 < mI \leqslant A, B \leqslant MI$ for some scalars $0 < m \leqslant M$ with $h = M/m$. Then for all $\alpha \in [0,1]$ and for the operator norm $\|\cdot\|$, we can write

$$\|A^r \sharp_\alpha B^r\| \leqslant K(h^2,\alpha)^{-r} \|(A \sharp_\alpha B)^r\| \leqslant K(h^2,\alpha)^{-r} \|(A \sharp_\alpha B)\|^r, \qquad 0 < r \leqslant 1,$$

and

$$\|A\sharp_\alpha B\|^r \leqslant \|(A\sharp_\alpha B)^r\| \leqslant K(h^{2r},\alpha)^{-1}\|A^r\sharp_\alpha B^r\|, \qquad r \geqslant 1.$$

These results are sharper than the corresponding inequalities presented in [10, Theorem 4, Corollary 5].

4. Reverse of the Golden-Thompson type inequality

In this section we reach to another estimate of the Golden-Thompson type inequality (4), by using results obtained in the preceding section as follows:

THEOREM 3. *Let H and K be Hermitian operators such that $H+sI \leqslant K \leqslant tI+H$ for some scalars $s \leqslant t$, and let $\alpha \in [0,1]$. Then*

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant K(e^{p(t-s)},\alpha)^{-\frac{1}{p}}\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad p>0, \tag{13}$$

where $K(w,\alpha)$ is the generalized Kantorovich constant defined as (5). In particular,

$$\lambda_k(e^{H+K}) \leqslant \frac{e^t+e^s}{2e^{\frac{t}{2}}e^{\frac{s}{2}}}\lambda_k(e^{2H}\sharp e^{2K}). \tag{14}$$

Proof. Replacing A and B by e^H and e^K in the inequality (12) of Theorem 2, we can deduce

$$\lambda_k(e^{qH}\sharp_\alpha e^{qK})^{\frac{1}{q}} \leqslant K(e^{p(t-s)},\alpha)^{-\frac{1}{p}}\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}, \qquad 0<q\leqslant p.$$

By [9, Lemma 3.3], we have

$$e^{(1-\alpha)H+\alpha K} = \lim_{q\to 0}(e^{qH}\sharp_\alpha e^{qK})^{\frac{1}{q}},$$

and hence it follows that for each $p>0$,

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leqslant K(e^{p(t-s)},\alpha)^{-\frac{1}{p}}\lambda_k(e^{pH}\sharp_\alpha e^{pK})^{\frac{1}{p}}.$$

This gives the first alleged inequality. For the second, it is enough to put $\alpha = \frac{1}{2}$ and $p=2$ in the inequality (13). Since a simple calculation shows that

$$K\left(w^p,\frac{1}{2}\right)^{-\frac{1}{p}} = \left(\frac{w^{\frac{p}{4}}+w^{-\frac{p}{4}}}{2}\right)^{\frac{1}{p}},$$

and hence

$$K\left(e^{2(t-s)},\frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{e^t+e^s}{2e^{\frac{t}{2}}e^{\frac{s}{2}}}\right)^{\frac{1}{2}}. \quad \square$$

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Mohammad Bagher Ghaemi
School of Mathematics
Iran University of Science and Technology
Narmak, Tehran 16846-13114, Iran
e-mail: mghaemi@iust.ac.ir

Venus Kaleibary
School of Mathematics
Iran University of Science and Technology
Narmak, Tehran 16846-13114, Iran
e-mail: v.kaleibary@gmail.com