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ON SUFFICIENT CONDITIONS FOR A POLYNOMIAL TO BE SIGN–INDEPENDENTLY HYPERBOLIC OR TO HAVE REAL SEPARATED ZEROS

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Abstract. The well-known Hutchinson's theorem states that if P be a polynomial with positive coefficients, $P(x)=\sum_{k=0}^{n}a_kx^k$, and $\frac{a_{k-1}^2}{a_{k-2}a_k}\geqslant 4$ for $k=2,3,\ldots,n$, then all the zeros of P are real. We obtain sufficient conditions for a real polynomial to be a sign-independently hyperbolic polynomial or to have real separated roots in the style of Hutchinson's theorem.

1. Introduction

To formulate our results we need some definitions and notations.

DEFINITION 1. A real polynomial P is called hyperbolic (or real-rooted) if all zeros of P are real.

Denote by $\mathcal{HP}\subset\mathbb{R}[x]$ the set of hyperbolic polynomials, and by $\mathcal{HP}_+$ the set of hyperbolic polynomials with all positive coefficients.

DEFINITION 2. A hyperbolic polynomial is called sign-independently hyperbolic if it remains hyperbolic after an arbitrary sign change of its coefficients (see [6] and [2]).

Obviously all coefficients of a sign-independently hyperbolic polynomial are non-vanishing.

For $P(x)=\sum_{k=0}^{n}a_kx^k\in\mathcal{HP}_+$ we use the following notations.

$$p_k=p_k(P):=\frac{a_{k-1}}{a_k},\quad 1\leqslant k\leqslant n; \tag{1}$$

$$q_k=q_k(P):=\frac{p_k}{p_{k-1}}=\frac{a_{k-1}^2}{a_{k-2}a_k},\quad 2\leqslant k\leqslant n.$$

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It is easy to check that

$$a_k = \frac{a_0}{p_1 p_2 \ldots p_k},\ k \geqslant 1\ ; \quad a_k = \frac{a_1}{q_2^{k-1} q_3^{k-2} \ldots q_{k-1}^2 q_k} \left(\frac{a_1}{a_0}\right)^{k-1}, \quad k \geqslant 2. \tag{2}$$

Note that the problem of finding whether or not a given polynomial has only real zeros is rather difficult and subtle. In 1926, Hutchinson ([5, p. 327]) extended the work of Petrovitch ([7]) and Hardy ([3] or [4, pp. 95–100]) and found the following sufficient condition for a polynomial (entire function) with positive coefficients to have only real zeros.

THEOREM A. ([5, p. 327]) *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *be an entire function with positive coefficients. Inequalities* $q_k(f) := \frac{a_{k-1}^2}{a_{k-2}a_k} \geqslant 4,\ \forall k \geqslant 2,$ *hold if and only if the following two properties are valid:*

(i) The zeros of f *are all real, simple and negative, and*

(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$*, formed by taking any number of consecutive terms of* f*, are all real and non-positive.*

For some extensions of Hutchinson's results see, for example, ([1, §4]).

We need two frequently used measures of zero separation for hyperbolic polynomials.

DEFINITION 3. Given a polynomial $P \in \mathcal{HP}$, $\deg P \geqslant 2$, denote by $\operatorname{mesh}(P)$ the minimal distance between its roots:

$$\operatorname{mesh}(P) := \min_{1 \leqslant j \leqslant n-1} (x_{j+1} - x_j)$$

for $P = C(x-x_1)(x-x_2)\cdot \ldots \cdot (x-x_n)$, where $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. (If P has a double real root, then $\operatorname{mesh}(P) = 0$).

DEFINITION 4. Given a polynomial $P \in \mathcal{HP}_+$, $\deg P \geqslant 2$, denote by $\operatorname{lmesh}(P)$ the minimal quotient between its roots:

$$\operatorname{lmesh}(P) := \min_{1 \leqslant j \leqslant n-1} \frac{x_{j+1}}{x_j}$$

for $P = C(x+x_1)(x+x_2)\cdot \ldots \cdot (x+x_n)$, where $0 < x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. (If P has a negative double root, then $\operatorname{lmesh}(P) = 1$).

A question on finding simple sufficient conditions for a polynomial to have a mesh (or logarithmic mesh) that is greater than a prescribed number was given to us by Mikhail Tyaglov. The following theorem gives such sufficient condition for a mesh.

THEOREM 1. *Let* $c \geqslant 0$ *be a given number.*

1. Suppose that $P(x) = \sum_{k=0}^{n} a_k x^k$ *be a polynomial with positive coefficients,* $n \geqslant 2$, *and* $p_{k+1}^2(P) - 4p_k(P)p_{k+1}(P) \geqslant c^2$ *for* $k = 1, 2, \ldots, n-1$. *Then* $P \in \mathcal{HP}_+$ *and* $\operatorname{mesh}(P) \geqslant c$.

2. *For every* $c > 0$, $\varepsilon > 0$, *and every* $n \geqslant 2$, *there exists a polynomial* $P_\varepsilon(x) = \sum_{k=0}^n a_k(\varepsilon)x^k \in \mathscr{H}\mathscr{P}_+$, *such that* $p_{k+1}^2(P_\varepsilon) - 4p_k(P_\varepsilon)p_{k+1}(P_\varepsilon) \geqslant c^2 - \varepsilon$ *for* $k = 1,2,\dots,n-1$, *but* $\mathrm{mesh}(P_\varepsilon) < c$.

Note that for $c = 0$, Theorem 1 reduces to Hutchinson's Theorem A.

The next theorem deals with logarithmic mesh.

THEOREM 2. *Let* $d \geqslant 1$ *be a given number.*

1. Suppose that $P(x) = \sum_{k=0}^n a_k x^k$ *is a polynomial with positive coefficients,* $n \geqslant 2$, *and* $q_k(P) \geqslant \frac{(d+1)^2}{d}$ *for* $k = 2,3,\dots,n$. *Then* $P \in \mathscr{H}\mathscr{P}_+$ *and* $\mathrm{lmesh}(P) \geqslant d$.

2. *For every* $d > 1$, $\varepsilon > 0$, *and every* $n \geqslant 2$, *there exists a polynomial* $P_\varepsilon(x) = \sum_{k=0}^n a_k(\varepsilon)x^k \in \mathscr{H}\mathscr{P}_+$, *such that* $q_j(P_\varepsilon) \geqslant \frac{(d+1)^2}{d} - \varepsilon$ *for* $i = 2,3,\dots,n$, *but* $\mathrm{lmesh}(P_\varepsilon) < d$.

We mention that for $d = 1$, Theorem 2 reduces to Hutchinson's Theorem A as well.

Applying the reasonings analogous to those used in the proofs of Theorems 1 and 2 we can obtain the following statement.

THEOREM 3. *Suppose that* $P(x) = \sum_{k=0}^n a_k x^k$ *is a polynomial with positive coefficients,* $n \geqslant 2$, *and* $\frac{a_{n-1}^2}{a_{n-2}a_n} \geqslant 4$, $\forall n \geqslant 2$. *Let* $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ *be the zeros of* $P(-x)$.

1. For $k = 1,2,\dots,n-1$, *denote by* $\Delta_k := p_{k+1}^2(P) - 4p_k(P)p_{k+1}(P)$. *Then* $x_{k+1} - x_k \geqslant \Delta_k$ *for every* k.

2. For $k = 1,2,\dots,n-1$, *denote by* $\delta_k := \frac{1}{2}\left(p_{k+2} - \sqrt{p_{k+2}^2(P) - 4p_{k+2}(P)p_{k+1}(P)} - p_k - \sqrt{p_k^2(P) - 4p_k(P)p_{k+1}(P)}\right)$. *Then* $x_{k+1} - x_k \leqslant \delta_k$ *for every* k.

To formulate our next theorem, we need one more notation. For $x > 1$, we consider the function $\varphi(x) = 1 - 2\sum_{k=1}^\infty x^{-\frac{k^2}{2}}$. We observe that φ is an increasing function in $(1;\infty)$, $\lim_{x\to 1+0}\varphi(x) = -\infty$ and $\lim_{x\to+\infty}\varphi(x) = 1$. So the equation $1 - 2\sum_{k=1}^\infty x^{-\frac{k^2}{2}} = 0$ has the unique positive root which we denote by a_∞. One can check that $a_\infty \approx 4.81058280$.

The following theorem answers the question posed by Boris Shapiro.

THEOREM 4. *1. Let* $f(x) = \sum_{k=0}^\infty a_k x^k$ *be an entire function with positive coefficients. Suppose that* $q_k(f) := \frac{a_{k-1}^2}{a_{k-2}a_k} \geqslant a_\infty$ *for all* $k \geqslant 2$. *Then for every* $n \in \mathbb{N}$, *the* n*-th section* $S_n(x) := \sum_{k=0}^n a_k x^k$ *is sign-independently hyperbolic.*

2. *For every* $\varepsilon > 0$, *there exists a real entire function* $g_\varepsilon(z) = \sum_{k=0}^\infty b_k(\varepsilon)z^k$ *with non-vanishing coefficients such that* $q_k(g_\varepsilon) > a_\infty - \varepsilon$ *for all* $k \geqslant 2$, *and all but a finite number of sections of* g_ε *are not hyperbolic.*

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. 1. Set $Q(x) := P(-x) = \sum_{k=0}^{n}(-1)^k a_k x^k$.

By assumption $p_{k+1}^2(P) - 4p_k(P)p_{k+1}(P) \geqslant 0$ for $k = 1, 2, \ldots, n-1$, whence $0 < p_1(P) < p_2(P) < \ldots < p_n(P)$.

For $x \in [0; p_1(P))$, we get $a_0 > a_1 x > a_2 x^2 > \ldots > a_n x^n$. Thus we have

$$Q(x) = (a_0 - a_1 x) + (a_2 x^2 - a_3 x^3) + \ldots > 0 \quad \text{for} \quad x \in [0; p_1(P)).$$

For $x > p_n(P)$, we get $a_0 < a_1 x < a_2 x^2 < \ldots < a_n x^n$. Thus we have

$$(-1)^n Q(x) = (a_n x^n - a_{n-1} x^{n-1}) + (a_{n-2} x^{n-2} - a_{n-3} x^{n-3}) + \ldots > 0 \quad \text{for} \quad x > p_n(P).$$

Let us fix $l, 1 \leqslant l \leqslant n-1$. For $x \in (p_l(P); p_{l+1}(P))$, we have

$$a_0 < a_1 x < a_2 x^2 < \ldots < a_{l-1} x^{l-1} < a_l x^l$$

and

$$a_l x^l > a_{l+1} x^{l+1} > a_{l+2} x^{l+2} > \ldots > a_n x^n.$$

Thus for $x \in (p_l(P); p_{l+1}(P))$, we get

$$\begin{aligned}(-1)^l Q(x) &= \sum_{j=0}^{l-2}(-1)^{l+j} a_j x^j + \left(-a_{l-1}x^{l-1} + a_l x^l - a_{l+1}x^{l+1}\right) + \sum_{j=l+2}^{n}(-1)^{l+j} a_j x^j \\ &=: \Sigma_1(x) + \left(-a_{l-1}x^{l-1} + a_l x^l - a_{l+1}x^{l+1}\right) + \Sigma_2(x).\end{aligned}$$

We observe that for all $x \in (p_l(P); p_{l+1}(P))$, summands in $\Sigma_1(x)$ are alternating in sign and their moduli are increasing. Analogously for all $x \in (p_l(P); p_{l+1}(P))$, summands in $\Sigma_2(x)$ are alternating in sign and their moduli are decreasing. So $\Sigma_1(x) \geqslant 0$, $\Sigma_2(x) \geqslant 0$ for all $x \in (p_l(P); p_{l+1}(P))$, and

$$(-1)^l Q(x) \geqslant -a_{l-1}x^{l-1} + a_l x^l - a_{l+1}x^{l+1} \quad \text{for} \quad x \in (p_l(P); p_{l+1}(P)).$$

The quadratic polynomial $-a_{l-1}x^{l-1} + a_l x^l - a_{l+1}x^{l+1}$ has all real roots since its discriminant is nonnegative: $D = a_l^2 - 4a_{l-1}a_{l+1} > 0$ by our assumptions. The roots of this quadratic polynomial are

$$x_1(l) := \frac{a_l - \sqrt{a_l^2 - 4a_{l-1}a_{l+1}}}{2a_{l+1}} = \frac{1}{2}\left(p_{l+1}(P) - \sqrt{p_{l+1}^2(P) - 4p_l(P)p_{l+1}(P)}\right)$$

and

$$x_2(l) := \frac{a_l + \sqrt{a_l^2 - 4a_{l-1}a_{l+1}}}{2a_{l+1}} = \frac{1}{2}\left(p_{l+1}(P) + \sqrt{p_{l+1}^2(P) - 4p_l(P)p_{l+1}(P)}\right).$$

Now we will check that $p_l(P) < x_1(l) \leqslant x_2(l) < p_{l+1}(P)$. We have

$$p_l(P) < x_1(l) \Leftrightarrow 2p_l(P) < p_{l+1}(P) - \sqrt{p_{l+1}^2(P) - 4p_l(P)p_{l+1}(P)} \Leftrightarrow$$

$$\sqrt{p_{l+1}^2(P)-4p_l(P)p_{l+1}(P)} < p_{l+1}(P)-2p_l(P).$$

By the assumption $p_{l+1}(P)-2p_l(P)>0,$ whence the last inequality is equivalent to

$$p_{l+1}^2(P)-4p_l(P)p_{l+1}(P) < (p_{l+1}(P)-2p_l(P))^2$$

which is obviously valid. Further we have

$$x_2(l) < p_{l+1}(P) \Leftrightarrow \sqrt{p_{l+1}^2(P)-4p_l(P)p_{l+1}(P)} < p_{l+1}(P),$$

and the last inequality is obviously valid.

We have proved that for every $l=1,2,\dots,n-1,$

$$(-1)^l Q(x) > 0 \text{ for all } x \in (x_1(l);x_2(l)) \subset (p_l(P);p_{l+1}(P)).$$

So that

$$Q(x) > 0 \text{ for all } x \in [0;p_1(P)).$$

$$Q(x) < 0 \text{ for all } x \in (x_1(1);x_2(1)) \subset (p_1(P);p_2(P)).$$

$$Q(x) > 0 \text{ for all } x \in (x_1(2);x_2(2)) \subset (p_2(P);p_3(P)).$$

$$Q(x) < 0 \text{ for all } x \in (x_1(3);x_2(3)) \subset (p_3(P);p_4(P)).$$

$$\vdots$$

Thus

$$\begin{gathered} \exists y_1 \in [p_1(P);x_1(1)] \text{ such that } Q(y_1)=0. \\ \exists y_2 \in [x_2(1);x_1(2)] \text{ such that } Q(y_2)=0. \\ \exists y_3 \in [x_2(2);x_1(3)] \text{ such that } Q(y_3)=0. \\ \vdots \\ \exists y_{n-1} \in [x_2(n-2);x_1(n-1)] \text{ such that } Q(y_{n-1})=0. \\ \exists y_n \in [x_2(n-1);+\infty) \text{ such that } Q(y_n)=0. \end{gathered} \tag{3}$$

We have proved that $Q \in \mathscr{HP}$, so $P \in \mathscr{HP}_+$. Moreover,

$$\operatorname{mesh}(P) = \operatorname{mesh}(Q) \geqslant \min_{1\leqslant j\leqslant n-1}(x_2(j)-x_1(j))$$

$$= \min_{1\leqslant j\leqslant n-1}\sqrt{p_{j+1}^2(P)-4p_j(P)p_{j+1}(P)} \geqslant c$$

by the assumption. The first statement of Theorem 1 is proved.

2. Let $c>0,$ $\varepsilon>0$ and $n\geqslant 2$ are given. Denote by $k:=\max(\frac{c^2}{2},c^2-\varepsilon)>0,$ $\lambda:=2+\sqrt{4+k}>4.$ Consider the polynomial $Q_2(x):=1+x+\frac{x^2}{\lambda}.$ We observe

that $p_1(Q_2) = 1, p_2(Q_2) = \lambda$. Since $D = 1 - \frac{4}{\lambda} > 0$ the polynomial Q_2 is hyperbolic. We have: $\operatorname{mesh}(Q_2) = \frac{\sqrt{D}}{\frac{1}{\lambda}} = \sqrt{\lambda^2 - 4\lambda} = \sqrt{k} < c$. We also have $p_2^2(Q_2) - 4p_2(Q_2)p_1(Q_2) = \lambda^2 - 4\lambda = k \geqslant c^2 - \varepsilon$.

For $\varepsilon_1 > 0$, consider the polynomial $Q_{3,\varepsilon_1}(x) := Q_2(x) + \varepsilon_1 x^3$. Since $Q_2 \in \mathscr{H}\mathscr{P}_+$ we can choose ε_1 small enough so that $Q_{3,\varepsilon_1} \in \mathscr{H}\mathscr{P}_+$. Moreover, since $\operatorname{mesh}(Q_2) < c$ by Hurwitz's theorem we can choose ε_1 small enough so that $\operatorname{mesh}(Q_{3,\varepsilon_1}) < c$. Note that $p_1(Q_{3,\varepsilon_1}) = 1, p_2(Q_{3,\varepsilon_1}) = \lambda, p_3(Q_{3,\varepsilon_1}) = \frac{1}{\lambda\varepsilon_1}$. Finally we choose $\varepsilon_1 > 0$ small enough so that $p_3(Q_{3,\varepsilon_1})^2 - 4p_3(Q_{3,\varepsilon_1})p_2(Q_{3,\varepsilon_1}) \geqslant c^2 - \varepsilon$.

For $\varepsilon_2 > 0$, consider the polynomial $Q_{4,\varepsilon_2}(x) := Q_{3,\varepsilon_1}(x) + \varepsilon_2 x^4$. We can choose ε_2 small enough so that $Q_{4,\varepsilon_2} \in \mathscr{H}\mathscr{P}_+$ and $\operatorname{mesh}(Q_{4,\varepsilon_2}) < c$. Note that $p_1(Q_{4,\varepsilon_2}) = 1, p_2(Q_{4,\varepsilon_2}) = \lambda, p_3(Q_{4,\varepsilon_2}) = \frac{1}{\lambda\varepsilon_1}, p_4(Q_{4,\varepsilon_2}) = \frac{\varepsilon_1}{\varepsilon_2}$. We choose $\varepsilon_2 > 0$ small enough so that $p_4(Q_{4,\varepsilon_2})^2 - 4p_4(Q_{4,\varepsilon_2})p_3(Q_{4,\varepsilon_2}) \geqslant c^2 - \varepsilon$.

Continuing the construction in the given manner we obtain the required polynomial.

The second statement of Theorem 1 is proved. □

Proof of Theorem 2. 1. The proof is analogous to the proof of Theorem 1, but is shorter. Set $Q(x) := P(-x)$. By assumption $q_k(P) \geqslant \frac{(d+1)^2}{d} \geqslant 4$ for $k = 2, 3, \ldots, n$, thus $p_{k+1}^2(P) - 4p_k(P)p_{k+1}(P) \geqslant 0$ for $k = 1, 2, \ldots, n-1$. So all the arguments from the proof of the first statement of Theorem 1 remain valid. We obtain that Q has positive zeros $y_1, y_2, \ldots, y_n$ satisfying conditions (3) (we use the same notations

$$x_1(l) := \frac{1}{2}\left(p_{l+1}(P) - \sqrt{p_{l+1}^2(P) - 4p_l(P)p_{l+1}(P)}\right)$$

and

$$x_2(l) := \frac{1}{2}\left(p_{l+1}(P) + \sqrt{p_{l+1}^2(P) - 4p_l(P)p_{l+1}(P)}\right),$$

$l = 1, 2, \ldots, n-1$). So we have proved that $P \in \mathscr{H}\mathscr{P}_+$. Since the zeros of P are $-y_n, -y_{n-1}, \ldots, -y_2, -y_1$, we get

$$\operatorname{lmesh}(P) \geqslant \min_{1 \leqslant j \leqslant n-1} \frac{x_2(j)}{x_1(j)} = \min_{1 \leqslant j \leqslant n-1} \frac{p_{j+1}(P) + \sqrt{p_{j+1}^2(P) - 4p_j(P)p_{j+1}(P)}}{p_{j+1}(P) - \sqrt{p_{j+1}^2(P) - 4p_j(P)p_{j+1}(P)}}$$

$$= \min_{1 \leqslant j \leqslant n-1} \frac{q_{j+1}(P) + \sqrt{q_{j+1}^2(P) - 4q_{j+1}(P)}}{q_{j+1}(P) - \sqrt{q_{j+1}^2(P) - 4q_{j+1}(P)}}.$$

Solving the inequality $\frac{q_{j+1}(P) + \sqrt{q_{j+1}^2(P) - 4q_{j+1}(P)}}{q_{j+1}(P) - \sqrt{q_{j+1}^2(P) - 4q_{j+1}(P)}} \geqslant d$ we obtain $q_{j+1}(P) \geqslant \frac{(d+1)^2}{d}$, $1 \leqslant j \leqslant n-1$, and we are done. The first statement of Theorem 2 is proved.

2. We use the same reasoning as in the proof of the second statement of Theorem 1. □

3. Proof of Theorem 4

1. Let $f(x)=\sum_{k=0}^{\infty} a_k x^k$ be an entire function with positive coefficients. Without loss of generality we can assume that $a_0 = 1$ and $a_1 = 1$. For $q_k := q_k(f) = \frac{a_{k-1}^2}{a_{k-2}a_k}$, $k \geqslant 2$, we have $f(x) = 1+x+\sum_{k=2}^{\infty} \frac{x^k}{q_2^{k-1}q_3^{k-2}\ldots q_{k-1}^2 q_k}$ (see (2)).

Let $n \geqslant 2$ be any natural number, and $(\sigma_k)_{k=0}^{n}$ be any sequence such that $\sigma_k \in \{-1,1\}$ for all k. We have to prove that the polynomial $S_n(z) = \sum_{k=0}^{n} \sigma_k a_k z^k$ is hyperbolic. We consider the following sequence of radii

$$R_1 = \sqrt{q_2}, \quad R_j = q_2 q_3 \cdot \ldots \cdot q_j \sqrt{q_{j+1}}, \quad j = 2,3,\ldots,n-1.$$

By our assumptions $R_1 < R_2 < \ldots < R_{n-1}$ (since $\frac{R_{j+1}}{R_j} = \sqrt{q_j q_{j+1}} > 1$).

For every fixed $j = 1,2,\ldots,n-1$, we put

$$S_n(z) = \frac{\sigma_j}{q_2^{j-1}q_3^{j-2}\ldots q_{j-1}^2 q_j} z^j + \left(\sigma_0 + \sigma_1 z + \sum_{k=2}^{j-1} \frac{\sigma_k}{q_2^{k-1}q_3^{k-2}\ldots q_{k-1}^2 q_k} z^k\right) + \left(\sum_{k=j+1}^{n} \frac{\sigma_k}{q_2^{k-1}q_3^{k-2}\ldots q_{k-1}^2 q_k} z^k\right) =: \Sigma_0(z) + \Sigma_1(z) + \Sigma_2(z).$$

For any $\theta \in [0;2\pi]$, we have

$$|\Sigma_0(R_j e^{i\theta})| = \left| \frac{(q_2 q_3 \cdot \ldots \cdot q_j \sqrt{q_{j+1}})^j}{q_2^{j-1}q_3^{j-2}\ldots q_{j-1}^2 q_j} \right| = q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} \sqrt{q_{j+1}^j}.$$

Let us estimate $\Sigma_1(z)$ and $\Sigma_2(z)$ from above for $z = R_j e^{i\theta}$. We obtain

$$\begin{aligned}
|\Sigma_2(R_j e^{i\theta})| &\leqslant \sum_{k=j+1}^{n} \frac{(q_2 q_3 \cdot \ldots \cdot q_j \sqrt{q_{j+1}})^k}{q_2^{k-1}q_3^{k-2}\ldots q_{k-1}^2 q_k} \\
&= q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} \sqrt{q_{j+1}^j} \sum_{k=j+1}^{n} \frac{1}{q_{j+1}^{(k-j)/2} q_{j+2}^{k-j-1} q_{j+3}^{k-j-2} \ldots q_{k-1}^2 q_k} \\
&\leqslant q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} \sqrt{q_{j+1}^j} \sum_{k=j+1}^{n} \frac{1}{a_\infty^{(k-j)/2} \cdot a_\infty^{1+2+\ldots+(k-j-1)}} \\
&= q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} \sqrt{q_{j+1}^j} \sum_{k=j+1}^{n} \frac{1}{\sqrt{a_\infty}^{(k-j)^2}} \\
&= q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} \sqrt{q_{j+1}^j} \sum_{k=1}^{n} \frac{1}{\sqrt{a_\infty}^{k^2}} \\
&< q_2 q_3^2 \cdot \ldots \cdot q_j^{j-1} \sqrt{q_{j+1}^j} \sum_{k=1}^{\infty} \frac{1}{\sqrt{a_\infty}^{k^2}}.
\end{aligned}$$

Analogously we get

$$|\Sigma_1(R_je^{i\theta})| \leqslant \sum_{k=0}^{j-1} \frac{(q_2q_3\cdot\ldots\cdot q_j\sqrt{q_{j+1}})^k}{q_2^{k-1}q_3^{k-2}\ldots q_{k-1}^2q_k} = \sum_{k=0}^{j-1} q_2q_3^2\cdot\ldots\cdot q_k^{k-1}q_{k+1}^kq_{k+2}^k\cdot\ldots\cdot q_j^k\sqrt{q_{j+1}^k}$$

$$= q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j}\sum_{k=0}^{j-1}\frac{1}{q_{k+2}q_{k+3}^2\cdot\ldots\cdot q_j^{j-1-k}\sqrt{q_{j+1}^{j-k}}}$$

$$\leqslant q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j}\sum_{k=0}^{j-1}\frac{1}{a_\infty^{1+2+\ldots+(j-k-1)}\cdot a_\infty^{\frac{j-k}{2}}}$$

$$= q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j}\sum_{k=0}^{j-1}\frac{1}{a_\infty^{\frac{(j-k)^2}{2}}}$$

$$= q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j}\sum_{s=1}^{j-1}\frac{1}{a_\infty^{\frac{s^2}{2}}} < q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j}\sum_{s=1}^{\infty}\frac{1}{a_\infty^{\frac{s^2}{2}}}.$$

Therefore we obtain that for every $\theta\in[0;2\pi]$,

$$|\Sigma_0(R_je^{i\theta})| = q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j} = q_2q_3^2\cdot\ldots\cdot q_j^{j-1}\sqrt{q_{j+1}^j}\cdot 2\sum_{s=1}^{\infty}\frac{1}{a_\infty^{\frac{s^2}{2}}}$$

$$> |\Sigma_1(R_je^{i\theta})| + |\Sigma_2(R_je^{i\theta})|,$$

since by definition of a_∞ we have $1 = 2\sum_{k=1}^{\infty} a_\infty^{-\frac{k^2}{2}}$.

Using the Rouché theorem we conclude that for every $j = 1,2,\ldots,n-1,$ the number of zeros of $S_n(z)$ inside the circle $\{z : |z| < R_j\}$ is equal to the number of zeros of $\Sigma_0(z) = \frac{\sigma_j}{q_2^{j-1}q_3^{j-2}\ldots q_{j-1}^2q_j}z^j$ in this circle. Thus S_n has exactly j zeros inside the circle $\{z : |z| < R_j\}$ for each j. So S_n has one zero in $\{z : |z| < R_1\}$ and since S_n is a real polynomial this zero is real. Next we observe that S_n has two zeros in $\{z : |z| < R_2\}$ and since one of the zeros is real we get that both zeros are real. Arguing similarly we obtain that the polynomial S_n of degree n has $n-1$ real zeros. Therefore S_n is hyperbolic.

2. Let us choose an arbitrary $\varepsilon > 0$ and denote by $b := \max(\frac{a_\infty+1}{2}, a_\infty - \frac{\varepsilon}{2})$. Since $b < a_\infty$ we have $\varphi(b) < 0$. So there exists $n\in\mathbb{N}$ such that

$$1 - 2\sum_{k=1}^{n} b^{-\frac{k^2}{2}} < 0.$$

Consider the real entire function

$$g_b(z) := \sum_{k=0}^{n-1}\frac{z^k}{b^{\frac{k(k-1)}{2}}} - \frac{z^n}{b^{\frac{n(n-1)}{2}}} + \sum_{k=n+1}^{\infty}\frac{z^k}{b^{\frac{k(k-1)}{2}}}.$$

We show that for all $m \geqslant 2n$, the m-th section of this function

$$S_m(z) := \sum_{k=0}^{n-1}\frac{z^k}{b^{\frac{k(k-1)}{2}}} - \frac{z^n}{b^{\frac{n(n-1)}{2}}} + \sum_{k=n+1}^{m}\frac{z^k}{b^{\frac{k(k-1)}{2}}}$$

is not hyperbolic. At first we consider the polynomial S_{2n}. Suppose that this polynomial is hyperbolic. The coefficients of S_{2n} are not of the same sign, therefore S_{2n} could not have all negative zeros. So if this polynomial is hyperbolic it necessarily has positive zeros. But we prove that $S_{2n}(x) > 0$ for all $x > 0$.

For $x > 0$, we have

$$\begin{aligned}
S_{2n}(x) &:= \sum_{k=0}^{n-1} \frac{x^k}{b^{\frac{k(k-1)}{2}}} - \frac{x^n}{b^{\frac{n(n-1)}{2}}} + \sum_{k=n+1}^{2n} \frac{x^k}{b^{\frac{k(k-1)}{2}}} \\
&= \sum_{k=0}^{n-1} \left(\frac{x^k}{b^{\frac{k(k-1)}{2}}} + \frac{x^{2n-k}}{b^{\frac{(2n-k)(2n-k-1)}{2}}} \right) - \frac{x^n}{b^{\frac{n(n-1)}{2}}} \\
&\geqslant \sum_{k=0}^{n-1} 2\sqrt{\frac{x^k}{b^{\frac{k(k-1)}{2}}} \cdot \frac{x^{2n-k}}{b^{\frac{(2n-k)(2n-k-1)}{2}}}} - \frac{x^n}{b^{\frac{n(n-1)}{2}}} \\
&= \frac{x^n}{b^{\frac{n(n-1)}{2}}} \left(\sum_{k=0}^{n-1} \frac{1}{b^{\frac{(n-k)^2}{2}}} - 1 \right) = \frac{x^n}{b^{\frac{n(n-1)}{2}}} \left(\sum_{j=1}^{n} \frac{1}{b^{\frac{j^2}{2}}} - 1 \right) > 0
\end{aligned}$$

by our choice of n. Hence S_{2n} does not have positive zeros whence it is not hyperbolic. For every $m \geqslant 2n$ we have $S_m(x) \geqslant S_{2n}(x)$ for all $x > 0$, so S_m is not hyperbolic also.

Theorem 4 is proved. □

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