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BERNSTEIN–MARKOV TYPE INEQUALITIES AND OTHER INTERESTING ESTIMATES FOR POLYNOMIALS ON CIRCLE SECTORS

P. JIMÉNEZ-RODRÍGUEZ, G. A. MUÑOZ-FERNÁNDEZ,
D. PELLEGRINO AND J. B. SEOANE-SEPÚLVEDA

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Abstract. In this paper we study various polynomial inequalities for 2-homogeneous polynomials on the circular sector $\{re^{i\theta} : r \in [0,1], \theta \in [0, \frac{\pi}{2}]\}$. In particular, we obtain sharp Bernstein and Markov inequalities for such polynomials, we calculate the polarization constant of the space formed by those polynomials and, finally, we provide the unconditional basis constant of the canonical basis of that polynomial space.

1. Preliminaries

A straightforward consequence of the well-known Krein-Milman Theorem states that any convex function $f : K \to \mathbb{R}$ defined on a convex body (i.e., a convex and compact subset with not empty interior) $K \subset \mathbb{R}^n$ attains its maximum at the extreme points of K. This idea, which will be referred to, from now on, as the Krein-Milman approach, has been used repeatedly in the past by many authors in order to obtain a wide range of sharp inequalities. In this paper we apply the Krein-Milman approach in a very specific setting, namely, the space of 2-homogeneous polynomials on the circular sector $\{re^{i\theta} : r \in [0,1], \theta \in [0, \frac{\pi}{2}]\}$, or $D\left(\frac{\pi}{2}\right)$ for short. If P is a 2-homogeneous polynomial on $\mathbb{R}^2$, we define its norm $\|P\|_{D(\frac{\pi}{2})}$ by

$$\|P\|_{D(\frac{\pi}{2})} := \sup\left\{|P(\mathbf{x})| : \mathbf{x} \in D\left(\frac{\pi}{2}\right)\right\}.$$

Similarly, if L is a bilinear form on $\mathbb{R}^2$, we also define

$$\|L\|_{D(\frac{\pi}{2})} := \sup\left\{|L(\mathbf{x},\mathbf{y})| : \mathbf{x},\mathbf{y} \in D\left(\frac{\pi}{2}\right)\right\}.$$

The spaces of 2-homogeneous polynomials, bilinear forms and symmetric bilinear forms on $\mathbb{R}^2$ endowed with the above norms will be represented, respectively, by $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$, $\mathscr{L}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ and $\mathscr{L}^s\left({}^2D\left(\frac{\pi}{2}\right)\right)$. The unit ball and the unit sphere of

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$\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ will be represented by $\mathsf{B}_{D(\frac{\pi}{2})}$ and $\mathsf{S}_{D(\frac{\pi}{2})}$ respectively, whereas the extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ will be denoted by $\operatorname{ext}\left(\mathsf{B}_{D(\frac{\pi}{2})}\right)$. The mapping that assigns to every $(a,b,c)\in\mathbb{R}^3$ the polynomial $P(x,y)=ax^2+by^2+cxy$ allows us to identify $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ with $(\mathbb{R}^3,\|\cdot\|_{D(\frac{\pi}{2})})$, where $\|(a,b,c)\|_{D(\frac{\pi}{2})}=\|P\|_{D(\frac{\pi}{2})}$. This latter representation together with the usual polynomial representation of $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ will be interchanged throughout the paper.

In [28] the authors provide a formula for $\|\cdot\|_{D(\frac{\pi}{2})}$ and an explicit description of $\operatorname{ext}\left(\mathsf{B}_{D(\frac{\pi}{2})}\right)$ (among the description of polynomial spaces on circular sectors of different amplitudes from $\frac{\pi}{2}$). More specifically:

LEMMA 1. (Theorem 3.1, [28]) *If $a,b,c\in\mathbb{R}$ and $P(x,y)=ax^2+by^2+cxy$ then*

$$\|P\|_{D(\frac{\pi}{2})}=\max\left\{|a|,|b|,\frac{1}{2}|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}|\right\}.$$

LEMMA 2. (Theorem 5.2, [28]) *The set of extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ is given by*

$$\operatorname{ext}\left(\mathsf{B}_{D(\frac{\pi}{2})}\right)=\{\pm P_t,\pm Q_t\,:\,-1\leqslant t\leqslant 1\}\cup\{\pm(1,1,0)\},$$

where

$$P_t=(t,1,-2\sqrt{2(1+t)}),$$
$$Q_t=(1,t,-2\sqrt{2(1+t)}).$$

Using this geometrical information in combination with the Krein-Milman approach we obtain

1. sharp Bernstein-Markov inequalities in $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$,
2. the exact polarization constant of the polynomial space $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$, and
3. the precise value of the unconditional constant of the canonical basis of the space $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$.

This article complements three previous works, namely [1], [14] and [34], where similar questions are addressed for 2-polynomials on the circular sector of amplitude $\pi/4$, $D\left(\frac{\pi}{2}\right)$, the unit square $\square$ and the simplex Δ (the three in $\mathbb{R}^2$) respectively.

In fact, this paper, together with [1], [14] and [34] are just four among many other research works where the Krein-Milman approach can be used in order to obtain sharp polynomial inequalities of all kinds (see for instance [21, 29, 30, 32] just to mention a few). Actually, the list of potential applications of the Krein-Milman approach can be substantially enlarged if one takes into consideration the vast literature devoted to the study of the geometry of polynomial spaces (see for instance [2, 7, 8, 9, 15, 16, 17, 18, 19, 20, 23, 27, 33, 36]).

The first of the three questions studied in this paper deals with estimates on the derivative of polynomials. We are talking about the so called Bernstein and Markov inequalities. This kind of estimates have been studied since the chemist D. Medeleev posed the first Markov-type problem back in 1889. Mendeleev wanted to know how large the derivative of a polynomial of degree 2 can get in comparison with its sup norm on some interval. Mendeleev's question was generalized a few years later by the brothers V. Markov and A. Markov, who finally proved that for a real polynomial p of arbitrary degree $\leqslant n$ we have that

$$\|p^{(k)}\| \leqslant T_n^{(k)}(1)\|p\|, \tag{1}$$

where $p^{(k)}$ stands for the k-th derivative of p, $\|\cdot\|$ is the sup norm over the unit interval and $T_n(x) = \cos n \arccos x$, for $|x| \leqslant 1$, is the n-th Chebyshev polynomial of the first kind. Inequalities such as (1) are known as Markov-type inequalities, whereas pointwise estimates on the derivative of a polynomial are widely known as Bernstein-type inequalities. Bernstein and Markov inequalities have been studied in the general setting of Banach spaces (see [22, 31, 39, 40] and the references therein), where most of the classical results in one variable can also be proved. Bernstein and Markov-type inequalities have also been studied for polynomials on a convex body without central symmetry (see, e.g., [4, 24, 26, 34, 35]). In this particular setting, it is customary to find estimates on the Euclidean length of the gradient of the polynomials rather than estimates on the norm of the differential. In Section 2 we provide sharp estimates on the length of the gradient of 2-homogeneous polynomials on $D\left(\frac{\pi}{2}\right)$.

There is another question that arises from the study of Markov inequalities for homogeneous polynomials on a Banach space that is treated in Section 3. If E is a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with closed unit ball B_E and $n \in \mathbb{N}$, then we will denote by $\mathscr{P}(^nE)$ and $\mathscr{L}^s(^nE)$ the space of all continuous $n-$homogeneous polynomials on E and the space of all continuous symmetric $n-$linear forms on E^n endowed respectively with the norms given by

$$\|P\| = \sup\{\|P(x)\| : x \in \mathsf{B}_E\},$$
$$\|L\| = \sup\{\|L(x_1,\dots,x_n)\| : x_1,\dots,x_n \in \mathsf{B}_E\}.$$

According to an old and well-known algebraic result, for every $P \in \mathscr{P}(^nE)$ there exists a unique $\check{P} \in \mathscr{L}^s(^nE)$ such that $P(x) = \check{P}(x,\overset{n}{\dots},x)$, for every $x \in E$ and the mapping $\mathscr{P}(^nE) \ni P \mapsto \check{P} \in \mathscr{L}^s(^nE)$ is an algebraic isomorphism. Furthermore, Martin [25] proved that

$$\|P\| \leqslant \|\check{P}\| \leqslant \frac{n^n}{n!}\|P\|, \tag{2}$$

for every $P \in \mathscr{P}(^nE)$. It can be shown that the constant $\frac{n^n}{n!}$ cannot be improved in general since equality is achieved for ℓ_1^n and the polynomial defined by $\Phi(x_1,\dots,x_n) = x_1 \cdots x_n$ for every $(x_1,\dots,x_n) \in \mathbb{K}^n$. However, for a specific space E the constant $\frac{n^n}{n!}$ in (2) can be improved. The best constant $\mathbb{K}(n;E)$ in the inequality

$$\|\check{P}\| \leqslant M\|P\|,$$

for every $P \in \mathscr{P}(^nE)$ is called the nth polarization constant of E.

In Section 3 we give a version of (2) for 2-homogeneous polynomials on $D\left(\frac{\pi}{2}\right)$.

Finally, Section 4 is devoted to the calculation of the unconditional constant of the canonical basis of the spaces $\mathscr{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$. The problem can be stated in more general terms as follows: Let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$, where $\mathbf{x}=(x_1,\ldots,x_m)\in\mathbb{R}^m$ and $\alpha=(\alpha_1,\ldots,\alpha_m)$ with $\alpha_k\in\mathbb{N}\cup\{0\}$, $1\leqslant k\leqslant m$. If $P(\mathbf{x})=\sum_{|\alpha|\leqslant n}a_\alpha\mathbf{x}^\alpha$ is a polynomial of degree n on $\mathbb{R}^m$, we define its modulus $|P|$ by $|P|(\mathbf{x})=\sum_{|\alpha|\leqslant n}|a_\alpha|\mathbf{x}^\alpha$. If $\mathsf{B}\subset\mathbb{R}^m$ is a convex body, we let $\mathscr{P}_n(\mathsf{B})$ and $\mathscr{P}(^n\mathsf{B})$ represent, respectively, the space of polynomials of degree at most n and the space of n-homogeneous polynomials on $\mathbb{R}^m$ endowed with the norm

$$\|P\|_{\mathsf{B}}=\sup\{|P(x)|:\ x\in\mathsf{B}\}.$$

Now let $\mathscr{B}_n=\{\mathbf{x}^\alpha:|\alpha|\leqslant n\}$ be the canonical basis of $\mathscr{P}_n(\mathsf{B})$ and $\mathscr{S}\subset\mathscr{B}_n$. Then it is easy to see that the unconditional constant of $\mathscr{S}$ coincides with the best possible constant $C_{\mathsf{B},\mathscr{S}}$ in the inequality

$$\||P|\|_{\mathsf{B}}\leqslant C_{\mathsf{B},\mathscr{S}}\|P\|_{\mathsf{B}}, \tag{3}$$

for every P in the space generated by $\mathscr{S}$. In particular, if $\mathscr{S}=\{\mathbf{x}^\alpha:|\alpha|=n\}$, then $C_{\mathsf{B},\mathscr{S}}$ would be the unconditional constant of the canonical basis of $\mathscr{P}(^n\mathsf{B})$.

This type of inequalities have been studied since a long time ago. Of special importance are the results obtained by Bohr in 1914 for infinite complex power series [6] since they are the starting point of a prolific research line that continues to be productive nowadays (see for instance [3, 5, 10, 11, 12, 13]). It is interesting to note that the relationship between unconditional constants in polynomial spaces and inequalities of the type (3) was already noticed in [11].

2. Bernstein and Markov-type inequalities for polynomials on sectors

Using the Krein-Milman approach in combination with Lemma 2, we can obtain the following Bernstein type estimate:

PROPOSITION 1. *For every $(x,y)\in D\left(\frac{\pi}{2}\right)$ and $P\in\mathscr{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$ we have that*

$$\|\nabla P(x,y)\|_2\leqslant\Phi(x,y)\|P\|_{D\left(\frac{\pi}{2}\right)},$$

with

$$\Phi(x,y)=\begin{cases}\sqrt{16(x-y)^2+4(x^2+y^2)} & \text{if } 0\leqslant y\leqslant\frac{x}{2},\\ \sqrt{\frac{x^4}{y^2}+4(x^2+y^2)} & \text{if } 0<\frac{x}{2}<y\leqslant x,\\ \sqrt{\frac{y^4}{x^2}+4(x^2+y^2)} & \text{if } 0<x<y\leqslant 2x,\\ \sqrt{16(y-x)^2+4(x^2+y^2)} & \text{if } 2x<y\leqslant 1.\end{cases}$$

Proof. We want to calculate

$$\sup\left\{\|\nabla P(x,y)\|_2 : P \in \operatorname{ext}\left(B_{D(\frac{\pi}{2})}\right)\right\}.$$

For $P = (1,1,0)$, we have $\|\nabla P(x,y)\|_2^2 = 4(x^2+y^2)$.

For the rest of polynomials, the case $xy = 0$ is trivial and is left to the reader, so assume that both $x \neq 0$ and $y \neq 0$. First, consider $P_t = (t,1,-2\sqrt{2(1+t)})$. Then,

$$\|\nabla P_t(x,y)\|_2^2 = 4t^2x^2 + 8(1+t)y^2 - 8t\sqrt{2(1+t)}xy + 4y^2 + 8(1+t)x^2 - 8\sqrt{2(1+t)}xy.$$

Make now the change $u = \sqrt{2(1+t)}$ (so $u \in [0,2]$) to have

$$g_{x,y}(u) := \|\nabla P_u(x,y)\|_2^2 = x^2u^4 - 4xyu^3 + 4y^2u^2 + 4(x^2+y^2).$$

The critical points for $g_{x,y}$ are $u = 0$, $u = \frac{2y}{x}$ and $u = \frac{y}{x}$. Notice $g''_{x,y}(\frac{2y}{x}) > 0$, so we are in a relative minimum and therefore this point shall not be taken into consideration. Also,

$$\begin{aligned} g_{x,y}(0) &= 4(x^2+y^2),\\ g_{x,y}\left(\frac{y}{x}\right) &= \frac{y^4}{x^2} + 4(x^2+y^2),\\ g_{x,y}(2) &= 16(x-y)^2 + 4(x^2+y^2). \end{aligned}$$

Hence,

$$\begin{aligned}\sup\left\{\|\nabla P_t(x,y)\|_2^2 : -1 \leqslant t \leqslant 1\right\} &= \begin{cases} \max\left\{g_{x,y}(0), g_{x,y}\left(\frac{y}{x}\right), g_{x,y}(2)\right\} & \text{if } 0 \leqslant \frac{y}{x} \leqslant 2,\\ \max\left\{g_{x,y}(0), g_{x,y}(2)\right\} & \text{otherwise.}\end{cases}\\ &= \begin{cases} \max\left\{g_{x,y}\left(\frac{y}{x}\right), g_{x,y}(2)\right\} & \text{if } 0 \leqslant \frac{y}{x} \leqslant 2,\\ g_{x,y}(2) & \text{otherwise.}\end{cases}\end{aligned}$$

Since $g_{x,y}(2) \leqslant g_{x,y}\left(\frac{y}{x}\right)$ if $0 \leqslant \frac{y}{x} \leqslant 2$, we conclude that

$$\sup\left\{\|\nabla P_t(x,y)\|_2^2 : -1 \leqslant t \leqslant 1\right\} = \begin{cases} \frac{y^4}{x^2} + 4(x^2+y^2) & \text{if } 0 \leqslant \frac{y}{x} \leqslant 2,\\ 16(x-y)^2 + 4(x^2+y^2) & \text{otherwise.}\end{cases}$$

Since $Q_t(x,y) = P_t(y,x)$, by symmetry, we obtain

$$\sup\left\{\|\nabla Q_t(x,y)\|_2^2 : -1 \leqslant t \leqslant 1\right\} = \begin{cases} \frac{x^4}{y^2} + 4(x^2+y^2) & \text{if } 0 \leqslant \frac{x}{y} \leqslant 2,\\ 16(x-y)^2 + 4(x^2+y^2) & \text{otherwise.}\end{cases}$$

Putting all together we have that $\sup_{-1 \leqslant t \leqslant 1}\left\{\|\nabla P_t(x,y)\|_2^2, \|\nabla Q_t(x,y)\|_2^2\right\}$ is given by

$$\begin{cases} \max\left\{\frac{y^4}{x^2} + 4(x^2+y^2), 16(x-y)^2 + 4(x^2+y^2)\right\} & \text{if } 0 \leqslant \frac{y}{x} \leqslant \frac{1}{2},\\ \max\left\{\frac{y^4}{x^2} + 4(x^2+y^2), \frac{x^4}{y^2} + 4(x^2+y^2)\right\} & \text{if } \frac{1}{2} \leqslant \frac{y}{x} \leqslant 2,\\ \max\left\{\frac{x^4}{y^2} + 4(x^2+y^2), 16(x-y)^2 + 4(x^2+y^2)\right\} & \text{if } 2 \leqslant \frac{y}{x}.\end{cases}$$

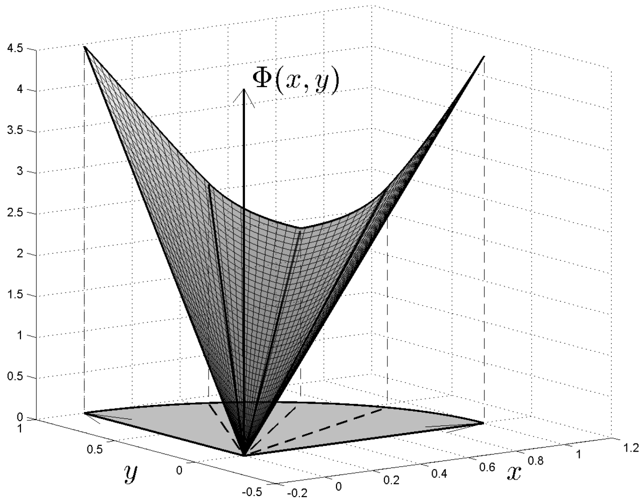

Figure 1: *Graph of the mapping* $\Phi(x,y)$

Taking the latter into account, we can conclude that

$$\|\nabla P(x,y)\|_2 \leqslant \Phi(x,y)\|P\|_{D(\frac{\pi}{2})},$$

for any $P \in D(\frac{\pi}{2})$, where

$$\Phi(x,y) = \begin{cases} \sqrt{16(x-y)^2+4(x^2+y^2)} & \text{if } 0 \leqslant y \leqslant \frac{x}{2}, \\ \sqrt{\frac{x^4}{y^2}+4(x^2+y^2)} & \text{if } 0 < \frac{x}{2} < y \leqslant x, \\ \sqrt{\frac{y^4}{x^2}+4(x^2+y^2)} & \text{if } 0 < x < y \leqslant 2x, \\ \sqrt{16(y-x)^2+4(x^2+y^2)} & \text{if } 2x < y \leqslant 1. \end{cases}$$

□

Using the previous inequality we can derive the following Markov type estimate:

COROLLARY 1. *For every* $(x,y) \in D\left(\frac{\pi}{2}\right)$*, we have*

$$\|DP(x,y)\|_2 \leqslant 2\sqrt{5}\|P\|_{D(\frac{\pi}{2})}, \tag{4}$$

with equality attained for $\pm P_1(x,y) = \pm Q_1(x,y) = \pm(x^2+y^2-4xy)$.

Proof. It suffices to check that

$$\max_{(x,y)\in D\left(\frac{\pi}{2}\right)} \Phi(x,y) = 2\sqrt{5},$$

being the maximum attained at the points $(1,0)$ and $(0,1)$. An inspection of the proof of Proposition 1 reveals that equality in (4) holds for the extreme polynomials $\pm P_1 = \pm Q_1$, or in other words $\pm(x^2+y^2-4xy)$. $\square$

3. Polarization constants for polynomials on sectors

The main result of this section provides a sharp estimate on the norm of a polynomial in $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ in terms of the norm of its polar as an element of $\mathscr{L}^s({}^2D\left(\frac{\pi}{2}\right)$. The constant thus obtained is known as the polarization constant of the space $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$. The following easily verified lemma will be useful in the calculations to come.

LEMMA 3. *Let $f(t) = a\cos t + b\sin t$ be defined for $0 \leqslant t \leqslant \frac{\pi}{2}$. Then,*

$$\max_{0\leqslant\theta\leqslant\frac{\pi}{2}} |f(t)| = \begin{cases} \max\{|a|,|b|\} & \text{if } ab \leqslant 0, \\ \sqrt{a^2+b^2} & \text{otherwise.} \end{cases}$$

Now, as a consequence of the Krein-Milman approach in combination with Lemma 2 we have the following:

THEOREM 1. *For every $(x,y) \in D(\frac{\pi}{2})$ and $P \in \mathscr{P}({}^2D(\frac{\pi}{2}))$ we have that*

$$\|DP(x,y)\|_{D(\frac{\pi}{2})} \leqslant \Psi(x,y)\|P\|_{D(\frac{\pi}{2})}, \tag{5}$$

where

$$\Psi(x,y) = \begin{cases} 2(2x-y) & \text{if } 0 \leqslant y < \frac{x}{2}, \\ 2\left(y+\frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \leqslant y < x, \\ 2\left(x+\frac{y^2}{2x}\right) & \text{if } x \leqslant y < 2x, \\ 2(2y-x) & \text{if } y \geqslant 2x. \end{cases}$$

Moreover, inequality (5) is optimal for each $(x,y) \in D(\frac{\pi}{2})$.

Proof. In order to calculate $\Psi(x,y) := \sup\{\|DP(x,y)\|_{D(\frac{\pi}{2})} : \|P\|_{D(\frac{\pi}{2})}) \leqslant 1\}$, by the Krein-Milman approach, it suffices to calculate

$$\sup\{\|DP(x,y)\|_{D(\frac{\pi}{2})} : P \in \mathrm{ext}(B_{D(\frac{\pi}{2})})\}.$$

The easy case $xy = 0$ is left to the reader, so assume that both $x \neq 0$ and $y \neq 0$. Let us deal first with the polynomials $Q_t = (1, t, -2\sqrt{2(1+t)})$, with $t \in [-1,1]$.

Since $Q_t(x,y) = x^2 + ty^2 - 2\sqrt{2(1+t)}xy$, then

$$\nabla Q_t(x,y) = \left(2x - 2\sqrt{2(1+t)}y, 2ty - 2\sqrt{2(1+t)}x\right).$$

Therefore,

$$\|DQ_t(x,y)\|_{D(\frac{\pi}{2})} = \sup_{(h,k)\in D\left(\frac{\pi}{2}\right)} \left|\left(2x-2\sqrt{2(1+t)}y\right)h+\left(2ty-2\sqrt{2(1+t)}x\right)k\right|.$$

In order to calculate the above supremum we can restrict attention to the extreme points of $D\left(\frac{\pi}{2}\right)$ (except for the point $(h,k)=(0,0)$ that does not contribute anything to the supremum). Thus, putting $(h,k)=(\cos\theta,\sin\theta)$ with $0\leqslant\theta\leqslant\frac{\pi}{2}$ and $\lambda=\frac{y}{x}$,

$$\sup_{-1\leqslant t\leqslant 1}\|\nabla Q_t(x,y)\|_{D(\frac{\pi}{2})} = 2x\sup_{(t,\theta)\in C}\left|\left(1-\sqrt{2(1+t)}\lambda\right)\cos\theta+\left(t\lambda-\sqrt{2(1+t)}\right)\sin\theta\right|,$$

where $C=[-1,1]\times[0,\frac{\pi}{2}]$.

Define

$$f_\lambda(t,\theta)=\left(1-\sqrt{2(1+t)}\lambda\right)\cos\theta+\left(t\lambda-\sqrt{2(1+t)}\right)\sin\theta,$$

and consider the following cases:

- $0<\theta<\frac{\pi}{2}$, $-1<t<1$.

  The critical points of f_λ in the interior of C are the solutions of the equations:

$$\frac{\partial f_\lambda}{\partial t}(t_0,\theta_0)=\frac{-\sqrt{2}\lambda}{2\sqrt{1+t_0}}\cos\theta_0+\left(\lambda-\frac{\sqrt{2}}{2\sqrt{1+t_0}}\right)\sin\theta_0=0, \tag{6}$$

$$\frac{\partial f_\lambda}{\partial\theta}(t_0,\theta_0)=-\left(1-\sqrt{2(1+t_0)}\lambda\right)\sin\theta_0+\left(t_0\lambda-\sqrt{2(1+t_0)}\right)\cos\theta_0=0. \tag{7}$$

Working with equation (6), we get to the next expression:

$$\sin\theta_0=\frac{\frac{\sqrt{2}}{2\sqrt{1+t_0}}\lambda}{\lambda-\frac{\sqrt{2}}{2\sqrt{1+t_0}}}\cos\theta_0=\frac{\sqrt{2}\lambda}{2\lambda\sqrt{1+t_0}-\sqrt{2}}\cos\theta_0 \tag{8}$$

and, plugging the expression in (8) into equation (7) we obtain

$$\left[\left(t_0\lambda-\sqrt{2(1+t_0)}\right)-\frac{\sqrt{2}\lambda\left(1-\sqrt{2(1+t_0)}\lambda\right)}{2\lambda\sqrt{1+t_0}-\sqrt{2}}\right]\cos\theta_0=\left[t_0\lambda-\sqrt{2(1+t_0)}+\lambda\right]\cos\theta_0$$
$$=0.$$

Now, since $0<\theta_0<\frac{\pi}{2}$ we can have $\cos\theta_0\neq 0$ and hence,

$$\lambda(1+t_0)=\sqrt{2(1+t_0)},$$

from which $t_0=\frac{2}{\lambda^2}-1$.

If we now apply the condition $-1 < t_0 < 1$ we get the restriction $\lambda > 1$, that is, we will only have critical points in the interior of C when $y > x$.

Now, plugging t_0 in (8), we obtain $\tan\theta_0 = \lambda$, from which

$$\sin\theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}} \quad \text{and} \quad \cos\theta_0 = \frac{1}{\sqrt{1+\lambda^2}}.$$

Then,

$$\begin{aligned} 2x|f_\lambda(t_0,\theta_0)| &= 2x\left|\left(1-\sqrt{\frac{4}{\lambda^2}}\lambda\right)\frac{1}{\sqrt{1+\lambda^2}} + \left[\left(\frac{2}{\lambda^2}-1\right)\lambda - \sqrt{\frac{4}{\lambda^2}}\right]\frac{\lambda}{\sqrt{1+\lambda^2}}\right| \\ &= 2x\left|\frac{-1}{\sqrt{1+\lambda^2}} - \frac{\lambda^2}{\sqrt{1+\lambda^2}}\right| = 2x\sqrt{1+\lambda^2} \end{aligned} \tag{9}$$

- $t = -1$ and $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Using lemma 3, we may conclude that

$$2x \max_{0\leqslant\theta\leqslant\frac{\pi}{2}} |f_\lambda(-1,\theta)| = 2x\max\{1,\lambda\}. \tag{10}$$

- $t = 1$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

In this case we shall study the expression $2x|f_\lambda(1,\theta)| = 2x|(1-2\lambda)\cos\theta + (\lambda - 2)\sin\theta|$. Again, by lemma 3, we have the following:

$$\max_{0\leqslant\theta\leqslant\frac{\pi}{2}} 2x|f_\lambda(1,\theta)| = \begin{cases} 2x\max\{|1-2\lambda|,|\lambda-2|\} & \text{if } 0\leqslant\lambda<\frac{1}{2} \text{ or } \lambda>2, \\ 2x\sqrt{(2\lambda-1)^2+(2-\lambda)^2} & \text{if } \frac{1}{2}\leqslant\lambda\leqslant 2. \end{cases} \tag{11}$$

It can be easily checked that the expression we have arrived at in (11) is greater than (9).

- $\theta = 0$, $-1 \leqslant t \leqslant 1$.

We need to calculate

$$\begin{aligned} 2x \max_{-1\leqslant t\leqslant 1} |f_\lambda(t,0)| &= 2x \max_{-1\leqslant t\leqslant 1}\left|1-\sqrt{2(1+t)}\lambda\right| = 2x\max\{1,|1-2\lambda|\} \\ &= \begin{cases} 2x & \text{if } 0\leqslant\lambda<1, \\ 2x(2\lambda-1) & \text{if } \lambda\geqslant 1. \end{cases} \end{aligned} \tag{12}$$

Observe that the latter is always greater that (10).

- $\theta = \frac{\pi}{2}$, $-1 \leqslant t \leqslant 1$.

We have to calculate

$$2x \max_{-1\leqslant t\leqslant 1} |t\lambda - \sqrt{2(1+t)}|,$$

for which we define $h(t) = t\lambda - \sqrt{2(1+t)}$, for $t \in [-1,1]$. The critical points of h satisfy

$$h'(t) = \lambda - \frac{\sqrt{2}}{2\sqrt{1+t}} = 0,$$

from which

$$t_1 = \frac{1}{2\lambda^2} - 1.$$

Observe that $t_0 \in [-1,1]$ if and only if $|\lambda| \geqslant \frac{1}{2}$.

To summarize we have equation

$$2x \max_{-1\leqslant t\leqslant 1} |t\lambda - \sqrt{2(1+t)}| = \begin{cases} 2x \max\{|h(-1)|, |h(1)|, |h(t_1)|\} & \text{if } \lambda \geqslant \frac{1}{2}, \\ 2x \max\{|h(-1)|, |h(1)|\} & \text{if } 0 \leqslant \lambda < \frac{1}{2}, \end{cases}$$
$$= \begin{cases} 2x\left(\lambda + \frac{1}{2\lambda}\right) & \text{if } \lambda \geqslant \frac{1}{2}, \\ 2x(2-\lambda) & \text{if } 0 \leqslant \lambda < \frac{1}{2}. \end{cases} \tag{13}$$

Since we have already discarded (9) and (10), putting together (11), (12) and (13) we arrive at the fact that $2x \sup_{(t,\theta)\in C} |f_\lambda(t,\theta)|$ is given by

$$2x \begin{cases} \max\{|1-2\lambda|, |2-\lambda|, 1\} & \text{if } 0 \leqslant \lambda \leqslant \frac{1}{2}, \\ \max\left\{\sqrt{(2\lambda-1)^2 + (2-\lambda)^2}, 1, \left(\lambda + \frac{1}{2\lambda}\right)\right\} & \text{if } \frac{1}{2} \leqslant \lambda \leqslant 1, \\ \max\left\{\sqrt{(2\lambda-1)^2 + (2-\lambda)^2}, 2\lambda - 1, \left(\lambda + \frac{1}{2\lambda}\right)\right\} & \text{if } 1 \leqslant \lambda \leqslant 2, \\ \max\left\{\left(\lambda + \frac{1}{2\lambda}\right), |1-2\lambda|, |2-\lambda|\right\} & \text{if } \lambda \geqslant 2. \end{cases}$$

The only comparisons in the previous expression with some difficulty are

$$\lambda + \frac{1}{2\lambda} \quad \text{and} \quad \sqrt{(2\lambda-1)^2 + (\lambda-2)^2}.$$

Through standard calculations, we deduce that $\lambda + \frac{1}{2\lambda} \geqslant \sqrt{(2\lambda-1)^2 + (\lambda-2)^2}$ whenever $\lambda \leqslant \frac{1+\sqrt{2}}{2}$. Since we shall consider this situation when $\lambda \geqslant \frac{1}{2}$, we conclude

$$\sup_{-1\leqslant t\leqslant 1} \|\nabla Q_t(x,y)\|_{D(\frac{\pi}{2})} = \Psi_1(x,y),$$

where

$$\Psi_1(x,y) = \begin{cases} 2(2x-y) & \text{if } 0 \leqslant y < \frac{x}{2}, \\ 2\left(y + \frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \leqslant y < \frac{1+\sqrt{2}}{2}x, \\ 2\sqrt{(2x-y)^2 + (2y-x)^2} & \text{if } \frac{1+\sqrt{2}}{2}x \leqslant y < 2x, \\ 2(2y-x) & \text{if } y \geqslant 2x. \end{cases}$$

Using the symmetry $P_t(x,y) = Q_t(y,x)$, we may see

$$\sup_{-1\leqslant t\leqslant 1} \|\nabla P_t(x,y)\|_{D(\frac{\pi}{2})} = \Psi_2(x,y),$$

where

$$\Psi_2(x,y) = \Psi_1(y,x) = \begin{cases} 2(2x-y) & \text{if } 0 \leqslant y < \frac{x}{2}, \\ 2\sqrt{(2x-y)^2+(2y-x)^2} & \text{if } \frac{x}{2} \leqslant y < \left(2\sqrt{2}-2\right)x, \\ 2\left(x+\frac{y^2}{2x}\right) & \text{if } \left(2\sqrt{2}-2\right)x \leqslant y < 2x, \\ 2(2y-x) & \text{if } y \geqslant 2x. \end{cases}$$

Therefore, we conclude

$$\begin{aligned} \Psi(x,y) &= \max\{\Psi_1(x,y),\Psi_2(x,y)\} \\ &= \begin{cases} 2(2x-y) & \text{if } 0 \leqslant y < \frac{x}{2}, \\ \max\left\{2\sqrt{(2x-y)^2+(2y-x)^2}, 2\left(y+\frac{x^2}{2y}\right)\right\} & \text{if } \frac{x}{2} \leqslant y \leqslant \left(2\sqrt{2}-2\right)x, \\ \max\left\{2\left(y+\frac{x^2}{2y}\right), 2\left(x+\frac{y^2}{2x}\right)\right\} & \text{if } \left(2\sqrt{2}-2\right)x \leqslant y \leqslant \frac{1+\sqrt{2}}{2}x, \\ \max\left\{2\left(x+\frac{y^2}{2x}\right), 2\sqrt{(2x-y)^2+(2y-x)^2}\right\} & \text{if } \frac{1+\sqrt{2}}{2}x \leqslant y \leqslant 2x, \\ 2(2y-x) & \text{if } y \geqslant 2x. \end{cases} \\ &= \begin{cases} 2(2x-y) & \text{if } 0 \leqslant y < \frac{x}{2}, \\ 2\left(y+\frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \leqslant y < x, \\ 2\left(x+\frac{y^2}{2x}\right) & \text{if } x \leqslant y < 2x, \\ 2(2y-x) & \text{if } y \geqslant 2x. \end{cases} \end{aligned}$$

□

Taking the maximum of $\Psi(x,y)$ with $(x,y) \in D\left(\frac{\pi}{2}\right)$ we can obtain the polarization constant of $\mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$:

COROLLARY 2. *Let $P \in \mathscr{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ and assume $L \in \mathscr{L}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ is the polar of P. Then*

$$\|L\|_{D(\frac{\pi}{2})} \leqslant 2\|P\|_{D(\frac{\pi}{2})}.$$

Moreover, equality is achieved for $P_1(x,y) = x^2+y^2-4xy$.

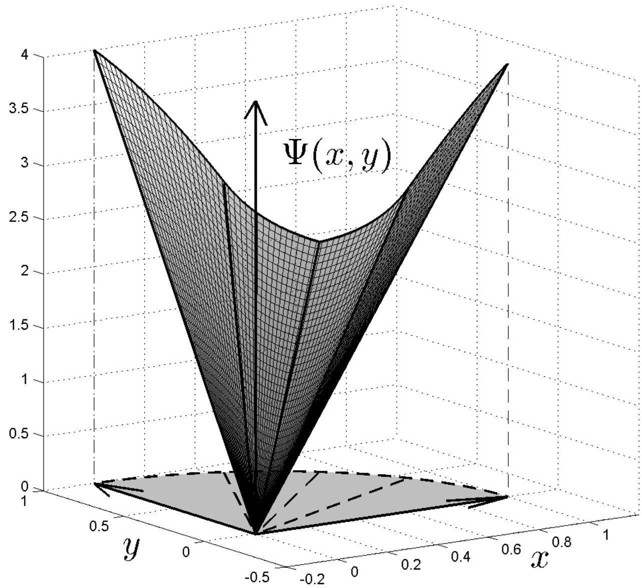

Figure 2: *Graph of the mapping* $\Psi(x,y)$

4. Unconditional constants for polynomials on sectors

As in the preceding sections, a combination of the Krein-Milman approach and Lemmas 1 and 2 allows us to proof the following:

PROPOSITION 2. *The unconditional constant of the canonical basis of the space* $\mathscr{P}(^2D(\frac{\pi}{2}))$ *is 3. In other words, the inequality*

$$|||ax^2+by^2+cxy|||_{D(\frac{\pi}{2})} \leqslant 3\|ax^2+by^2+cxy\|_{D(\frac{\pi}{2})},$$

holds for all $a,b,c\in\mathbb{R}$ *and 3 is optimal since equality is achieved for the polynomials* $\pm(x^2+y^2-4xy)$.

Proof. Observe that the extreme polynomials in the unit ball of $\mathscr{P}(^2D(\frac{\pi}{2}))$ are

$$P_t(x) = tx^2+y^2-2\sqrt{2(1+t)}xt,$$
$$Q_t(x) = x^2+ty^2-2\sqrt{2(1+t)}xt,$$

with $t\in[-1,1]$. If we plug these polynomials in Lemma 1, due to the symmetry of the problem we end up with the maximum of

$$\max\left\{|t|,1,\frac{1}{2}\left||t|+1+\sqrt{(|t|-1)^2+8(1+t)}\right|\right\}.$$

The latter function attains its maximum at 1 and turns out to be 3. $\square$

5. Conclusions

Comparing the results obtained in [21] and [34] for polynomials on the simplex Δ, in [1] for polynomials on the sector $D\left(\frac{\pi}{4}\right)$, in [14] for polynomials on the unit square $\square$ and the results obtained in the previous sections, we have the following:

	$\mathscr{P}(^2\Delta)$	$\mathscr{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$	$\mathscr{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$	$\mathscr{P}(^2\square)$
Markov constants	$2\sqrt{10}$	$2\sqrt{5}$	$4(13+8\sqrt{2})$	$\sqrt{13}$
Polarization constants	3	2	$2+\frac{\sqrt{2}}{2}$	$\frac{3}{2}$
Unconditional Constants	2	3	$5+4\sqrt{2}$	5

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

1. $\pm(x^2+y^2-6xy)$ for the simplex.
2. $\pm(x^2+y^2-4xy)$ for the sector $D\left(\frac{\pi}{2}\right)$.
3. $\pm\left(x^2+(5+4\sqrt{2})y^2-(4+4\sqrt{2})xy\right)$ for the sector $D\left(\frac{\pi}{4}\right)$.
4. $\pm(x^2+y^2-3xy)$ for the unit square.

Compare the previous table with similar results that hold for 2-homogeneous polynomials on the Banach spaces ℓ_1^2, ℓ_2^2 and ℓ_∞^2:

	$\mathscr{P}(^2\ell_1^2)$	$\mathscr{P}(^2\ell_2^2)$	$\mathscr{P}(^2\ell_\infty^2)$
Markov constants	4	2	$2\sqrt{2}$
Polarization constants	2	1	2
Unconditional Constants	$\frac{1+\sqrt{2}}{2}$	$\sqrt{2}$	$1+\sqrt{2}$

Observe that the Markov constants of the spaces $\mathscr{P}(^2\ell_1^2)$ and $\mathscr{P}(^2\ell_\infty^2)$ can be calculated taking into consideration the description of the geometry of those spaces given in [9]. Also, the Markov constant of $\mathscr{P}(^2\ell_2^2)$ is twice its polarization constant, or in other words, 2.

On the other hand, the constants appearing in the second line of the previous table are well-known results (see for instance [38]).

Finally, the unconditional constants corresponding to the third line of the previous table were calculated in Theorem 3.5, Theorem 3.19 and Theorem 3.6 of [21].

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P. Jiménez-Rodríguez
Department of Mathematical Sciences, Kent State University
1300 Lefton Esplanade, Kent, Ohio 44242, USA
e-mail: pjimene1@kent.edu

G. A. Muñoz-Fernández
Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid
Plaza de Ciencias 3, Madrid, 28040, Spain
e-mail: gustavo@ucm.es

D. Pellegrino
Departamento de Matemática, Universidade Federal da Paraíba
58.051-900 João Pessoa, Brazil
e-mail: pellegrino@pq.cnpq.br

J. B. Seoane-Sepúlveda
Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid
Plaza de Ciencias 3, Madrid, 28040, Spain
e-mail: jseoane@mat.ucm.es