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WEIGHTED COMPOSITION OPERATORS FROM ZYGMUND TYPE SPACES INTO BLOCH TYPE SPACES

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Abstract. In this paper, we investigate the boundedness and compactness of weighted composition operators from Zygmund type spaces to Bloch type spaces.

1. Introduction

Let μ be a positive continuous function on $[0,1)$. We say that μ is normal, if there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0,1)$ such that (see, for example, [23]).

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0;$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = \infty.$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let H^∞ denote the bounded analytic function space in $\mathbb{D}$.

Suppose ω is normal on $[0,1)$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch type space, denoted by $\mathscr{B}_\omega$, if

$$\|f\|_{\mathscr{B}_\omega} = |f(0)| + \sup_{z\in\mathbb{D}} \omega(|z|)|f'(z)| < \infty.$$

It is easy to see that $\mathscr{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathscr{B}_\omega}$. When $\omega(t) = 1-t^2$, we get the classical Bloch space, denoted by $\mathscr{B} = \mathscr{B}(\mathbb{D})$.

An $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ is said to belong to the Zygmund space, denoted by $\mathscr{Z}$, if

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

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where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. As it was noticed in [12], by Theorem 5.3 in [5] we have that $f \in \mathscr{Z}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty$.

Suppose μ is normal on $[0,1)$. The Zygmund type space, denoted by $\mathscr{Z}_\mu$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty.$$

It is also easy to see that $\mathscr{Z}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathscr{Z}_\mu}$. When $\mu(t) = 1-t^2$, we get the Zygmund space, which was introduced in [12].

Throughout this paper, $S(\mathbb{D})$ denotes the set of all analytic self-maps of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ, which is defined by

$$(C_\varphi f)(z) = f(\varphi(z))$$

for $f \in H(\mathbb{D})$. Let $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined on $H(\mathbb{D})$ as follows.

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

We refer to the books [4, 34] for the theory of composition operators.

The boundness, compactness and essential norm of composition operator and some related operators on Bloch type spaces with $\omega(t) = (1-t^2)^\alpha$ were studied, for example, in [2, 3, 7, 9, 11, 19, 20, 21, 22, 30, 31, 33, 35]. Composition operator and some other concrete operators from or into the Zygmund type spaces on various domains have attracted some attention and were studied, for example, in [1, 6, 8, 10, 13, 14, 15, 16, 17, 18, 24, 25, 26, 27, 28, 29].

In [15], Li and Stević studied the boundedness and compactness of weighted composition operators from the Zygmund space into the Bloch space. Motivated by [15], in this paper we investigate the weighted composition operator $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$. More precisely, we obtain some sufficient and necessary conditions for the boundedness and compactness of weighted composition operators from Zygmund type spaces $\mathscr{Z}_\mu$ to Bloch type spaces $\mathscr{B}_\omega$.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Auxiliary results

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

LEMMA 1. [7] *Suppose μ is normal on $[0,1)$, then there exists $\mu_* \in H(\mathbb{D})$, such that*

1. *For any $t \in [0,1)$, $\mu_*(t) \in \mathbb{R}_+$, $\mu_*(t)$ is increasing on $[0,1)$;*
2. *$\inf_{t\in[0,1)} \mu(t)\mu_*(t) > 0$;*

3. $\sup_{z\in\mathbb{D}} \mu(|z|)|\mu_*(z)| < \infty.$

REMARK 1. By Lemma 1, we have

1. $\mu \approx \frac{1}{\mu_*}$, on the interval $[0,1)$.

2. For any $z = |z|e^{i\theta} \in \mathbb{D}$,

$$\left|\int_0^z \mu_*(\eta)d\eta\right| = \left|\int_0^{|z|} \mu_*(te^{i\theta})dt\right| \lesssim \int_0^{|z|} \frac{1}{\mu(t)}dt \approx \int_0^{|z|} \mu_*(t)dt.$$

3. For any $z,\eta \in \mathbb{D}$, if $|\eta| \leqslant |z|$,

$$\begin{aligned}\mu(|z|)|\mu_*(\eta)| &\leqslant \mu(|z|) \max_{|\gamma|\leqslant|z|} |\mu_*(\gamma)| = \mu(|z|) \max_{|\gamma|=|z|} |\mu_*(\gamma)| \\ &\leqslant \sup_{\gamma\in\mathbb{D}} \mu(|\gamma|)|\mu_*(\gamma)| < \infty.\end{aligned}$$

LEMMA 2. *Suppose μ is normal on $[0,1)$, then the following statements hold.*

1. There exists a $\delta \in (0,1)$, such that μ is decreasing on $[\delta,1)$, $\lim_{t\to1} \mu(t) = 0$.

2. For any $\alpha > 1$, there exists a $\delta \in (0,1)$ such that $\mu(t) \approx \mu(t^\alpha)$ when $t \in [\delta^{\frac{1}{\alpha}},1)$.

Proof. By the definition of normal function, there exists an $a > 0$ and $\delta \in [0,1)$ such that $\frac{\mu(t)}{(1-t)^a}$ is decreasing on $[\delta,1)$. Since $\mu(t) = \frac{\mu(t)}{(1-t)^a}(1-t)^a$, we see that $\mu(t)$ is decreasing on $[\delta,1)$ and $\lim_{t\to1} \mu(t) = 0$.

Similarly, there is a $b > 0$, such that $\frac{\mu(t)}{(1-t)^b}$ is increasing on $[\delta,1)$. Since $\lim_{t\to1} \frac{1-t}{1-t^\alpha} = \frac{1}{\alpha} > 0$, for any $t \in [\delta^{\frac{1}{\alpha}},1)$,

$$1 > \frac{\mu(t)}{\mu(t^\alpha)} = \frac{\frac{\mu(t)}{(1-t)^b}}{\frac{\mu(t^\alpha)}{(1-t^\alpha)^b}} \frac{(1-t)^b}{(1-t^\alpha)^b} > \frac{(1-t)^b}{(1-t^\alpha)^b} > C.$$

Therefore $\mu(t) \approx \mu(t^\alpha)$. □

LEMMA 3. *Suppose ω and μ are normal on $[0,1)$. Then for every $z \in \mathbb{D}$, $f \in \mathscr{B}_\omega$, $g \in \mathscr{Z}_\mu$, we have*

$$|f(z)| \leqslant G_\omega(z)\|f\|_{\mathscr{B}_\omega} \qquad \text{and} \qquad |g(z)| \leqslant H_\mu(z)\|g\|_{\mathscr{Z}_\mu},$$

where

$$G_\omega(z) = 1 + \int_0^{|z|} \frac{1}{\omega(t)}dt, \quad H_\mu(z) = 1 + \int_0^{|z|} \frac{|z|-t}{\mu(t)}dt.$$

Proof. The first inequality is proved in [32]. For every $z\in\mathbb{D}$ and $g\in\mathscr{Z}_\mu$, we have

$$|g(z)| = \left|g(0)+zg'(0)+\int_0^1\int_0^t z^2g''(sz)dsdt\right| \\ \leqslant \|g\|_{\mathscr{Z}_\mu}+\left|\int_0^1\int_0^t z^2g''(sz)dsdt\right|,$$

and

$$\int_0^1\int_0^t |z^2g''(sz)|dsdt \leqslant \|g\|_{\mathscr{Z}_\mu}\int_0^1\int_0^t \frac{|z|^2}{\mu(|sz|)}dsdt \\ = \|g\|_{\mathscr{Z}_\mu}\int_0^1 \frac{|z|(|z|-|sz|)}{\mu(|sz|)}ds.$$

Therefore $|g(z)|\leqslant H_\mu(z)\|g\|_{\mathscr{Z}_\mu}$. □

REMARK 2. From the relationship of $\mathscr{Z}_\mu$ and $\mathscr{B}_\mu$, by Lemma 3, for every $z\in\mathbb{D}$ and $g\in\mathscr{Z}_\mu$,

$$|g'(z)|\leqslant G_\mu(z)\|g'\|_{\mathscr{B}_\mu}\leqslant G_\mu(z)\|g\|_{\mathscr{Z}_\mu}.$$

LEMMA 4. [32] *Suppose that μ is normal on $[0,1)$ such that $\int_0^1\frac{1}{\mu(t)}dt<\infty$. If $\{f_n\}$ is bounded in $\mathscr{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n(z)|=0.$$

LEMMA 5. *Suppose that μ is normal on $[0,1)$ such that $\lim\limits_{|z|\to1}\int_0^{|z|}\frac{|z|-t}{\mu(t)}dt<\infty$. If $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n(z)|=0.$$

Proof. Suppose $\|f_n\|_{\mathscr{Z}_\mu}\leqslant M$ for all $n\in\mathbb{N}^+$. Because $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, by Cauchy's estimate we see that $\{f_n'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. For any given $\varepsilon>0$, by the assumption, there is a $\delta\in(0,1)$ and $N\in\mathbb{N}^+$, such that

$$I(u,s)\overset{def}{=}\int_0^u\frac{u-t}{\mu(t)}dt-\int_0^s\frac{s-t}{\mu(t)}dt<\varepsilon \quad\text{and}\quad |f_n(z)|<\frac{\varepsilon}{2}, \tag{1}$$

when $\delta\leqslant s<u<1$, $n>N$, $|z|\leqslant(1+\delta)/2$. Then

$$\int_s^u\frac{u-t}{\mu(t)}dt\leqslant\int_s^u\frac{u-t}{\mu(t)}dt+\int_0^s\frac{u-t}{\mu(t)}dt-\int_0^s\frac{s-t}{\mu(t)}dt \\ =I(u,s)<\varepsilon. \tag{2}$$

If $|z| \geqslant \delta$, let $z = |z|e^{\mathrm{i}\theta}$, then

$$\begin{aligned}
|f_n(z) - f_n(\delta e^{\mathrm{i}\theta})| &= \left| \int_{\delta e^{\mathrm{i}\theta}}^{z} \Big(\int_{\delta e^{\mathrm{i}\theta}}^{t} f_n''(s)ds + f_n'(\delta e^{\mathrm{i}\theta}) \Big) dt \right| \\
&\leqslant \int_{\delta}^{|z|} \int_{\delta}^{t} \frac{\|f_n\|_{\mathscr{Z}_\mu}}{\mu(s)} ds dt + \int_{\delta}^{|z|} \left| f_n'(\delta e^{\mathrm{i}\theta}) \right| dt \\
&= \|f_n\|_{\mathscr{Z}_\mu} \int_{\delta}^{|z|} \frac{|z| - s}{\mu(s)} ds + (|z| - \delta) \left| f_n'(\delta e^{\mathrm{i}\theta}) \right|.
\end{aligned}$$

Since $\{f_n\}, \{f_n'\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$, by (2),

$$|f_n(z)| \leqslant \left| f_n(z) - f_n(\delta e^{\mathrm{i}\theta}) \right| + \left| f_n(\delta e^{\mathrm{i}\theta}) \right| \leqslant M\varepsilon + \left| f_n'(\delta e^{\mathrm{i}\theta}) \right| + \varepsilon \leqslant C\varepsilon$$

as $n \to \infty$. Since ε is an arbitrary positive number, we get the desired result. □

To study the compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [4]).

LEMMA 6. *Suppose $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is bounded. Then $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is compact if and only if whenever $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ then $\lim_{n\to\infty} \|uC_\varphi f_n\|_{\mathscr{B}_\omega} = 0$.*

3. Main results and proofs

In this section, we formulate and prove the main results in this paper.

THEOREM 1. *Suppose $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Let*

$$E(z) := \omega(|z|)|u'(z)|, \quad F(z) := \omega(|z|)|u(z)\varphi'(z)|.$$

Then $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} E(z)H_\mu(\varphi(z)) < \infty, \quad \sup_{z\in\mathbb{D}} F(z)G_\mu(\varphi(z)) < \infty. \tag{3}$$

Proof. Suppose that (3) holds. For any $f \in \mathscr{Z}_\mu$, using Lemma 3 and Remark 2, we have

$$\begin{aligned}
\|uC_\varphi f\|_{\mathscr{B}_\omega} &= |u(0)f(\varphi(0))| + \sup_{z\in\mathbb{D}} \omega(|z|)|u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z))| \\
&\leqslant |u(0)|H_\mu(\varphi(0))\|f\|_{\mathscr{Z}_\mu} + \sup_{z\in\mathbb{D}} \omega(|z|)|u'(z)f(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} \omega(|z|)|u(z)\varphi'(z)f'(\varphi(z))| \\
&\leqslant C\|f\|_{\mathscr{Z}_\mu} + \sup_{z\in\mathbb{D}} E(z)H_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} + \sup_{z\in\mathbb{D}} F(z)G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \\
&\lesssim \|f\|_{\mathscr{Z}_\mu}.
\end{aligned}$$

Therefore $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is bounded.

Conversely, assume that $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is bounded. Let $f(z) = 1$. By the assumption, we see that $uC_\varphi f = u \in \mathscr{B}_\omega$. Then

$$\sup_{z\in\mathbb{D}} E(z) = \sup_{z\in\mathbb{D}} \omega(|z|)|u'(z)| = \|u\|_{\mathscr{B}_\omega} < \infty.$$

Similarly, let $f(z) = z$, then we get $u\varphi \in \mathscr{B}_\omega$. Therefore,

$$\begin{aligned}\sup_{z\in\mathbb{D}} F(z) &= \sup_{z\in\mathbb{D}} \omega(|z|)|u(z)\varphi'(z)| \\ &\leqslant \sup_{z\in\mathbb{D}} \omega(|z|)\left|u(z)\varphi'(z) + u'(z)\varphi(z)\right| + \sup_{z\in\mathbb{D}} \omega(|z|)|u'(z)\varphi(z)| \\ &\leqslant \|u\varphi\|_{\mathscr{B}_\omega} + \|u\|_{\mathscr{B}_\omega} < \infty.\end{aligned}$$

Let $\mu_*(z)$ denote the analytic function related to $\mu(t)$ in Lemma 1. For any $\eta \in \mathbb{D}$, we choose $a \neq 0$ and set

$$f_a(z) = \int_{\frac{|a|^3}{a}z}^{\frac{|a|^3}{a^2}z^2} ds \int_0^s \mu_*(t)dt.$$

After a calculation, we have $f_a \in H(\mathbb{D})$, $f_a(0) = 0, f_a(a) = 0$, $f_a'(0) = 0$,

$$f_a'(z) = \frac{2|a|^3}{a^2} z \int_0^{\frac{|a|^3}{a^2}z^2} \mu_*(t)dt - \frac{|a|^3}{a} \int_0^{\frac{|a|^3}{a}z} \mu_*(t)dt,$$

and

$$f_a''(z) = \frac{2|a|^3}{a^2} \int_0^{\frac{|a|^3}{a^2}z^2} \mu_*(t)dt + \frac{4|a|^6}{a^4} z^2 \mu_*\Big(\frac{|a|^3}{a^2}z^2\Big) - \frac{|a|^6}{a^2}\mu_*\Big(\frac{|a|^3}{a}z\Big).$$

By Remark 1 and Lemma 1,

$$\begin{aligned}\mu(|z|)|f_a''(z)| &\leqslant 2\mu(|z|)\left|\int_0^{\frac{|a|^3}{a^2}z^2} \mu_*(t)dt\right| + 4\mu(|z|)\left|\mu_*\Big(\frac{|a|^3}{a^2}z^2\Big)\right| + \mu(|z|)\left|\mu_*\Big(\frac{|a|^3}{a}z\Big)\right| \\ &\lesssim \int_0^{|az^2|} \mu(|z|)\mu_*(t)dt + C \lesssim C.\end{aligned}$$

Therefore $f_a(z) \in \mathscr{Z}_\mu$ and $\|f_a\|_{\mathscr{Z}_\mu} \lesssim C$.

If $|a| = |\varphi(\eta)| \leqslant \frac{1}{2}$, then

$$F(\eta)G_\mu(\varphi(\eta)) \leqslant \sup_{z\in\mathbb{D}} F(z)G_\mu\Big(\frac{1}{2}\Big) < C. \tag{4}$$

Since $\varphi(\eta) = a$, $f_a(a) = 0$ and

$$f_a'(a) = \frac{|a|^3}{a}\int_0^{|a|^3} \mu_*(t)dt,$$

we get

$$
\begin{aligned}
|a|^2F(\eta)\int_0^{|a|^3}\mu_*(t)dt &= \omega(|\eta|)\left|u(\eta)\varphi'(\eta)f_a'(a)\right| \\
&= \omega(|\eta|)\left|u'(\eta)f_a(\varphi(\eta))+u(\eta)\varphi'(\eta)f_a'(\varphi(\eta))\right| \\
&\leqslant \sup_{z\in\mathbb{D}}\omega(|z|)\left|(uC_\varphi f_a)'(z)\right| \\
&\leqslant \|uC_\varphi f_a\|_{\mathscr{B}_\omega} \leqslant \|uC_\varphi\|\cdot\|f_a\|_{\mathscr{Z}_\mu} \lesssim \|uC_\varphi\|. \qquad (5)
\end{aligned}
$$

If $|a|>\frac{1}{2}$, by Remark 1 and Lemma 2,

$$
\int_{\frac{1}{2}}^{|a|}\frac{1}{\mu(t)}dt \approx \int_{\frac{1}{2}}^{|a|}\frac{1}{\mu(t^3)}dt \approx |a|^2\int_{\frac{1}{8}}^{|a|^3}\frac{1}{\mu(t)}dt \lesssim |a|^2\int_0^{|a|^3}\mu_*(t)dt, \qquad (6)
$$

which together with (4) and (5) imply

$$
\begin{aligned}
F(\eta)G_\mu(\varphi(\eta)) &= F(\eta)+F(\eta)\int_0^{\frac{1}{2}}\frac{1}{\mu(t)}dt+F(\eta)\int_{\frac{1}{2}}^{|a|}\frac{1}{\mu(t)}dt \\
&\lesssim 1+|a|^2F(\eta)\int_0^{|a|^3}\mu_*(t)dt < C. \qquad (7)
\end{aligned}
$$

Here we used the fact that $\sup_{\eta\in\mathbb{D}}F(\eta)<\infty$. Therefore

$$
\sup_{\eta\in\mathbb{D}}F(\eta)G_\mu(\varphi(\eta))<\infty. \qquad (8)
$$

Let

$$
g_a(z)=\int_0^{\overline{a}z}\int_0^t\mu_*(s)dsdt, \quad z\in\mathbb{D}.
$$

By Remark 1 and Lemma 2,

$$
\mu(|z|)|g_a''(z)| = \mu(|z|)|\overline{a}^2\mu_*(\overline{a}z)| \leqslant \mu(|z|)|\mu_*(\overline{a}z)| < C.
$$

Hence $\|g_a\|_{\mathscr{Z}_\mu}<C$.

If $|a|=|\varphi(\eta)|\leqslant\frac{1}{2}$,

$$
E(\eta)H_\mu(\varphi(\eta)) \leqslant \sup_{z\in\mathbb{D}}E(z)H_\mu\left(\frac{1}{2}\right) < C. \qquad (9)
$$

If $|a|>\frac{1}{2}$, by Lemmas 1 and 2, we get

$$
\int_{\frac{1}{2}}^{|a|}\frac{|a|-s}{\mu(s)}ds \approx \int_{\frac{1}{4}}^{|a|^2}\frac{|a|-\sqrt{t}}{\mu(\sqrt{t})}dt \approx \int_{\frac{1}{4}}^{|a|^2}\frac{|a|^2-t}{\mu(t)}dt \lesssim g_a(a). \qquad (10)
$$

Since $\varphi(\eta)=a$, by Remark 2 and (8), we have

$$
\begin{aligned}
E(\eta)g_a(a) &= \omega(|\eta|)|u'(\eta)g_a(\varphi(\eta))+u(\eta)\varphi'(\eta)g_a'(a)-u(\eta)\varphi'(\eta)g_a'(a)| \\
&\leqslant \|uC_\varphi g_a\|_{\mathscr{B}_\omega}+F(\eta)|g_a'(a)| \\
&\leqslant \|uC_\varphi\|\|g_a\|_{\mathscr{Z}_\mu}+F(\eta)G_\mu(\varphi(\eta))\|g_a\|_{\mathscr{Z}_\mu} < C. \qquad (11)
\end{aligned}
$$

Hence by (10) and (11), we get

$$E(\eta)H_\mu(\varphi(\eta)) = E(\eta)+E(\eta)\int_0^{\frac{1}{2}} \frac{|a|-s}{\mu(s)}ds + E(\eta)\int_{\frac{1}{2}}^{|a|} \frac{|a|-s}{\mu(s)}ds$$
$$\lesssim E(\eta)+E(\eta)\int_0^{\frac{1}{2}} \frac{1-s}{\mu(s)}ds + E(\eta)g_a(a) < C, \tag{12}$$

when $|a| > \frac{1}{2}$. From (9) and (12), we get that

$$\sup_{\eta\in\mathbb{D}} E(\eta)H_\mu(\varphi(\eta)) < \infty,$$

finishing the proof of the theorem. □

THEOREM 2. *Suppose $u\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is bounded. Then the following statements hold.*

(i) When $\lim_{|z|\to 1} H_\mu(z) < \infty$, $\lim_{|z|\to 1} G_\mu(z) < \infty$, then $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is compact.

(ii) When $\lim_{|z|\to 1} H_\mu(z) = \infty$, $\lim_{|z|\to 1} G_\mu(z) = \infty$, then $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} F(z)G_\mu(\varphi(z)) = 0 \qquad \text{and} \qquad \lim_{|\varphi(z)|\to 1} E(z)H_\mu(\varphi(z)) = 0.$$

(iii) When $\lim_{|z|\to 1} H_\mu(z) < \infty$, $\lim_{|z|\to 1} G_\mu(z) = \infty$, then $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} F(z)G_\mu(\varphi(z)) = 0.$$

Proof. (i) By the boundedness of $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{B}_\omega$ we see that

$$\sup_{z\in\mathbb{D}} E(z) < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} F(z) < \infty.$$

Let $\{f_n\}$ be a bounded sequence in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. We have

$$\|uC_\varphi f_n\|_{\mathscr{B}_\omega} = |u(0)f_n(\varphi(0))| + \sup_{z\in\mathbb{D}} \omega(|z|)\left|u'(z)f_n(\varphi(z)) + u(z)\varphi'(z)f_n'(\varphi(z))\right|$$
$$\leqslant |u(0)f_n(\varphi(0))| + \sup_{z\in\mathbb{D}} E(z)|f_n(\varphi(z))| + \sup_{z\in\mathbb{D}} F(z)\left|f_n'(\varphi(z))\right|$$
$$\leqslant |u(0)f_n(\varphi(0))| + C\sup_{z\in\mathbb{D}}|f_n(\varphi(z))| + C\sup_{z\in\mathbb{D}}\left|f_n'(\varphi(z))\right|.$$

From Lemma 5, $\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n(z)| = 0$. Since $f_n \in \mathscr{Z}_\mu$, we have that $f_n' \in \mathscr{B}_\mu$ and $\|f_n'\|_{\mathscr{B}_\mu} = \|f_n\|_{\mathscr{Z}_\mu} - |f_n(0)|$. Then $\{f_n'\}$ is bounded in $\mathscr{B}_\mu$. Because $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Cauchy estimate, $\{f_n'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. From Lemma 4, $\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n'(z)| = 0$. Then

$\lim_{n\to\infty}\|uC_\varphi f_n\|_{\mathscr{B}_\omega}=0$. From Lemma 6, we see that $uC_\varphi:\mathscr{Z}_\mu\to\mathscr{B}_\omega$ is a compact operator.

(ii). *Sufficiency*. Assume that $\{f_n\}$ is a bounded sequence in $\mathscr{Z}_\mu$ and $f_n\to 0$ uniformly on compact subset of $\mathbb{D}$. Without loss of generality, we assume that $\|f_n\|_{\mathscr{Z}_\mu}\leqslant 1$. By the assumption, for any $\varepsilon>0$, there is a $\delta\in(0,1)$ such that

$$F(z)G_\mu(\varphi(z))<\varepsilon \quad\text{and}\quad E(z)H_\mu(\varphi(z))<\varepsilon$$

when $\delta<|\varphi(z)|<1$.

Set $K_\delta=\{w:|w|\leqslant\delta, w\in\mathbb{D}\}$. Then K_δ is a compact subset of $\mathbb{D}$. Because $f_n\to 0$ and $f_n'\to 0$ uniformly on compact subset of $\mathbb{D}$, we see that there is an $N\in\mathbb{N}^+$ such that

$$\sup_{w\in K_\delta}|f_n(w)|<\varepsilon,\qquad \sup_{w\in K_\delta}|f_n'(w)|<\varepsilon,\quad\text{when}\quad n>N.$$

If $|\varphi(z)|>\delta$, by Lemma 3

$$\begin{aligned}\omega(|z|)\left|(uC_\varphi f_n)'(z)\right| &= \omega(|z|)\left|u'(z)f_n(\varphi(z))+u(z)\varphi'(z)f_n'(\varphi(z))\right|\\ &\leqslant E(z)\,|f_n(\varphi(z))|+F(z)\left|f_n'(\varphi(z))\right|\\ &\leqslant E(z)H_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu}+F(z)G_\mu(\varphi(z))\|f_n'\|_{\mathscr{B}_\mu}\\ &\leqslant E(z)H_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu}+F(z)G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu}\\ &< 2\varepsilon.\end{aligned}$$

If $|\varphi(z)|\leqslant\delta$, then

$$\begin{aligned}\omega(|z|)\left|(uC_\varphi f_n)'(z)\right| &= \omega(|z|)\left|u'(z)f_n(\varphi(z))+u(z)\varphi'(z)f_n'(\varphi(z))\right|\\ &< \sup_{z\in\mathbb{D}}E(z)\varepsilon+\sup_{z\in\mathbb{D}}F(z)\varepsilon\\ &< C\varepsilon,\qquad\text{when}\quad n>N.\end{aligned}$$

Thus

$$\|uC_\varphi f_n\|_{\mathscr{B}_\omega}=|u(0)f_n(\varphi(0))|+\sup_{z\in\mathbb{D}}\omega(|z|)\left|(uC_\varphi f_n)'(z)\right|<C\varepsilon$$

when $n>N$. Therefore $\lim_{n\to\infty}\|uC_\varphi f_n\|_{\mathscr{B}_\omega}=0$. From Lemma 6, $uC_\varphi:\mathscr{Z}_\mu\to\mathscr{B}_\omega$ is a compact operator.

Necessity. By the boundedness of $uC_\varphi:\mathscr{Z}_\mu\to\mathscr{B}_\omega$ we see that $u\in\mathscr{B}_\omega$ and $u\varphi\in\mathscr{B}_\omega$. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty}|\varphi(z_n)|=1$. Let $a_n=\varphi(z_n)$,

$$M(z)=\int_0^z\int_0^t\mu_*(s)dsdt\qquad\text{and}\qquad N(z)=\int_0^z\mu_*(s)ds.$$

Here $\mu_*(z)$ denotes the analytic function related to $\mu(t)$ in Lemma 1. Set

$$f_n(z)=\frac{g_n(z)}{M(|a_n|)},$$

where

$$g_n(z) = M(\overline{a_n}z)\left(M\left(\frac{|a_n|^3 z^2}{a_n^2}\right) - M\left(\frac{|a_n|^3 z}{a_n}\right)\right).$$

Obviously, $f_n(0) = f_n(a_n) = 0$ and

$$\begin{aligned} g_n'(z) &= \overline{a_n}N(\overline{a_n}z)\left(M\left(\frac{|a_n|^3 z^2}{a_n^2}\right) - M\left(\frac{|a_n|^3 z}{a_n}\right)\right) \\ &\quad + M(\overline{a_n}z)\left(\frac{2|a_n|^3 z}{a_n^2}N\left(\frac{|a_n|^3 z^2}{a_n^2}\right) - \frac{|a_n|^3}{a_n}N\left(\frac{|a_n|^3 z}{a_n}\right)\right). \end{aligned}$$

Thus, $f_n'(0) = 0$, $f_n'(a_n) = \frac{g_n'(a_n)}{M(|a_n|)}$ and

$$g_n'(a_n) = \frac{|a_n|^3}{a_n} M(|a_n|^2) N(|a_n|^3). \tag{13}$$

Moreover,

$$\begin{aligned} g_n''(z) &= (\overline{a_n})^2 \mu_*(\overline{a_n}z)\left(M\left(\frac{|a_n|^3 z^2}{a_n^2}\right) - M\left(\frac{|a_n|^3 z}{a_n}\right)\right) \\ &\quad + 2\overline{a_n}N(\overline{a_n}z)\left(\frac{2|a_n|^3 z}{a_n^2}N\left(\frac{|a_n|^3 z^2}{a_n^2}\right) - \frac{|a_n|^3}{a_n}N\left(\frac{|a_n|^3 z}{a_n}\right)\right) \\ &\quad + M(\overline{a_n}z)\left(\frac{2|a_n|^3}{a_n^2}N\left(\frac{|a_n|^3 z^2}{a_n^2}\right) + \frac{4|a_n|^6 z^2}{a_n^4}\mu_*\left(\frac{|a_n|^3 z^2}{a_n^2}\right) - \frac{|a_n|^6}{a_n^2}\mu_*\left(\frac{|a_n|^3 z}{a_n}\right)\right). \end{aligned}$$

By Lemma 1 and Remark 1,

$$\sup_{z\in\mathbb{D}} \sup_{|\eta|\leqslant|z|} \mu(|z|)|\mu_*(\eta)| < C,$$

$$|N(\eta)| = \left|\int_0^\eta \mu_*(t)dt\right| \lesssim \int_0^{|\eta|} \mu_*(t)dt \leqslant \mu_*(|\eta|), \qquad \forall \eta \in \mathbb{D}.$$

Therefore

$$\mu(|z|)|g_n''(z)| \lesssim \int_0^{|a_n z|}\int_0^t \mu_*(s)dsdt + \mu(|z|)\left(\int_0^{|a_n z|}\mu_*(t)dt\right)^2. \tag{14}$$

By Lemma 2, there exists a $\delta_* \in (0,1)$ such that μ is decreasing on $(\delta_*, 1)$. For any $|z| > \sqrt{\delta_*}, |a_n| > \sqrt{\delta_*}$,

$$\frac{\mu(|z|)\left(\int_0^{|a_n z|}\mu_*(t)dt\right)^2}{\int_0^{|a_n|}\int_0^t \mu_*(s)dsdt} \leqslant \frac{\mu(|a_n z|)\left(\int_0^{|a_n z|}\mu_*(t)dt\right)^2}{\int_0^{|a_n z|}\int_0^t \mu_*(s)dsdt} \approx Q(|a_n z|), \tag{15}$$

where

$$Q(u) = \frac{(\int_0^u \mu_*(t)dt)^2}{\mu_*(u)\int_0^u (u-t)\mu_*(t)dt}, \qquad u \in (0,1).$$

We have known that μ is decreasing on $[\delta_*,1)$ and

$$\lim_{t\to 1}\mu_*(t)=\infty. \tag{16}$$

Since $\lim\limits_{|z|\to 1} H_\mu(z)=\infty$, $\lim\limits_{|z|\to 1} G_\mu(z)=\infty$, we have

$$\lim_{s\to 1}\int_0^s (s-t)\mu_*(t)dt=\infty \qquad \text{and} \qquad \lim_{s\to 1}\int_0^s \mu_*(t)dt=\infty. \tag{17}$$

From (16) and (17), there is a $\delta_1\in(\delta_*,1)$, such that if $u\in(\delta_1,1)$,

$$\frac{1}{2}\left(\int_0^u \mu_*(t)dt\right)^2 \leqslant \left(\int_0^u \mu_*(t)dt\right)^2-\left(\int_0^\delta \mu_*(t)dt\right)^2 \leqslant \left(\int_0^u \mu_*(t)dt\right)^2,$$

$$\frac{1}{2}\mu_*(u)\int_0^u (u-t)\mu_*(t)dt \leqslant \mu_*(u)\int_0^u (u-t)\mu_*(t)dt-\mu_*(\delta)\int_0^\delta (\delta-t)\mu_*(t)dt$$

and

$$\mu_*(u)\int_0^u (u-t)\mu_*(t)dt \geqslant \mu_*(u)\int_0^u (u-t)\mu_*(t)dt-\mu_*(\delta)\int_0^\delta (\delta-t)\mu_*(t)dt.$$

By Cauchy's differential mean value theorem, for any $u\in(\delta_1,1)$, there exists a $\eta\in(\delta,u)$ such that

$$\begin{aligned}
Q(u) &\approx \frac{\left(\int_0^u \mu_*(t)dt\right)^2-\left(\int_0^\delta \mu_*(t)dt\right)^2}{\mu_*(u)\int_0^u (u-t)\mu_*(t)dt-\mu_*(\delta)\int_0^\delta (\delta-t)\mu_*(t)dt}\\
&= \frac{2\mu_*(\eta)\int_0^\eta \mu_*(t)dt}{\mu_*'(\eta)\int_0^\eta (\eta-t)\mu_*(t)dt+\mu_*(\eta)\int_0^\eta \mu_*(t)dt}\\
&= 2\left(1+\frac{\mu_*'(\eta)\int_0^\eta (\eta-t)\mu_*(t)dt}{\mu_*(\eta)\int_0^\eta \mu_*(t)dt}\right)^{-1}.
\end{aligned}$$

From Lemma 1, $\mu_*(t)$ is increasing on $[0,1)$. Thus for any $t\in(0,1)$, $\mu_*'(t)\geqslant 0$. Therefore $Q(u)$ is bounded on $(\delta_1,1)$.

By (15), if $|a_nz|>\delta_1$, $|a_n|>\sqrt{\delta_*}$, $|z|>\sqrt{\delta_*}$, then

$$\frac{\mu(|z|)\left(\int_0^{|a_nz|}\mu_*(t)dt\right)^2}{M(|a_n|)}\leqslant C.$$

Otherwise, at least one of the inequalities $|a_nz|\leqslant\delta_1$, $|a_n|\leqslant\sqrt{\delta_*}$, $|z|\leqslant\sqrt{\delta_*}$ holds. Let $\delta_2=\max(\delta_1,\sqrt{\delta_*})$. Then $|a_nz|\leqslant\delta_2$. Since $\lim_{n\to\infty}|a_n|=1$, there exists an N such that $|a_n|>\frac{1}{2}$ when $n>N$. Because μ is a positive, continuous function on $[0,1)$ and $\lim\limits_{t\to 1}\mu(t)=0$, we have

$$\frac{\mu(|z|)\left(\int_0^{|a_nz|}\mu_*(t)dt\right)^2}{M(|a_n|)}\leqslant \frac{\mu(|z|)\left(\int_0^{\delta_2}\mu_*(t)dt\right)^2}{\int_0^{\frac{1}{2}}\int_0^t \mu_*(s)dsdt}<C,$$

i.e., we get

$$\mu(|z|)\left(\int_0^{|a_nz|}\mu_*(t)dt\right)^2 < CM(|a_n|)$$

when $n > N$. Therefore for any $z\in\mathbb{D}$ and $n>N$,

$$\mu(|z|)\left(\int_0^{|a_nz|}\mu_*(t)dt\right)^2 < CM(|a_n|). \tag{18}$$

Since

$$\int_0^{|a_nz|}\int_0^t \mu_*(s)dsdt < M(|a_n|), \tag{19}$$

by (14), (18), (19), we get that $\|f_n\|_{\mathscr{Z}_\mu} < C$ when $n>N$.

Since $M(|a_n|)\approx H_\mu(a_n)$, $\lim_{n\to\infty} H_\mu(a_n)=\infty$. Then $f_n\to 0$ uniformly on compact subsets of $\mathbb{D}$. Using Lemma 6, we obtain

$$\lim_{n\to\infty}\omega(|z_n|)\left|u'(z_n)f_n(\varphi(z_n))+u(z_n)\varphi'(z_n)f_n'(\varphi(z_n))\right|=0.$$

Because of $f_n(\varphi(z_n))=0$, we get

$$\lim_{n\to\infty}F(z_n)|f_n'(\varphi(z_n))|=0. \tag{20}$$

In addition,

$$\begin{aligned}\int_0^{|a_n|^2}\int_0^t \mu_*(s)dsdt &\gtrsim \int_{\frac14}^{|a_n|^2}(|a_n|^2-s)\mu_*(\sqrt{s})\frac{1}{2\sqrt{s}}ds\\ &\approx \int_{\frac12}^{|a_n|}(|a_n|-s)\mu_*(s)ds,\end{aligned} \tag{21}$$

$$\int_0^{|a_n|^3}\mu_*(t)dt \approx \int_0^{|a_n|^3}\frac{1}{\mu(t^3)}dt=\int_0^{|a_n|}\frac{1}{\mu(t)}\frac{1}{3\sqrt[3]{t^2}}dt > \frac13\int_0^{|a_n|}\frac{1}{\mu(t)}dt. \tag{22}$$

From (13) and (20), we get

$$\lim_{n\to\infty}\frac{F(z_n)M(|a_n|^2)N(|a_n|^3)}{M(|a_n|)}=0.$$

By (21) and (22), we obtain

$$\lim_{n\to\infty}\frac{F(z_n)\int_0^{|a_n|}\frac{1}{\mu(t)}dt\int_{\frac12}^{|a_n|}(|a_n|-s)\mu_*(s)ds}{\int_0^{|a_n|}(|a_n|-s)\mu_*(s)ds}=0.$$

Since $\lim_{|z|\to 1}\int_0^{|z|}\frac{|z|-t}{\mu(t)}dt=\infty$, we get

$$\lim_{n\to\infty}F(z_n)\int_0^{|a_n|}\frac{1}{\mu(t)}dt=0,$$

which implies that

$$\lim_{|\varphi(z)|\to 1} F(z)G_\mu(\varphi(z)) = 0.$$

Let $p_n(z) = \frac{q_n(z)}{M(|a_n|)}$, where $q_n(z) = (M(\overline{a_n}z))^2$. Then $q_n'(z) = 2\overline{a_n}M(\overline{a_n}z)N(\overline{a_n}z)$,

$$q_n''(z) = 2(\overline{a_n})^2(N(\overline{a_n}z))^2 + 2(\overline{a_n})^2\mu_*(\overline{a_n}z)M(\overline{a_n}z),$$

$$q_n(0) = 0,\ q_n'(0) = 0,\ p_n(a_n) \approx H_\mu(a_n) = H_\mu(\varphi_n(z_n)).$$

From Lemma 1,

$$\begin{aligned}\mu(|z|)|q_n''(z)| &\lesssim \mu(|z|)|\mu_*(\overline{a_n}z)||M(\overline{a_n}z)| + \mu(|z|)|N(\overline{a_n}z)|^2\\ &\leqslant CM(|a_n|) + \mu(|z|)\left(\int_0^{|a_n z|}\mu_*(s)ds\right)^2.\end{aligned}$$

By (18), we see that $\|p_n\|_{\mathscr{Z}_\mu} < C$ for any $n > N$. Moreover it is easy to see that $p_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim_{n\to\infty}\|uC_\varphi p_n\|_{\mathscr{B}_\omega} = 0$. Therefore, by Lemma 1

$$\begin{aligned}\|uC_\varphi p_n\|_{\mathscr{B}_\omega} &\geqslant E(z_n)p_n(\varphi(z_n)) - F(z_n)|p_n'(\varphi(z_n))|\\ &\geqslant CE(z_n)H_\mu(\varphi(z_n)) - F(z_n)G_\mu(\varphi(z_n))\|p_n\|_{\mathscr{Z}_\mu}\\ &\geqslant CE(z_n)H_\mu(\varphi(z_n)) - CF(z_n)G_\mu(\varphi(z_n)).\end{aligned}$$

Because $\lim_{|\varphi(z)|\to 1} F(z)G_\mu(\varphi(z)) = 0$, we get $\lim_{n\to\infty}E(z_n)H_\mu(\varphi(z_n)) = 0$, which implies

$$\lim_{|\varphi(z)|\to 1} E(z)H_\mu(\varphi(z)) = 0.$$

(iii). *Sufficiency*. Similarly to the proof of (i) and (ii), if $\lim_{|\varphi(z)|\to 1} F(z)G_\mu(\varphi(z)) = 0$, then uC_φ is a compact operator when $\lim_{|z|\to 1} H_\mu(z) < \infty$ and $\lim_{|z|\to 1} G_\mu(z) = \infty$.

Necessity. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty}|\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$. Set

$$k_n(z) = \frac{\int_0^{\overline{a_n}z}\left(\int_0^t \mu_*(s)ds\right)^2 dt}{\int_0^{|a_n|}\mu_*(t)dt}.$$

Then

$$k_n'(z) = \frac{\overline{a_n}\left(\int_0^{\overline{a_n}z}\mu_*(s)ds\right)^2}{\int_0^{|a_n|}\mu_*(s)ds} \quad \text{and} \quad k_n''(z) = \frac{2(\overline{a_n})^2\mu_*(\overline{a_n}z)\int_0^{\overline{a_n}z}\mu_*(s)ds}{\int_0^{|a_n|}\mu_*(s)ds}.$$

By Lemma 1 and Remark 1, we see that $k_n \in \mathscr{Z}_\mu$. Moreover, $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 6, $\lim_{n\to\infty}\|uC_\varphi k_n\|_{\mathscr{B}_\omega} = 0$. Since $\sup_{z\in\mathbb{D}} E(z) < \infty$, by Lemma 5 we get $\lim_{n\to\infty}E(z_n)|k_n(a_n)| = 0$. Since

$$\|uC_\varphi f_n\|_{\mathscr{B}_\omega} \geqslant F(z_n)|k_n'(a_n)| - E(z_n)|k_n(a_n)|,$$

we get

$$\lim_{n\to\infty} F(z_n)|k_n'(a_n)| = 0.$$

Since $\mu_*(t) \approx \frac{1}{\mu(t)}$, that is

$$\lim_{n\to\infty} F(z_n)\left|\frac{\left(\int_0^{|a_n|^2} \frac{1}{\mu(s)}ds\right)^2}{\int_0^{|a_n|} \frac{1}{\mu(s)}ds}\right| = 0. \tag{23}$$

For any $u \in (0,1)$,

$$\int_0^{u^2} \frac{1}{\mu(t)}dt = \int_0^u \frac{1}{\mu(\sqrt{t})}\frac{1}{2\sqrt{t}}dt > \frac{1}{2}\int_0^u \frac{1}{\mu(\sqrt{t})}dt \approx \int_0^u \frac{1}{\mu(t)}dt,$$

thus $\lim_{u\to 1} \int_0^{u^2} \frac{1}{\mu(t)}dt = \infty$. So there is an $r \in (\frac{1}{2}, 1)$, such that if $u > r$,

$$\frac{1}{2}\int_0^{u^2} \frac{1}{\mu(t)}dt \leqslant \int_0^{u^2} \frac{1}{\mu(t)}dt - \int_0^{\frac{1}{4}} \frac{1}{\mu(t)}dt \leqslant \int_0^{u^2} \frac{1}{\mu(t)}dt$$

and

$$\frac{1}{2}\int_0^{u} \frac{1}{\mu(t)}dt \leqslant \int_0^{u} \frac{1}{\mu(t)}dt - \int_0^{\frac{1}{2}} \frac{1}{\mu(t)}dt \leqslant \int_0^{u} \frac{1}{\mu(t)}dt.$$

From Cauchy's differential mean value theorem, for any $u \in (r,1)$, there exists an $\eta \in (\frac{1}{2}, u)$, such that

$$\frac{\int_0^{u^2} \frac{1}{\mu(t)}dt}{\int_0^{u} \frac{1}{\mu(t)}dt} \approx \frac{\int_0^{u^2} \frac{1}{\mu(t)}dt - \int_0^{\frac{1}{4}} \frac{1}{\mu(t)}dt}{\int_0^{u} \frac{1}{\mu(t)}dt - \int_0^{\frac{1}{2}} \frac{1}{\mu(t)}dt} = \frac{2\eta\frac{1}{\mu(\eta^2)}}{\frac{1}{\mu(\eta)}} \approx 1.$$

By (23), we get

$$\lim_{n\to\infty} F(z_n)\int_0^{|a_n|} \frac{1}{\mu(t)}dt = 0,$$

which implies that

$$\lim_{n\to\infty} F(z_n)G_\mu(\varphi(z_n)) = 0.$$

Therefore $\lim_{|\varphi(z)|\to 1} F(z)G_\mu(\varphi(z)) = 0$. The proof is completed. □

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