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SHARP BOUNDS FOR m–LINEAR HILBERT–TYPE OPERATORS ON THE WEIGHTED MORREY SPACES

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Abstract. On the product of m weighted Morrey spaces, some m-linear operators are shown to be bounded. The operator norm is calculated explicitly. It may be interesting to compare the results for the Hardy operator and the ones for the Hardy-Littlewood maximal operator. In the end of this article, some concrete examples are presented.

1. Introduction and main results

In this paper we show that weighted Morrey spaces are useful when we consider norm inequalities of m-linear operators in that the best constant can be attained. For example, the Hilbert inequality asserts that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y} dxdy < \pi \operatorname{cosec}\left(\frac{\pi}{p}\right) \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^{p'}(\mathbb{R}_+)} \tag{1}$$

holds for non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$ for $1 < p < \infty$. The parameters p and p' appearing in (1) are mutually conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. In addition, the constant $\pi \operatorname{cosec}\left(\frac{\pi}{p}\right)$ is the best possible in the sense that it can not be replaced with a smaller constant so that (1) still holds. The Hilbert inequality is one of the most interesting inequalities in mathematical analysis. Its applications have contributed so much in diverse fields of mathematics. At present, because of the requirement of higher-dimensional analysis and operator theory, multidimensional Hilbert-type inequalities have been studied. For more details about the Hilbert-type inequality, the reader is referred to [1, 2, 16, 18] as well as [20].

We present the definition of the weighted Morrey space $\mathscr{M}_q^p(\mathbb{R}_+, w_1, w_2)$ on $\mathbb{R}_+$, where $w_1, w_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are positive measurable functions.

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DEFINITION 1. Let $1 \leqslant q \leqslant p < \infty$. Then the weighted Morrey space $\mathscr{M}^p_q(\mathbb{R}_+, w_1, w_2)$ is the set of all $f \in L^q_{\rm loc}(\mathbb{R}_+)$ for which the norm

$$\|f\|_{\mathscr{M}^p_q(\mathbb{R}_+,w_1,w_2)} = \sup_I w_1(I)^{\frac{1}{p}-\frac{1}{q}} \left(\int_I |f(y)|^q w_2(y)\,dy\right)^{\frac{1}{q}}$$

is finite, where

$$w_1(I) = \int_I w_1(y)\,dy.$$

Here I moves over all intervals in $\mathbb{R}_+$. When $w_1 = 1$, abbreviate $\mathscr{M}^p_q(\mathbb{R}_+, w_1, w_2)$ to $\mathscr{M}^p_q(\mathbb{R}_+, w_2)$.

In the present paper, we are particularly interested in the norm

$$\|f\|_{\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_I \left(\int_I x^\alpha\,dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_I |f(y)|^q y^\beta\,dy\right)^{\frac{1}{q}}$$

It is easy to verify the following scaling law:

LEMMA 1. *Let $1 \leqslant q \leqslant p < \infty$ and $\alpha, \beta \in \mathbb{R}$. Let $t > 0$ and $f \in \mathscr{M}^p_q(\mathbb{R}_+, x^\alpha, x^\beta)$. Then we have*

$$\|f(t\cdot)\|_{\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = t^{-\frac{1}{p}-\frac{\beta}{q}-\alpha\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)}. \tag{2}$$

Motivated by (2), we define the *dilation index* for $\mathscr{M}^p_q(\mathbb{R}_+, x^\alpha, x^\beta)$ by

$$d(p,q,\alpha,\beta) = \frac{1}{p} + \frac{\beta}{q} + \alpha\left(\frac{1}{p} - \frac{1}{q}\right).$$

As the relation $\mathscr{M}^p_p(\mathbb{R}_+, u, w) = L^p(w)$ with norm coincidence implies, weighted Morrey spaces may be considered as an extension of weighted Lebesgue spaces. Furthermore unlike the weighted Lebesgue space $L^p(x^\alpha)$ the weighted Morrey space $\mathscr{M}^p_q(\mathbb{R}_+, x^\alpha, x^\beta)$ contains $x^{-d(p,q,\alpha,\beta)}$ when $\alpha \neq -1$. More precisely,

LEMMA 2. *Let $p > q \geqslant 1$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \left(1 - \frac{q}{p}\right)^{-\frac{1}{q}} |\alpha+1|^{-\frac{1}{p}}. \tag{3}$$

Thus, $x^{-d(p,q,\alpha,\beta)} \in \mathscr{M}^p_q(\mathbb{R}_+, x^\alpha, x^\beta)$ if and only if $\alpha \neq -1$.

As we establish in this paper, such a function attains the best constant in many occasions. Thus it is natural and important to study the boundedness of the operator on weighted Morrey spaces.

In this paper, placing ourselves mainly in the setting of $\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)$, we obtain the operator norm of m-linear Hilbert operator as well as the one for the Hardy operator given by

$$Hf(x) = \frac{1}{x}\int_0^x f(y)dy \quad (x>0).$$

We obtain the corresponding new operator norm inequalities as well.

We shall prove the following results.

THEOREM 1. *Let $m \in \mathbb{N}$. Suppose we have real parameters $\alpha,\beta,\beta_j,p,q,p_j,q_j$ for $j=1,\cdots,m$ satisfying*

$$1 \leqslant q \leqslant p < \infty, \quad 1 < q_j \leqslant p_j < \infty$$

for $j=1,2,\ldots,m$. Assume

$$\beta = \beta_1 + \cdots + \beta_m, \tag{4}$$

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} \tag{5}$$

and

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}. \tag{6}$$

Set

$$C(p_j,q,q_j,\alpha,\beta_j) = d\left(p_j,q_j,\alpha,\frac{\beta_j q_j}{q}\right) = \frac{1}{p_j} + \frac{\beta_j}{q} + \alpha\left(\frac{1}{p_j} - \frac{1}{q_j}\right).$$

Furthermore, let $K : \mathbb{R}_+^{m+1} \to [0,\infty)$ be a measurable function homogeneous of degree $-m$ satisfying

$$M = \int_{\mathbb{R}_+^m} K(1,y_1,y_2,\ldots,y_m)\prod_{j=1}^m y_j^{-C(p_j,q,q_j,\alpha,\beta_j)}\,dy_1\,dy_2\cdots dy_m < \infty. \tag{7}$$

Then the m-linear Hilbert-type operator

$$T(f_1,f_2,\ldots,f_m)(x) = \int_{\mathbb{R}_+^m} K(x,y_1,y_2,\ldots,y_m)\prod_{j=1}^m f_j(y_j),dy_1\cdots dy_m \quad (x>0)$$

is a bounded linear operator from $\prod_{j=1}^m \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})$ to $\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)$ with the operator norm less than or equal to M.

Moreover, if $p_j > q_j$, $\alpha \neq -1$ and

$$\frac{q_1}{p_1} = \frac{q_2}{p_2} = \cdots = \frac{q_m}{p_m} = \frac{q}{p}, \tag{8}$$

then

$$\|T\|_{\prod_{j=1}^m \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})\to\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)} = M. \tag{9}$$

By letting $q_j \to p_j$, $j=1,2,\cdots,m$ we recover the results on Lebesgue spaces.

COROLLARY 1. *Let $m \in \mathbb{N}$. Suppose we have real parameters β, β_j, p, p_j for $j = 1, \cdots, m$ satisfying*

$$1 \leqslant p < \infty, \quad 1 < p_j < \infty, \quad \beta_j < p\left(1 - \frac{1}{p_j}\right), \quad \beta > -1$$

for $j = 1, 2, \ldots, m$. Assume

$$\beta = \beta_1 + \cdots + \beta_m, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}.$$

Furthermore, let $K : \mathbb{R}_+^{m+1} \to [0, \infty)$ be a measurable function homogeneous of degree $-m$ satisfying

$$\int_{\mathbb{R}_+^m} K(1, y_1, y_2, \ldots, y_m) \prod_{j=1}^m y_j^{a_j} \, dy_1 \, dy_2 \cdots dy_m < \infty,$$

for $a_1, a_2, \cdots, a_m > -1$ and $a_1 + a_2 + \cdots + a_m < 0$. Then the operator T is a bounded from $\prod_{j=1}^m L^{p_j}(\mathbb{R}_+, x^{\beta_j p_j / p})$ to $L^p(\mathbb{R}_+, x^\beta)$ with the operator norm less than or equal to

$$M_L = \int_{\mathbb{R}_+^m} K(1, y_1, y_2, \ldots, y_m) \prod_{j=1}^m y_j^{-\frac{1}{p_j} - \frac{\beta_j}{p}} \, dy_1 \, dy_2 \cdots dy_m.$$

Moreover,

$$\|T\|_{\prod_{j=1}^m L^{p_j}(\mathbb{R}_+, x^{\beta_j p_j / p}) \to L^p(\mathbb{R}_+, x^\beta)} = M_L. \tag{10}$$

The proof of Theorem 1 hinges upon the Hölder inequality for weighted Morrey spaces. In fact, (4), (5) and (6) yield

$$\|f_1 f_2 \cdots f_m\|_{\mathscr{M}_q^p(\mathbb{R}_+, x^\alpha, x^\beta)} \leqslant \prod_{j=1}^m \|f_j\|_{\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+, x^\alpha, x^{\beta_j q_j / q})} \tag{11}$$

for all $(f_1, f_2, \ldots, f_m) \in \prod_{j=1}^m \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+, x^\alpha, x^{\beta_j q_j / q})$, since

$$f_1(x)^q f_2(x)^q \cdots f_m(x)^q x^\beta = f_1(x)^q x^{\beta_1} f_2(x)^q x^{\beta_2} \cdots f_m(x)^q x^{\beta_m}.$$

The next theorem concerns the Hardy operator.

THEOREM 2. *Let $1 \leqslant q \leqslant p < \infty$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq -1$. Then H is bounded if and only if*

$$d(p, q, \alpha, \beta) = \frac{1}{p} + \frac{\beta}{q} + \alpha\left(\frac{1}{p} - \frac{1}{q}\right) < 1. \tag{12}$$

In this case, the following equality holds:

$$\|H\|_{\mathscr{M}_q^p(\mathbb{R}_+, x^\alpha, x^\beta) \to \mathscr{M}_q^p(\mathbb{R}_+, x^\alpha, x^\beta)} = \frac{1}{1 - d(p, q, \alpha, \beta)}. \tag{13}$$

See [4, 6, 8, 9, 10, 14, 15, 17, 19, 22, 23, 31, 32] for the Hardy operator on Morrey spaces in various settings.

Mixing Theorems 1 and 2, we obtain the following inequality:

THEOREM 3. *With the assumptions as Theorem 1, if*

$$C(p_j,q,q_j,\alpha,\beta_j)=\frac{1}{p_j}+\frac{\beta}{q_j}+\alpha\left(\frac{1}{p_j}-\frac{1}{q_j}\right)<0 \tag{14}$$

and each f_j is a differentiable function such that $f_j(0)=0,\ j=1,2,\cdots,m$. Assume in addition that M given by (7) is finite. Then

$$\begin{aligned}\|T(f_1,f_2,\ldots,f_m)\|_{\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)}\\ \leqslant M\prod_{j=1}^m|C(p_j,q,q_j,\alpha,\beta_j)|^{-1}\|\mathscr{D}_+f_j\|_{\mathscr{M}^{p_j}_{q_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q+q_j})}.\end{aligned} \tag{15}$$

Moreover, if $p_j>q_j$, $\alpha\neq-1$ and $\frac{q_1}{p_1}=\frac{q_2}{p_2}=\cdots=\frac{q_m}{p_m}=\frac{q}{p}$, then the constant factor

$$M\prod_{j=1}^m|C(p_j,q,q_j,\alpha,\beta_j)|^{-1} \tag{16}$$

is the best possible.

Similarly, letting $q_j\to p_j$, $j=1,2,\cdots,m$, we obtain the following corollary.

COROLLARY 2. *With the assumptions as Corollary 1, if $C(p_j,p,p_j,\alpha,\beta_j)<0$ and f_j is a differentiable function such that $f_j(0)=0,\ j=1,2,\cdots,m$. Then*

$$\begin{aligned}\|T(f_1,f_2,\ldots,f_m)\|_{L^p(\mathbb{R}_+,x^\beta)}\\ \leqslant M\prod_{j=1}^m|C(p_j,p,p_j,\alpha,\beta_j)|^{-1}\|\mathscr{D}_+f_j\|_{L^{p_j}(\mathbb{R}_+,x^{\beta_j p_j/p+p_j})},\end{aligned} \tag{17}$$

where the constant factor $M\prod_{j=1}^m|C(p_j,p,p_j,\alpha,\beta_j)|^{-1}$ is the best possible.

For some related Hilbert-type inequalities involving some operators on weighted Lebesgue spaces, the reader is referred to the following references: [1] and [2].

2. Proof of the main results

First, we shall show the scaling law in the weighted Morrey space $\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)$ for all $1\leqslant q\leqslant p<\infty$ and $\alpha,\beta\in\mathbb{R}$. We prove Lemma 1.

Proof of Lemma 1. We calculate that

$$\begin{aligned}\|f(t\cdot)\|_{\mathscr{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)}&=\sup_I\left(\int_I y^\alpha\,dy\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_I|f(ty)|^q y^\beta\,dy\right)^{\frac{1}{q}}\\&=t^{-\frac{1}{q}-\frac{\beta}{q}}\sup_I\left(\int_I y^\alpha\,dy\right)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{tI}|f(y)|^q y^\beta\,dy\right)^{\frac{1}{q}}\end{aligned}$$

$$= t^{-\frac{1}{q}-\frac{\beta}{q}-(\alpha+1)\left(\frac{1}{p}-\frac{1}{q}\right)} \sup_I \left(\int_I (ty)^\alpha t\,dy\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{tI} |f(y)|^q y^\beta\,dy\right)^{\frac{1}{q}}$$
$$= t^{-\frac{1}{p}-\frac{\beta}{q}-\alpha\left(\frac{1}{p}-\frac{1}{q}\right)} \sup_I \left(\int_{tI} y^\alpha\,dy\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{tI} |f(y)|^q y^\beta\,dy\right)^{\frac{1}{q}}$$
$$= t^{-\frac{1}{p}-\frac{\beta}{q}-\alpha\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)}. \qquad \square$$

Proof of Lemma 2. Writing out the norm fully, we have

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_{t_2>t_1>0} \left(\int_{t_1}^{t_2} x^\alpha\,dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{t_1}^{t_2} x^{-d(p,q,\alpha,\beta)q+\beta}\,dx\right)^{\frac{1}{q}}.$$

Notice that

$$(\alpha+1)\left(\frac{1}{p}-\frac{1}{q}\right) - d(p,q,\alpha,\beta) + \frac{\beta+1}{q} = 0. \tag{18}$$

Thus, by the scaling argument, we have

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_{t>1} \left(\int_1^t x^\alpha\,dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_1^t x^{-d(p,q,\alpha,\beta)q+\beta}\,dx\right)^{\frac{1}{q}}.$$

Assume first that $\alpha > -1$. In this case, we have

$$\frac{\beta+1}{q} - d(p,q,\alpha,\beta) = (\alpha+1)\left(\frac{1}{q}-\frac{1}{p}\right) > 0.$$

If we calculate the integral, then we have

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)}$$
$$= \frac{(\alpha+1)^{\frac{1}{q}-\frac{1}{p}}}{(\beta+1-d(p,q,\alpha,\beta)q)^{\frac{1}{q}}} \sup_{t>1} \left(t^{\alpha+1}-1\right)^{\frac{1}{p}-\frac{1}{q}} \left(t^{-d(p,q,\alpha,\beta)q+\beta+1}-1\right)^{\frac{1}{q}}.$$

From (18), we deduce

$$(\alpha+1)^{\frac{1}{q}-\frac{1}{p}}(\beta+1-d(p,q,\alpha,\beta)q)^{-\frac{1}{q}} = \left(1-\frac{q}{p}\right)^{-\frac{1}{q}}(\alpha+1)^{-\frac{1}{p}}.$$

Again from (18), we have

$$-d(p,q,\alpha,\beta)q+\beta+1 = q\left(\frac{1}{q}-\frac{1}{p}\right)(\alpha+1).$$

As the relation

$$t^{ab}-1-(t^a-1)^b = b\int_{t^a-1}^{t^a} s^{b-1}\,ds - 1 \leqslant b\int_0^1 s^{b-1}\,ds - 1 = 0$$

for $a > 0$, $0 < b < 1$ and $t \geqslant 1$ implies

$$t^{-d(p,q,\alpha,\beta)q+\beta+1} - 1 = t^{(\alpha+1)\left(1-\frac{q}{p}\right)} - 1 \leqslant (t^{\alpha+1} - 1)^{1-\frac{q}{p}}.$$

Thus,

$$\sup_{t>1} \left(t^{\alpha+1} - 1\right)^{\frac{1}{p}-\frac{1}{q}} \left(t^{-d(p,q,\alpha,\beta)q+\beta+1} - 1\right)^{\frac{1}{q}} = 1.$$

Hence (3) holds.

If $\alpha < -1$, then

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_{0<t<1} \left(\int_t^1 x^\alpha\,dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_t^1 x^{-d(p,q,\alpha,\beta)q+\beta}\,dx\right)^{\frac{1}{q}}$$

by the scaling argument. Calculating the integral, we obtain

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_{0<t<1} \left(\frac{t^{\alpha+1} - 1}{-(\alpha+1)}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{t^{-d(p,q,\alpha,\beta)q+\beta} - 1}{d(p,q,\alpha,\beta)q - \beta - 1}\right)^{\frac{1}{q}}.$$

By the change of the variables, we have

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_{T>1} \left(\frac{T^{-(\alpha+1)} - 1}{-(\alpha+1)}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{T^{d(p,q,\alpha,\beta)q-\beta} - 1}{d(p,q,\alpha,\beta)q - \beta - 1}\right)^{\frac{1}{q}}.$$

Going through a similar argument, we learn (3) holds.

Finally, let $\alpha = -1$. Then we have

$$\|x^{-d(p,q,\alpha,\beta)}\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} = \sup_{t>1} \left(\int_1^t x^{-1}\,dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_1^t x^{-1}\,dx\right)^{\frac{1}{q}} = \infty$$

again by the dilation argument. □

Proof of Theorem 1. By the change of variables, we have

$$T(f_1, f_2, \dots, f_m)(x) = x^m \int_{\mathbb{R}^m_+} K(x, xy_1, xy_2, \dots, xy_m) \prod_{j=1}^m f_j(xy_j)\,dy_1\,dy_2 \cdots dy_m.$$

Since K is homogeneous of degree $-m$, we obtain

$$T(f_1, f_2, \dots, f_m)(x) = \int_{\mathbb{R}^m_+} K(1, y_1, y_2, \dots, y_m) \prod_{j=1}^m f_j(xy_j)\,dy_1\,dy_2 \cdots dy_m.$$

By the triangle inequality, we have

$$\begin{aligned}
&\|T(f_1, f_2, \dots, f_m)\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} \\
&\leqslant \int_{\mathbb{R}^m_+} K(1, y_1, y_2, \dots, y_m) \left\| \prod_{j=1}^m f_j(y_j\cdot) \right\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} dy_1\,dy_2 \cdots dy_m.
\end{aligned}$$

By the Hölder inequality (11), we have

$$\|T(f_1,f_2,\ldots,f_m)\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} \leqslant \int_{\mathbb{R}^m_+} K(1,y_1,y_2,\ldots,y_m)\prod_{j=1}^m \|f_j(y_j\cdot)\|_{\mathcal{M}^{p_j}_{q_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})}\,dy_1\,dy_2\cdots dy_m.$$

Using Lemma 1, we obtain

$$\prod_{j=1}^m \|f_j(y_j\cdot)\|_{\mathcal{M}^{p_j}_{q_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})} = \prod_{j=1}^m {y_j}^{-C(p_j,q,q_j,\alpha,\beta_j)}\|f_j\|_{\mathcal{M}^{p_j}_{q_j}(\mathbb{R}_+,y^\alpha,y^{\beta_j q_j/q})}$$

and hence

$$\|T(f_1,f_2,\ldots,f_m)\|_{\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)} \leqslant M\prod_{j=1}^m \|f_j\|_{\mathcal{M}^{p_j}_{q_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})}. \tag{19}$$

As a result it follows that the integral operator T is a bounded linear operator from $\prod_{j=1}^m \mathcal{M}^{p_j}_{q_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})$ to $\mathcal{M}^p_q(\mathbb{R}_+,x^\alpha,x^\beta)$.

Now, we will show that the operator norm of T is exactly M. Taking

$$f_j(x) = x^{-C(p_j,q,q_j,\alpha,\beta_j)}, \quad j=1,\cdots,m,$$

we calculate that

$$T(f_1,f_2,\ldots,f_m)(x) = Mx^{-d(p,q,\alpha,\beta)}. \tag{20}$$

Thus we see that (9) holds. □

Proof of Corollary 1. By letting $q_j \to p_j$, $j=1,2,\cdots,m$ on the inequality (19), we have

$$\|T(f_1,f_2,\ldots,f_m)\|_{L^p(\mathbb{R}_+,x^\beta)} \leqslant M_L\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}_+,x^{\beta_j p_j/p})}. \tag{21}$$

Now, we prove that inequality (21) involves the best possible constant factor on its right-hand side i.e.

$$\|T\|_{\prod_{j=1}^m L^{p_j}(\mathbb{R}_+,x^{\beta_j p_j/p})\to L^p(\mathbb{R}_+,x^\beta)} = M_L.$$

First, suppose that there exists a positive constant C smaller than M_L such that the inequality

$$\|T(f_1,f_2,\ldots,f_m)\|_{L^p(\mathbb{R}_+,x^\beta)} \leqslant C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}_+,x^{\beta_j p_j/p})}. \tag{22}$$

Let $\varepsilon > 0$ be a sufficiently small number. We set

$$a_j = -\frac{1}{p_j} - \frac{\beta_j}{p} - \frac{\varepsilon}{p_j},$$

for $j = 1,2,\cdots,m$. Considering the inequality (22) with functions f_j^ε defined by

$$f_j^\varepsilon(x) = x^{a_j}\chi_{(1,\infty)} \quad (j = 1,2,\cdots,m),$$

we learn that the right-hand side reduces to

$$\prod_{j=1}^{m} \|f_j^\varepsilon\|_{L^{p_j}(\mathbb{R}_+,x^{\beta_j p_j/p})} = \prod_{j=1}^{m}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p_j}} = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}. \tag{23}$$

Let $p > 1$. We calculate that

$$\begin{aligned}
&\|T(f_1^\varepsilon, f_2^\varepsilon,\ldots,f_m^\varepsilon)\|_{L^p(\mathbb{R}_+,x^\beta)}\\
&= \left(\int_0^\infty \left(\int_{\mathbb{R}_+^m} K(x,y_1,y_2,\ldots,y_m)\prod_{j=1}^m f_j^\varepsilon(y_j)dy_1\cdots dy_m\right)^p x^\beta dx\right)^{\frac{1}{p}}\\
&= \left(\int_0^\infty \left(\int_{(1,\infty)^m} K(x,y_1,y_2,\ldots,y_m)\prod_{j=1}^m y_j^{a_j}dy_1\cdots dy_m\right)^p x^\beta dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Then using the Hölder inequality, we have

$$\begin{aligned}
&\|T(f_1^\varepsilon, f_2^\varepsilon,\ldots,f_m^\varepsilon)\|_{L^p(\mathbb{R}_+,x^\beta)}\\
&\geqslant \left(\int_1^\infty \left(\int_{(1,\infty)^m} K(x,y_1,y_2,\ldots,y_m)\prod_{j=1}^m y_j^{a_j}dy_1\cdots dy_m\right)^p x^\beta dx\right)^{\frac{1}{p}}\\
&\geqslant \left(\int_1^\infty x^{-1-\varepsilon}dx\right)^{-\frac{1}{p'}}\int_1^\infty\left(\int_{(1,\infty)^m} K(x,y_1,y_2,\ldots,y_m)\prod_{j=1}^m y_j^{a_j}dy_1\cdots dy_m\right)x^{\frac{\beta}{p}-\frac{1}{p'}-\frac{\varepsilon}{p'}}dx\\
&= \left(\frac{1}{\varepsilon}\right)^{-\frac{1}{p'}}\cdot \mathrm{I},
\end{aligned}$$

where

$$\mathrm{I} = \int_1^\infty x^{-1-\varepsilon}\left(\int_{(1/x,\infty)^m} K(1,y_1,y_2,\ldots,y_m)\prod_{j=1}^m y_j^{a_j}dy_1\cdots dy_m\right)dx.$$

In the case $p = 1$, we can get a similar inequality as above without using the Hölder inequality.

Let $j = 1,\cdots,m$. We write

$$\mathbb{D}_j = \mathbb{D}_j(x) = \{(y_1,y_2,\ldots,y_m); 0 < y_j \leqslant 1/x, y_i > 0, i \neq j\}.$$

We also set

$$I_j(x) = \int_{\mathbb{D}_j(x)} K(1,y_1,y_2,\ldots,y_m)\prod_{k=1}^m y_k^{a_k}dy_1\cdots dy_m.$$

In view of the overlapping of the $\mathbb{D}_j(x)$'s, we have

$$I \geqslant \int_1^\infty \frac{dx}{x^{1+\varepsilon}} \left(\int_{\mathbb{R}_+^m} K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j}, dy_1 \cdots dy_m \right) - \int_1^\infty x^{-1-\varepsilon} \sum_{j=1}^m I_j(x)dx$$
$$\geqslant \frac{1}{\varepsilon} \int_{\mathbb{R}_+^m} K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m - \int_1^\infty \sum_{j=1}^m \frac{I_j(x)}{x} dx, \tag{24}$$

Without loss of generality, it suffices to find the appropriate estimate for the integral

$$I_1(x) = \int_{\mathbb{D}_1} K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m.$$

To this end, we choose γ such that $0 < \gamma < 1 + a_1$, so that

$$-y_1^\gamma \log y_1 \leqslant \frac{1}{e\gamma} \tag{25}$$

for $y_1 \in (0,1]$. Then by virtue of the Fubini theorem, we have

$$\begin{aligned}
&\int_1^\infty x^{-1} I_1(x)dx \\
&= \int_1^\infty x^{-1} \int_{\mathbb{D}_1} K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m dx \\
&= \int_1^\infty x^{-1} \left[\int_{\mathbb{R}_+^{m-1}} \int_0^{1/x} K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m \right] dx \\
&= \int_{\mathbb{R}_+^{m-1}} \int_0^1 K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} \left(\int_1^{1/y_1} x^{-1} dx \right) dy_1 \cdots dy_m \\
&= \int_{\mathbb{R}_+^{m-1}} \int_0^1 (-\log y_1) K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m.
\end{aligned}$$

From (25), we deduce

$$\begin{aligned}
\int_1^\infty x^{-1} I_1(x)dx &\leqslant \frac{1}{e\gamma} \int_{\mathbb{R}_+^{m-1}} \int_0^1 K(1,y_1,y_2,\ldots,y_m) y_1^{-\gamma} \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m \\
&\leqslant \frac{1}{e\gamma} \int_{\mathbb{R}_+^m} K(1,y_1,y_2,\ldots,y_m) y_1^{-\gamma} \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m < \infty.
\end{aligned}$$

Hence by (24), we have

$$\begin{aligned}
&\|T(f_1^\varepsilon, f_2^\varepsilon, \ldots, f_m^\varepsilon)\|_{L^p(\mathbb{R}_+, x^\beta)} \\
&\geqslant \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} \int_{\mathbb{R}_+^m} K(1,y_1,y_2,\ldots,y_m) \prod_{j=1}^m y_j^{a_j} dy_1 \cdots dy_m - \left(\frac{1}{\varepsilon}\right)^{-\frac{1}{p'}} \cdot O(1).
\end{aligned}$$

Moreover, (22) and (23) imply that

$$C \geqslant \int_{\mathbb{R}_+^m} K(1,y_1,y_2,\dots,y_m)\prod_{j=1}^m y_j^{a_j}dy_1\cdots dy_m - \varepsilon\cdot O(1).$$

Obviously, letting $\varepsilon \to 0^+$, it follows that $C \geqslant M_L$, which contradicts our assumption. Hence, M_L is the best possible in (21). □

Proof of Theorem 2. Let us suppose $d(p,q,\alpha,\beta) < 1$. By the change of variables, we have

$$Hf(x) = \frac{1}{x}\int_0^x f(y)dy = \int_0^1 f(xy)dy.$$

Using the triangle inequality and the scaling law, we have

$$\begin{aligned}
\|Hf\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)} &= \left\|\int_0^1 f(xy)dy\right\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)} \\
&\leqslant \int_0^1 \|f(y\cdot)\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)}dy \\
&= \frac{1}{1-d(p,q,\alpha,\beta)}\|f\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)}.
\end{aligned}$$

To calculate the operator norm, we define $f(x) = x^{-d(p,q,\alpha,\beta)}$. Then

$$\begin{aligned}
\|Hf\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)} &= \frac{1}{1-d(p,q,\alpha,\beta)}\|x^{-d(p,q,\alpha,\beta)}\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)} \\
&= \frac{1}{1-d(p,q,\alpha,\beta)}\|f\|_{\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)}.
\end{aligned}$$

If $d(p,q,\alpha,\beta) \geqslant 1$, then $Hf = \infty$ for $f(x) = x^{-d(p,q,\alpha,\beta)} \in \mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)$.

The proof is now complete. □

Proof of Theorem 3. In order to prove (15) we will rewrite the right-hand side of inequality (19) in a form that is more suitable for the application of the inequality (13). Namely, since

$$xH(\mathscr{D}_+f)(x) = \int_0^x f'(t)dt = f(x) - f(0) = f(x),$$

we have that

$$M\prod_{j=1}^m \|f_j\|_{\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})} = M\prod_{j=1}^m \|xH(\mathscr{D}_+f_j)\|_{\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})}. \tag{26}$$

Now, due to the inequality (13), it follows that

$$\begin{aligned}
&\|xH(\mathscr{D}_+ f_j)\|_{\mathscr{M}^{p_j}_{q_j}(\mathbb{R}_+, x^\alpha, x^{\beta_j q_j/q})} \\
&= \|H(\mathscr{D}_+ f_j)\|_{\mathscr{M}^{p_j}_{q_j}(\mathbb{R}_+, x^\alpha, x^{\beta_j q_j/q+q_j})} \\
&\leqslant \frac{1}{1-d(p_j,q_j,\alpha,\beta_j q_j/q+q_j)} \|\mathscr{D}_+ f_j\|_{\mathscr{M}^{p_j}_{q_j}(\mathbb{R}_+, x^\alpha, x^{\beta_j q_j/q+q_j})} \\
&= \frac{1}{|C(p_j,q,q_j,\alpha,\beta_j)|} \|\mathscr{D}_+ f_j\|_{\mathscr{M}^{p_j}_{q_j}(\mathbb{R}_+, x^\alpha, x^{\beta_j q_j/q+q_j})}. \qquad (27)
\end{aligned}$$

Hence, the inequality (15) holds due to (19), (26) and (27).

It remains to show that $M\prod_{j=1}^m |C(p_j,q,q_j,\alpha,\beta_j)|^{-1}$ is the best possible constant in (15) when $p_j > q_j$, $\alpha \neq -1$ and (8) holds. Now, for each $j = 1,2,\dots,m$ consider the function $f_j(x) = x^{-C(p_j,q,q_j,\alpha,\beta_j)}$. Then

$$\mathscr{D}_+ f_j(x) = -C(p_j,q,q_j,\alpha,\beta_j)x^{C(p_j,q,q_j,\alpha,\beta_j)-1}.$$

Thus we see that the constant factor (14) is the best possible. □

Proof of Corollary 2. The proof is similar to the proof of Corollary 1. We leave the details to the reader. □

3. Examples

In this section, we discuss our main results with regard to some particular choices of kernels.

3.1. Three convergent integrals

We give the necessary and sufficient conditions for the multiple integral in these theorems to converge.

We calculate

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbb{R}^n_+} \frac{{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}}{\max(1,y_1,\dots,y_m)^m}\, dy_1\, dy_2 \cdots dy_m, \\
\mathrm{J} &= \int_{\mathbb{R}^n_+} \frac{{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}}{(1+y_1+y_2+\cdots+y_m)^m}\, dy_1\, dy_2 \cdots dy_m, \\
\mathrm{K} &= \int_{\mathbb{R}^n_+} \frac{{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}}{(1+{y_1}^2+{y_2}^2+\cdots+{y_m}^2)^{\frac{m}{2}}}\, dy_1\, dy_2 \cdots dy_m.
\end{aligned}$$

LEMMA 3. *Let $a_1,a_2,\dots,a_m \in \mathbb{R}$. Then the following are equivalent:*

1. $a_1,a_2,\dots,a_m > -1$ *and* $a_1+a_2+\cdots+a_m < 0$.

2. $\mathrm{I} < \infty$.

3. $\mathrm{J} < \infty$.

4. $\mathrm{K} < \infty$.

Proof. Clearly I and J are simultaneously finite since

$$\max(1, y_1, y_2, \ldots, y_m) \leqslant 1 + y_1 + y_2 + \cdots + y_m \leqslant m \max(1, y_1, y_2, \ldots, y_m).$$

Likewise I and K are simultaneously finite. By the polar coordinate K is finite if and only if

$$\mathrm{L} = \int_{1 \leqslant y_1{}^2 + y_2{}^2 + \cdots + y_m{}^2 \leqslant 4} \frac{y_1{}^{a_1} y_2{}^{a_2} \cdots y_m{}^{a_m}}{(1 + y_1{}^2 + y_2{}^2 + \cdots + y_m{}^2)^{\frac{m}{2}}} \, dy_1 \, dy_2 \cdots dy_m$$

and

$$\mathrm{M} = \int_0^\infty \frac{r^{a_1 + a_2 + \cdots + a_m + m - 1}}{(1 + r^2)^{\frac{m}{2}}} \, dr$$

are both finite. Thus, I is finite if and only if $a_1, a_2, \ldots, a_m > -1$ and $a_1 + a_2 + \cdots + a_m < 0$. □

LEMMA 4. *Let $a_1, a_2, \ldots, a_m \in (-1, \infty)$ satisfy $a_1 + a_2 + \cdots + a_m < 0$.*

1. $\mathrm{I} = \dfrac{-m}{a_1 + a_2 + \cdots + a_m} \displaystyle\prod_{j=1}^{m} \frac{1}{a_j + 1}$.

2. $\mathrm{J} = \dfrac{\Gamma(-a_1 - a_2 - \cdots - a_m)\Gamma(a_1 + 1)\Gamma(a_2 + 1) \cdots \Gamma(a_m + 1)}{\Gamma(m)}$.

3. $\mathrm{K} = \dfrac{\Gamma\left(\frac{a_1 + a_2 + \cdots + a_m + m}{2}\right)\Gamma\left(\frac{m - a_1 - a_2 - \cdots - a_m}{2}\right)}{2^m \Gamma(m) \Gamma\left(\frac{m}{2}\right)} \displaystyle\prod_{j=1}^{m} \Gamma\left(\frac{a_j + 1}{2}\right)$.

Proof. (1) This integral is calculated in [3, Claim 1]. Here for the sake of convenience for readers we supply the proof. We calculate

$$\mathrm{I}_m = \int_{0 < y_1, y_2, \ldots, y_{m-1} \leqslant y_m < \infty} \frac{y_1{}^{a_1} y_2{}^{a_2} \cdots y_m{}^{a_m}}{\max(1, y_1, \ldots, y_m)^m} \, dy_1 \, dy_2 \cdots dy_m.$$

By simplifying the expression, we have

$$\mathrm{I}_m = \int_{0 < y_1, y_2, \ldots, y_{m-1} \leqslant y_m < \infty} \frac{y_1{}^{a_1} y_2{}^{a_2} \cdots y_m{}^{a_m}}{\max(1, y_m)^m} \, dy_1 \, dy_2 \cdots dy_m.$$

If we integrate against $y_1, y_2, \ldots, y_{m-1}$, we have

$$\mathrm{I}_m = \frac{1}{(a_1 + 1)(a_2 + 1) \cdots (a_{m-1} + 1)} \int_{\mathbb{R}_+} \frac{y_m{}^{m-1+a_1+a_2+\cdots+a_m}}{\max(1, y_m)^m} \, dy_1 \, dy_2 \cdots dy_m.$$

Since

$$\begin{aligned}&\int_{\mathbb{R}_+} \frac{{y_m}^{m-1+a_1+a_2+\cdots+a_m}}{\max(1,y_m)^m}\,dy_1\,dy_2\cdots dy_m\\&=\frac{1}{m+a_1+a_2+\cdots+a_m}-\frac{1}{a_1+a_2+\cdots+a_m}\\&=\frac{-m}{(m+a_1+a_2+\cdots+a_m)(a_1+a_2+\cdots+a_m)}\end{aligned}$$

(2) By the change of variables, we have

$$\begin{aligned}&\int_{y_1+y_2+\cdots+y_m<R, y_1,y_2,\ldots,y_m>0} \frac{{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}}{(1+y_1+y_2+\cdots+y_m)^m}\,dy_1\,dy_2\cdots dy_m\\&=R^{m+a_1+a_2+\cdots+a_m}\\&\quad\times\int_{y_1+y_2+\cdots+y_m<1, y_1,y_2,\ldots,y_m>0} \frac{{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}}{(1+Ry_1+Ry_2+\cdots+Ry_m)^m}\,dy_1\,dy_2\cdots dy_m.\end{aligned}$$

The above integral is known as the Dirichlet integral. Thus,

$$\begin{aligned}&\int_{y_1+y_2+\cdots+y_m<R, y_1,y_2,\ldots,y_m>0} \frac{{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}}{(1+y_1+y_2+\cdots+y_m)^m}\,dy_1\,dy_2\cdots dy_m\\&=R^{m+a_1+a_2+\cdots+a_m}\cdot\frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_m+1)}{\Gamma(m+a_1+a_2+\cdots+a_m)}\int_0^1\frac{t^{m-1+a_1+a_2+\cdots+a_m}}{(1+Rt)^m}\,dt\\&=\frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_m+1)}{\Gamma(m+a_1+a_2+\cdots+a_m)}\int_0^R\frac{t^{m-1+a_1+a_2+\cdots+a_m}}{(1+t)^m}\,dt.\end{aligned}$$

Since

$$\int_0^\infty t^A(1+t)^{-B}\,dt=\frac{\Gamma(1+A)\Gamma(B-A-1)}{\Gamma(B)} \tag{28}$$

for $A>-1$ and $B>-A-1$, we have the desired result.

(3) By the polar coordinate, we have

$$\begin{aligned}\mathrm{K}&=\int_{S^{m-1}}{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}\,d\sigma(y)\times\int_0^\infty\frac{r^{a_1+a_2+\cdots+a_m+n-1}}{(1+r^2)^{m/2}}\,dr\\&=\frac{1}{2}\int_{S^{m-1}}{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}\,d\sigma(y)\times\int_0^\infty\frac{r^{a_1/2+a_2/2+\cdots+a_m/2+n/2-1}}{(1+r)^{m/2}}\,dr\end{aligned}$$

In view of (28), we have

$$\begin{aligned}\mathrm{K}&=\frac{1}{2\Gamma(m)}\int_{S^{m-1}}{y_1}^{a_1}{y_2}^{a_2}\cdots{y_m}^{a_m}\,d\sigma(y)\\&\quad\times\Gamma\left(\frac{a_1+a_2+\cdots+a_m+m}{2}\right)\Gamma\left(\frac{m-a_1-a_2-\cdots-a_m}{2}\right).\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{S^{m-1}} {y_1}^{a_1}{y_2}^{a_2}\cdots {y_m}^{a_m}\,d\sigma(y)\times\int_0^\infty e^{-r^2}r^{m-1}\,dr\\
&=\int_{\mathbb{R}_+{}^m} {y_1}^{a_1}{y_2}^{a_2}\cdots {y_m}^{a_m}e^{-{y_1}^2-{y_2}^2-\cdots-{y_m}^2}\,dy_1\,dy_2\cdots dy_m\\
&=\frac{1}{2^m}\int_{\mathbb{R}_+{}^m} y_1^{\frac{a_1-1}{2}}y_2^{\frac{a_2-1}{2}}\cdots y_m^{\frac{a_m-1}{2}}e^{-y_1-y_2-\cdots-y_m}\,dy_1\,dy_2\cdots dy_m\\
&=\frac{1}{2^m}\prod_{j=1}^m\Gamma\left(\frac{a_j+1}{2}\right).
\end{aligned}$$

Thus,

$$\int_{S^{m-1}} {y_1}^{a_1}{y_2}^{a_2}\cdots {y_m}^{a_m}\,d\sigma(y)=\frac{1}{2^{m-1}}\Gamma\left(\frac{m}{2}\right)^{-1}\prod_{j=1}^m\Gamma\left(\frac{a_j+1}{2}\right).\quad\square$$

3.2. The kernel $K(x,y_1,y_2,\ldots,y_m)=(\max\{x,y_1,\ldots,y_m\})^{-m}$

An interesting example of a homogeneous kernel with degree $-m$, is the function

$$K(x,y_1,y_2,\ldots,y_m)=\frac{1}{(\max\{x,y_1,\ldots,y_m\})^m}.$$

COROLLARY 3. *Let $m\in\mathbb{N}$. Suppose we have real parameters $\alpha,\beta,\beta_j,p,q,p_j,q_j$ for $j=1,\cdots,m$ satisfying*

$$1\leqslant q\leqslant p<\infty,\quad 1<q_j\leqslant p_j<\infty$$

for $j=1,2,\ldots,m$. Assume

$$C(p,q,q,\alpha,\beta)<0,\quad C(p_j,q,q_j,\alpha,\beta_j)>-1,\quad j=1,2,\cdots,m,$$

and

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m},\quad \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m},\quad \beta=\beta_1+\cdots+\beta_m.$$

Then the m-linear Hardy-Littlewood-Pólya operator

$$T_H(f_1,f_2,\ldots,f_m)(x)=\int_{\mathbb{R}_+^m}\frac{1}{(\max\{x,y_1,\ldots,y_m\})^m}\prod_{j=1}^m f_j(y_j)\,dy_1dy_2\cdots dy_m\quad(x>0)$$

is a bounded linear operator from $\prod_{j=1}^m\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_jq_j/q})$ to $\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)$.

Moreover, if $p_j>q_j$, $\alpha\neq-1$ and (8) holds, then

$$\begin{aligned}
&\|T_H\|_{\prod_{j=1}^m\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_jq_j/q})\to\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)}\\
&=\frac{m}{-C(p,q,q,\alpha,\beta)}\prod_{j=1}^m\frac{1}{1+C(p_j,q,q_j,\alpha,\beta_j)}.
\end{aligned}\tag{29}$$

Proof. We have only to calculate

$$M_H = \int_{\mathbb{R}_+^m} \frac{1}{(\max\{1,y_1,\ldots,y_m\})^m} \prod_{j=1}^m y_j{}^{-C(p_j,q,q_j,\alpha,\beta_j)}\, dy_1\, dy_2 \cdots dy_m. \quad \square$$

3.3. The kernel $K(x,y_1,y_2,\ldots,y_m) = (x+y_1+y_2+\cdots+y_m)^{-m}$

The standard example of a homogeneous kernel with the negative degree of homogeneity is the function, defined by

$$K(x,y_1,y_2,\ldots,y_m) = (x+y_1+y_2+\cdots+y_m)^{-m}.$$

The constant M, appearing in (7), can be expressed in terms of the usual Gamma function Γ.

For this kernel, Theorem 1 yields the following.

COROLLARY 4. *Let $m \in \mathbb{N}$. Suppose we have real parameters $\alpha,\beta,\beta_j,p,q,p_j,q_j$ for $j=1,\cdots,m$ satisfying*

$$1 \leqslant q \leqslant p < \infty, \quad 1 < q_j \leqslant p_j < \infty$$

for $j=1,2,\ldots,m$. Assume

$$\beta = \beta_1+\cdots+\beta_m, \quad \frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}, \quad \frac{1}{q} = \frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}.$$

Assume that

$$C(p,q,q,\alpha,\beta) < 0, \quad C(p_j,q,q_j,\alpha,\beta_j) > -1,$$

for $j=1,2,\cdots,m$. Then the m-linear Hilbert operator

$$T_\oplus(f_1,f_2,\ldots,f_m)(x) = \int_{\mathbb{R}_+^m} \frac{1}{(x+y_1+y_2+\cdots+y_m)^m} \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \quad (x>0)$$

is a bounded linear operator from $\prod_{j=1}^m \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})$ to $\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)$.

Moreover, if $p_j > q_j$, $\alpha \neq -1$ and (8) holds, then

$$\begin{aligned} &\|T_\oplus\|_{\prod_{j=1}^m \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q}) \to \mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)} \\ &\quad = \frac{\Gamma(-C(p,q,q,\alpha,\beta))}{\Gamma(m)} \prod_{j=1}^m \Gamma(1+C(p_j,q,q_j,\alpha,\beta_j)). \end{aligned} \tag{30}$$

Proof. We have only to calculate

$$M_\oplus = \int_{\mathbb{R}_+^m} \frac{1}{(1+\sum_{j=1}^m y_j)^m} \prod_{j=1}^m y_j{}^{-C(p_j,q,q_j,\alpha,\beta_j)}\, dy_1\, dy_2 \cdots dy_m. \quad \square$$

It should be noticed here that for $q_j \to p_j$, $j=1,2,\cdots,m$, Corollary 3 reduces to the weighted version of [3, Claim 3.3].

3.4. The kernel $K(x,y_1,y_2,\ldots,y_m)=(x^2+{y_1}^2+{y_2}^2+\cdots+{y_m}^2)^{-m/2}$

For this kernel, Theorem 1 yields the following.

COROLLARY 5. *Let $m\in\mathbb{N}$. Suppose we have real parameters $\alpha,\beta,\beta_j,p,q,p_j,q_j$ for $j=1,\cdots,m$ satisfying*

$$1\leqslant q\leqslant p<\infty,\quad 1<q_j\leqslant p_j<\infty$$

for $j=1,2,\ldots,m$. Assume

$$\beta=\beta_1+\cdots+\beta_m,\quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m},\quad \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}.$$

Assume that

$$C(p,q,q,\alpha,\beta)<0,\quad C(p_j,q,q_j,\alpha,\beta_j)>-1,$$

for $j=1,2,\cdots,m$. Then the m-linear Hilbert-type operator

$$T_\dagger(f_1,f_2,\ldots,f_m)(x)=\int_{\mathbb{R}_+^m}\frac{1}{(x^2+y_1^2+y_2^2+\cdots+y_m^2)^{m/2}}\prod_{j=1}^m f_j(y_j)\,dy_1\cdots dy_m\quad (x>0)$$

is a bounded linear operator from $\prod_{j=1}^m\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})$ to $\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)$.

Moreover, if $p_j>q_j$, $\alpha\neq-1$ and (8) holds, then

$$\|T_\oplus\|_{\prod_{j=1}^m\mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+,x^\alpha,x^{\beta_j q_j/q})\to\mathscr{M}_q^p(\mathbb{R}_+,x^\alpha,x^\beta)}$$
$$=\frac{\Gamma\left(\frac{m-C(p,q,q,\alpha,\beta)}{2}\right)\Gamma\left(\frac{m+C(p,q,q,\alpha,\beta)}{2}\right)}{2^m\Gamma(m)\Gamma\left(\frac{m}{2}\right)}\prod_{j=1}^m\Gamma\left(\frac{-C(p_j,q,q_j,\alpha,\beta_j)+1}{2}\right).\quad (31)$$

4. Remarks

4.1. Weighted Lebesgue spaces

It may be interesting to compare these results with existing ones.

First, our results are located as extensions of the earlier results. In 2006, Bényi and Oh obtained the following result [3, Claim 1]:

THEOREM 4. *Let $m\in\mathbb{N}$, $1<p,p_1,\cdots,p_m<\infty$ be such that $\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. Furthermore, let $K:\mathbb{R}_+^{m+1}\to[0,\infty)$ be a measurable function homogeneous of degree $-m$ satisfying*

$$C_m=\int_{\mathbb{R}_+^m}K(1,y_1,y_2,\ldots,y_m)\prod_{j=1}^m y_j^{-\frac{1}{p_j}}\,dy_1\cdots dy_m<\infty.$$

Then the operator T is a bounded linear operator from $\prod_{j=1}^m L^{p_j}(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+)$. Moreover,

$$\|T\|_{\prod_{j=1}^m L^{p_j}(\mathbb{R}_+)\to L^p(\mathbb{R}_+)}=C_m.\quad (32)$$

In particular,

$$\|T_{\oplus}\|_{\prod_{j=1}^{m} L^{p_j}(\mathbb{R}_+)\to L^p(\mathbb{R}_+)} = \frac{1}{\Gamma(m)} \prod_{j=1}^{m} \Gamma\left(\frac{1}{p_j'}\right). \tag{33}$$

Note that Corollary 1 recaptures Theorem 4 as a special case of $\beta = \beta_1 = \beta_2 = \cdots = \beta_m = 0$.

In 2012, Fu et al. proved the following weighted partial extension of (33) [7, Theorem 3].

THEOREM 5. *Let $m \in \mathbb{N}$, $1 \leqslant p < \infty, 1 < p_1, \cdots, p_m < \infty$ and $\beta_1, \beta_2, \ldots, \beta_m$ satisfy*

$$-p\left(1+\frac{1}{p_j}\right) < \beta_j < p\left(1-\frac{1}{p_j}\right), \quad j = 1,2,\ldots,m. \tag{34}$$

Assume that p and β satisfy

$$\beta = \beta_1 + \cdots + \beta_m, \quad \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Then the operator $T_{\oplus}$ is bounded from $\prod_{j=1}^{m} L^{p_j}(\mathbb{R}_+, x^{\beta_j p_j/p})$ to $L^p(\mathbb{R}_+, x^{\beta})$. Moreover,

$$\|T_{\oplus}\|_{\prod_{j=1}^{m} L^{p_j}(\mathbb{R}_+, x^{\beta_j p_j/p})\to L^p(\mathbb{R}_+, x^{\beta})} = \frac{1}{\Gamma(m)} \Gamma\left(\frac{1+\beta}{p}\right) \prod_{j=1}^{m} \Gamma\left(\frac{1}{p_j'} - \frac{\beta_j}{p}\right).$$

We need

$$\beta_j < p\left(1-\frac{1}{p_j}\right)$$

in order that (7) holds. It seems that the left inequality in (34) is not necessary in the light of Corollary 1.

As the following remark shows, the weight w_2 is less significant role in our generalized setting.

REMARK 1. If

$$L(x, y_1, y_2, \ldots, y_m) = K(x, y_1, y_2, \ldots, y_m) {y_1}^{-\beta_1/q} {y_2}^{-\beta_2/q} \cdots {y_m}^{-\beta_m/q},$$

then

$$T(f_1, f_2, \ldots, f_m)(x) = \int_{\mathbb{R}_+^m} K(x, y_1, y_2, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j)\, dy_1\, dy_2 \cdots dy_m \quad (x > 0)$$

is a bounded linear operator from $\prod_{j=1}^{m} \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+, x^{\alpha}, x^{\beta_j q_j/q})$ to $\mathscr{M}_q^p(\mathbb{R}_+, x^{\alpha}, x^{\beta})$ if and only if

$$S(f_1, f_2, \ldots, f_m)(x) = \int_{\mathbb{R}_+^m} L(x, y_1, y_2, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j)\, dy_1\, dy_2 \cdots dy_m \quad (x > 0)$$

is a bounded linear operator from $\prod_{j=1}^{m} \mathscr{M}_{q_j}^{p_j}(\mathbb{R}_+, x^{\alpha}, x^{\beta_j q_j/q})$ to $\mathscr{M}_q^p(\mathbb{R}_+, x^{\alpha})$.

4.2. Hardy operator v.s. Hardy-Littlewood maximal operator

Next, let us compare Theorem 2 with the boundedness of the Hardy-Littlewood maximal operator.

Let

$$Mf(x) = \sup_I \frac{\chi_I(x)}{|I|} \int_I |f(y)|\,dy$$

be the Hardy-Littlewood maximal operator. We now place ourselves in the setting of $\mathbb{R}$ instead of the half line $\mathbb{R}_+$, so that we consider positive measurable functions $w_1, w_2 : \mathbb{R} \to \mathbb{R}_+$. Then the weighted Morrey space $\mathscr{M}_q^p(\mathbb{R}_+, w_1, w_2)$ is the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}_+)$ for which the norm

$$\|f\|_{\mathscr{M}_q^p(\mathbb{R}_+,w_1,w_2)} = \sup_I w_1(I)^{\frac{1}{p}-\frac{1}{q}} \left(\int_I |f(y)|^q w_2(y)\,dy \right)^{\frac{1}{q}}$$

is finite, where

$$w_1(I) = \int_I w_1(y)\,dy.$$

Here I moves over all intervals in $\mathbb{R}$. The problem of finding the necessary and sufficient condition for M to be bounded on $\mathscr{M}_q^p(\mathbb{R}, w_1, w_2)$ is open. We have the following:

THEOREM 6. *Let $1 < q < p < \infty$ and $-n < \alpha, \beta < \infty$. The Hardy-Littlewood maximal operator M on $\mathscr{M}_q^p(\mathbb{R}, |x|^\alpha, |x|^\beta)$ is bounded if and only if*

$$-\frac{q}{p}\left(1+\alpha\left(1-\frac{p}{q}\right)\right) \leqslant \beta < q - \frac{q}{p}\left(1+\alpha\left(1-\frac{p}{q}\right)\right). \tag{35}$$

Note that (35) consists of two parts; (12) and another condition. In both cases, (12) comes about because of the test function $x^{-d(p,q,\alpha,\beta)}$ for the Hardy operator and $|x|^{-d(p,q,\alpha,\beta)}$ for the Hardy-Littlewood maximal operator.

4.3. Fefferman-Phong inequality

According to Theorem 3, we have

$$\|x^{-\beta} f\|_{L^p(\mathbb{R}_+)} = \|Hf'\|_{L^p(\mathbb{R}_+, x^{p(1-\beta)})} \leqslant \left(\beta - \frac{1}{p}\right)^{-1} \|f'\|_{L^p(\mathbb{R}_+)}$$

with sharp constant together with the Morrey counterpart, which is nowadays called the Olsen inequality [21, Theorem 2]. Recall that we define the fractional integral operator I_α with $0 < \alpha < n$ by;

$$I_\alpha f(x) \equiv \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy$$

for all suitable functions f on $\mathbf{R}^n$. Olsen's inequality is the one of the form

$$\|g \cdot I_\alpha f\|_Z \leqslant C \|f\|_X \|g\|_Y,$$

where X,Y,Z are suitable Banach spaces. There is a vast amount of literatures on Olsen inequalities; see [5, 24, 25, 26, 27, 28, 29, 30, 33] for theoretical aspects and [11, 12, 13, 21] for applications to PDEs.

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