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SOME NEW GENERALIZED FORMS OF HARDY'S TYPE INEQUALITY ON TIME SCALES

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Abstract. In this paper, we prove some new dynamic inequalities from which some known dynamic inequalities on time scales, some integral and discrete inequalities due to Hardy, Copson, Chow, Levinson, Pachpatte Yang and Hwang will be deduced as special cases. Also, some new corresponding integral and discrete inequalities will be formulated. The results will be proved by employing the chain rule, integration by parts formula, Hölder's inequality and Jensen's inequality on time scales.

1. Introduction

In 1920 Hardy [10] proved the following discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n),\quad p>1, \tag{1.1}$$

where $a(n)>0$ for $n\geqslant 1$.

In 1925 Hardy [11] established the continuous version of the inequality (1.1) by using the calculus of variations. In particular, he proved that if $f(t)$ is a positive integrable function over any finite interval $(0,t)$, f^p is an integrable function over $(0,\infty)$ and $p>1$, then

$$\int_0^{\infty}\left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt\leqslant\left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(t)dt. \tag{1.2}$$

The constant $(p/(p-1))^p$ in (1.1) and (1.2) is the best possible.

For generalizations, extensions and applications of these inequalities in literature, we refer the reader to the papers [3, 4, 6, 7, 8, 9, 11, 12, 13, 17] and the books [14, 15, 16, 19]. In the following, we present some of these results that serve and motivate the contents of this paper. We begin with the development of the discrete inequality.

In 1925 Hardy [11] generalized (1.1) and proved that if $p>1$, $a(n)>0$, $\lambda(n)>0$, for $n\geqslant 1$ and $\Lambda(n)=\sum_{i=1}^{n}\lambda(i)$, then

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{\Lambda^p(n)}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^p\leqslant\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\lambda(n)a^p(n). \tag{1.3}$$

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In 1928 Copson [7] generalized (1.3) and proved that if $p \geqslant c > 1$, $a(n) > 0$ and $\lambda(n) > 0$ for $n \geqslant 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leqslant \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n), \tag{1.4}$$

where $\Lambda(n) = \sum_{i=1}^{n} \lambda(i)$, and if $0 \leqslant c < 1 < p$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \leqslant \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{1.5}$$

In 1970 Leindler [17] proved that if $\sum_{i=n}^{\infty} \lambda(i) < \infty$, $p > 1$ and $0 \leqslant c < 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda^*(n))^c} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^p \leqslant \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) (\Lambda^*(n))^{p-c} a^p(n), \tag{1.6}$$

where $\Lambda^*(n) = \sum_{i=n}^{\infty} \lambda(i)$.

In 1987 Bennett [3] proved that if $\sum_{i=n}^{\infty} \lambda(i) < \infty$ and $1 < c \leqslant p$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda^*(n))^c} \left(\sum_{i=n}^{\infty} \lambda(i) a(i) \right)^p \leqslant \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) (\Lambda^*(n))^{p-c} a^p(n). \tag{1.7}$$

In 1990 Pachpatte [23] applied Jensen's inequality for convex functions and established an interesting generalization of Hardy's type inequality (1.3). In particular, he proved that if $\varphi(u)$ is a real-valued positive convex function defined for $u > 0$ and $p > 1$, then

$$\sum_{n=1}^{\infty} \lambda(n) \varphi^p \left(\frac{A(n)}{\Lambda(n)} \right) \leqslant \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \varphi^p(a(n)), \tag{1.8}$$

where

$$\Lambda(n) = \sum_{i=1}^{n} \lambda(i), \text{ and } A(n) = \sum_{i=1}^{n} a(i) \lambda(i).$$

Now, we recall some results for integral inequalities. In 1928 Hardy [12] generalized the inequality (1.2) and proved that if $f(t) > 0$ is a positive integrable function over any finite interval $(0, t)$, $f^p(t)$ is an integrable function over $(0, \infty)$ and p, $\gamma > 1$, then

$$\int_0^{\infty} \frac{1}{t^{\gamma}} \left(\int_0^t f(s) ds \right)^p dt \leqslant \left(\frac{p}{\gamma-1} \right)^p \int_0^{\infty} \frac{1}{t^{\gamma-p}} f^p(t) dt, \tag{1.9}$$

and for $p > 1$, $0 < \gamma \leqslant 1$, Hardy proved that

$$\int_0^{\infty} \frac{1}{t^{\gamma}} \left(\int_t^{\infty} f(s) ds \right)^p dt \leqslant \left(\frac{p}{1-\gamma} \right)^p \int_0^{\infty} \frac{1}{t^{\gamma-p}} f^p(t) dt. \tag{1.10}$$

In 1964 Levinson [18] employed Jensen's inequality to extend (1.2). Precisely, he proved that, if $\phi(u)$ is a real-valued positive convex function for $u > 0$, $p > 1$, $f(t) > 0$, $\lambda(t) > 0$ for $t > 0$, and there exists a constant $K > 0$ such that

$$p - 1 + \frac{\lambda'(t) \Lambda(t)}{\lambda^2(t)} \geqslant \frac{p}{K}, \quad \text{for all } t > 0, \tag{1.11}$$

then

$$\int_0^\infty \phi\left(\frac{1}{\Lambda(t)}\int_0^t \lambda(s) f(s) ds\right) dt \leqslant K^p \int_0^\infty \phi(f(t)) dt, \tag{1.12}$$

where $\Lambda(t) = \int_0^t \lambda(s) ds$. As a special case of (1.12), if $q(t) = 1$ and $\phi(u) = u^p$ we get the classical Hardy's inequality (1.2).

In 1976 Copson [8] proved the integral forms of his inequalities (1.4) and (1.5) which can be considered as generalizations of the inequalities (1.9) and (1.10). In particular, Copson proved that if $p \geqslant 1$ and $\gamma > 1$, then

$$\int_0^\infty \frac{\lambda(t)}{\Lambda^\gamma(t)}\Phi^p(t)dt \leqslant \left(\frac{p}{\gamma-1}\right)^p \int_0^\infty \frac{\lambda(t)}{\Lambda^{\gamma-p}(t)} f^p(t)dt, \tag{1.13}$$

where

$$\Lambda(t) = \int_0^t \lambda(s)ds, \text{ and } \Phi(t) = \int_0^t \lambda(s)f(s)ds,$$

and if $p \geqslant 1,\ 0 \leqslant \gamma < 1$, then

$$\int_0^\infty \frac{\lambda(t)}{\Lambda^\gamma(t)}\bar{\Phi}^p(t)dt \leqslant \left(\frac{p}{1-\gamma}\right)^p \int_0^\infty \frac{\lambda(t)}{\Lambda^{\gamma-p}(t)} f^p(t)dt, \tag{1.14}$$

where

$$\bar{\Phi}(t) = \int_t^\infty \lambda(s)f(s)ds.$$

In 1999 Yang and Hwang [34] generalized the inequality (1.12) due to Levinson and proved that, if $p > 1,\ \lambda(t),\ q(t),\ f(t)$ are nonnegative functions and there exists a constant $K > 0$ such that

$$p - 1 + \frac{q'(t)\Lambda(t)}{q^2(t)\lambda(t)} \geqslant \frac{p}{K}, \quad \text{for all } t > 0,$$

then

$$\int_0^\infty \lambda(t)\left(\frac{\Phi(t)}{\Lambda(t)}\right)^p dt \leqslant K^p \int_0^\infty \lambda(t) f^p(t) dt, \tag{1.15}$$

where

$$\Phi(t) = \int_0^t \lambda(s)q(s)f(s)ds \text{ and } \Lambda(t) = \int_0^t \lambda(s)f(s)ds.$$

In recent years the study of dynamic inequalities on time scales has been considered by several authors. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary closed subset of the real numbers $\mathbb{R}$. For developing of dynamic inequalities on time scales, we refer the reader to the book [2] and the papers [1, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32].

We assume throughout that $\mathbb{T}$ has the topology which inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} :\ s > t\}$, and $\rho(t) := \sup\{s \in \mathbb{T} :\ s < t\}$, respectively, where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$

and $t > \inf\mathbb{T}$, is right-dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided that f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist finitely. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$.

The (delta) integral can be defined as follows. If $F^{\Delta}(t) = f(t)$, then the Cauchy (delta) integral of f is defined by $\int_a^t f(s)\Delta s := F(t) - F(a)$. It can be shown (see [5]) that if $f \in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t) := \int_{t_0}^t f(s)\Delta s$ exists, t_0, $t \in \mathbb{T}$, and satisfies $F^{\Delta}(t) = f(t)$, $t \in \mathbb{T}$. An infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

The integration on discrete time scales is defined by

$$\int_a^b f(t)\triangle t = \sum_{t\in[a,b)} \mu(t)f(t).$$

Throughout the rest of the paper, we will assume that the functions in the statements of the theorems are positive, rd-continuous functions and the integrals considered are assumed to exist. Without loss of generality, we assume that $\sup\mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$.

For completeness, we recall some results for dynamic inequalities that motivate the contents of this paper. In [24] the author applied the technique used by Elliott [9] and established the time scales version of the Hardy inequality (1.2). In particular he proved that if $p > 1$, f is a nonnegative rd-continuous function and the delta integral $\int_0^{\infty} f^p(t)\Delta t$ exists as a finite number, then

$$\int_a^{\infty} \left(\frac{1}{\sigma(t) - a} \int_a^{\sigma(t)} f(s)\Delta s \right)^p \Delta t \leqslant \left(\frac{p}{p-1} \right)^p \int_a^{\infty} f^p(t)\Delta t. \tag{1.16}$$

If in addition $\mu(t)/t \to 0$ as $t \to \infty$, then the constant is the best possible. In the proof of the inequality (1.16) the author assumed that $\varphi^{\Delta}(t) > 0$ where $\varphi(t) = \int_a^t f(s)\Delta s/(t - a)$.

In [27] the authors proved the time scale version of (1.9) which is given by

$$\int_a^{\infty} \frac{1}{t^{\gamma}} \left(\int_a^{\sigma(t)} f(s)\Delta s \right)^p \Delta t \leqslant \left(\frac{pK^{\gamma}}{\gamma - 1} \right)^p \int_a^{\infty} \frac{1}{t^{\gamma-p}} f^p(t)\Delta t, \tag{1.17}$$

where $p > 1$, $\gamma > 1$ and there exists a constant $K > 0$, with $t/\sigma(t) \geqslant 1/K$, for $t \in \mathbb{T}$. They also proved that if $p > 1$, and $\gamma < 1$, then

$$\int_a^{\infty} \frac{1}{\sigma^{\gamma}(t)} \left(\int_t^{\infty} f(s)\Delta s \right)^p \Delta t \leqslant \left(\frac{p}{1-\gamma} \right)^p \int_a^{\infty} \frac{1}{\sigma^{\gamma-p}(t)} f^p(t)\Delta t, \tag{1.18}$$

which is the time scale version of (1.10) . Notice the difference between (1.17) and (1.18), where (1.17) contains the head $\int_a^{\sigma(t)} f(s)\Delta s$ while (1.18) contains the tail $\int_t^{\infty} f(s)\Delta s$.

In [30] the authors extended the inequality (1.16) and proved that if $p>1$ and $\gamma>1$, then

$$\int_a^{\infty}\frac{1}{(\sigma(t)-a)^{\gamma}}\left(\int_a^{\sigma(t)}f(s)\Delta s\right)^p\Delta t\leqslant\left(\frac{p}{\gamma-1}\right)^p\int_a^{\infty}\frac{(\sigma(t)-a)^{\gamma(p-1)}}{(t-a)^{(\gamma-1)p}}f^p(t)\Delta t. \tag{1.19}$$

In [29] the authors proved the time scale versions of the Copson inequalities (1.4) and (1.5). In particular, they proved that if $p,\ \gamma>1$, then

$$\int_a^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{\gamma}}(\Phi^{\sigma}(t))^p\Delta t\leqslant\left(\frac{p}{\gamma-1}\right)^p\int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{\gamma(p-1)}}{(\Lambda(t))^{(\gamma-1)p}}\lambda(t)f^p(t)\Delta t, \tag{1.20}$$

where

$$\Lambda(t):=\int_a^t\lambda(s)\Delta s \text{ and } \Phi(t):=\int_a^t\lambda(s)f(s)\Delta s,\quad \text{for any } t\in[a,\infty)_{\mathbb{T}},$$

and if $p>1$, $0\leqslant\gamma<1$, then

$$\int_a^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{\gamma}}\left(\bar{\Phi}(t)\right)^p\Delta t\leqslant\left(\frac{p}{1-\gamma}\right)^p\int_a^{\infty}(\Lambda^{\sigma}(t))^{p-\gamma}\lambda(t)f^p(t)\Delta t, \tag{1.21}$$

where

$$\bar{\Phi}(t):=\int_t^{\infty}\lambda(s)f(s)\Delta s,\ \text{for any } t\in[a,\infty)_{\mathbb{T}}.$$

Also in [29] the authors proved some generalizations of the inequalities (1.6) and (1.7) of Leindler's and Bennett's type on time scales. They proved that if $p>1$ and $0\leqslant\gamma<1$, then

$$\int_a^{\infty}\frac{\lambda(t)}{\Omega^{\gamma}(t)}(\Phi^{\sigma}(t))^p\Delta t\leqslant\left(\frac{p}{1-\gamma}\right)^p\int_a^{\infty}\frac{\lambda(t)}{\Omega^{\gamma-p}(t)}f^p(t)\Delta t, \tag{1.22}$$

where

$$\Omega(t):=\int_t^{\infty}\lambda(s)\Delta s,\ \text{for any } t\in[a,\infty)_{\mathbb{T}},$$

and if $p\geqslant\gamma>1$, then

$$\int_a^{\infty}\frac{\lambda(t)}{\Omega^{\gamma}(t)}\left(\bar{\Phi}(t)\right)^p\Delta t\leqslant\left(\frac{p}{\gamma-1}\right)^p\int_a^{\infty}\frac{\lambda(t)}{\Omega^{\gamma-p}(t)}f^p(t)\Delta t. \tag{1.23}$$

Following this trend, to develop the study of dynamic inequalities on time scales, we will prove some new dynamic inequalities of Hardy, Copson, Pachpatte, Yang and Hwang types. The inequalities as special cases contain the inequalities (1.19), (1.20) and (1.21) on time scales. Also, as a special case, when $\mathbb{T}=\mathbb{R}$, the results contain the integral inequalities (1.9), (1.10), (1.12), (1.13), (1.14) and (1.15) and as a special case when $\mathbb{T}=\mathbb{N}$ the results contain the discrete inequalities (1.4) and (1.5). The results will be proved by employing the chain rule, integration by parts formula, Hölder's inequality and Jensen's inequality on time scales which will be stated later in Section 3.

2. Generalizations of Yang and Hwang's inequality

In this section, we will prove some new generalizations of Yang and Hwang's inequality (1.15) on time scales. As special cases from these general inequalities, we will derive several inequalities of Hardy's and Copson's types.

To prove the main results, we will make use of the following derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two delta differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

The chain rule formula on time scales, that will be used throughout the paper, is given by

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1} dh x^{\Delta}(t), \tag{2.2}$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90]. The integration by parts formula on time scales is given by

$$\int_a^b u(t)\upsilon^{\Delta}(t)\Delta t = [u(t)\upsilon(t)]_a^b - \int_a^b u^{\Delta}(t)\upsilon^{\sigma}(t)\Delta t. \tag{2.3}$$

The Hölder's inequality on time scales [5, Theorem 6.13] is given by

$$\int_a^b |u(t)\upsilon(t)|\Delta t \leqslant \left[\int_a^b |u(t)|^q \Delta t\right]^{\frac{1}{q}} \left[\int_a^b |\upsilon(t)|^p \Delta t\right]^{\frac{1}{p}}, \tag{2.4}$$

where $p > 1$ and $1/p + 1/q = 1$.

Now, we are in a position to state and prove the main results in this section. For simplicity, we define the operators

$$\Phi(t) := \int_a^t \lambda(s)q(s)f(s)\Delta s, \; \Lambda(t) := \int_a^t \lambda(s)q^{\sigma}(s)\Delta s, \; t \in [a,\infty)_{\mathbb{T}}. \tag{2.5}$$

THEOREM 2.1. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_{\mathbb{T}}$, $1 < \gamma \leqslant p$ and $q(t)$ be an increasing function on $[a,\infty)_{\mathbb{T}}$. Furthermore, assume that there exists a constant $K > 0$ such that*

$$\gamma - 1 + \frac{q^{\Delta}(t)\Lambda^{\sigma}(t)\Phi^{p}(t)}{\lambda(t)\left(q^{\sigma}(t)\right)^2 \left(\Phi^{\sigma}(t)\right)^p} \geqslant \frac{p}{K}, \quad \text{for } t \in [a,\infty)_{\mathbb{T}}. \tag{2.6}$$

Then

$$\int_a^{\infty} \frac{\lambda(t)}{\left(\Lambda^{\sigma}(t)\right)^{\gamma}} \left(\Phi^{\sigma}(t)\right)^p \Delta t \leqslant K^p \int_a^{\infty} \frac{\left(\Lambda^{\sigma}(t)\right)^{\gamma(p-1)}}{\left(\Lambda(t)\right)^{(\gamma-1)p}} \lambda(t) f^p(t)\Delta t. \tag{2.7}$$

Proof. Integrating the left hand side of (2.7) by parts formula (2.3) with $\upsilon^{\sigma}(t) = (\Phi^{\sigma}(t))^{p}/q^{\sigma}(t)$, and $u^{\Delta}(t) = \lambda(t)q^{\sigma}(t)(\Lambda^{\sigma}(t))^{-\gamma}$, we obtain

$$\begin{aligned}
&\int_a^{\infty} \lambda(t)(\Lambda^{\sigma}(t))^{-\gamma}(\Phi^{\sigma}(t))^{p}\Delta t \\
&= \int_a^{\infty} \lambda(t)q^{\sigma}(t)(\Lambda^{\sigma}(t))^{-\gamma}\left(\frac{(\Phi^{\sigma}(t))^{p}}{q^{\sigma}(t)}\right)\Delta t \\
&= u(t)\left.\frac{\Phi^{p}(t)}{q(t)}\right|_a^{\infty} + \int_a^{\infty}(-u(t))\left(\frac{\Phi^{p}(t)}{q(t)}\right)^{\Delta}\Delta t, \qquad (2.8)
\end{aligned}$$

where $u(t) = -\int_t^{\infty}\lambda(s)q^{\sigma}(s)(\Lambda^{\sigma}(s))^{-\gamma}\Delta s$. Using the facts that $\Phi(a) = 0$ and $u(\infty) = 0$, we get from (2.8) that

$$\int_a^{\infty}\lambda(t)(\Lambda^{\sigma}(t))^{-\gamma}(\Phi^{\sigma}(t))^{p}\Delta t = \int_a^{\infty}(-u(t))\left(\frac{\Phi^{p}(t)}{q(t)}\right)^{\Delta}\Delta t. \qquad (2.9)$$

Applying the chain rule (2.2) and using the fact that $\Lambda^{\Delta}(t) = \lambda(t)q^{\sigma}(t) > 0$, we see that

$$\begin{aligned}
\left(\Lambda^{-\gamma+1}(t)\right)^{\Delta} &= -(\gamma-1)\int_0^1 [h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)]^{-\gamma}dh\Lambda^{\Delta}(t) \\
&= -(\gamma-1)\int_0^1 \frac{dh}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)]^{\gamma}}\lambda(t)q^{\sigma}(t) \\
&\leqslant -(\gamma-1)\int_0^1 \frac{dh}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda^{\sigma}(t)]^{\gamma}}\lambda(t)q^{\sigma}(t) \\
&= -(\gamma-1)\lambda(t)q^{\sigma}(t)(\Lambda^{\sigma}(t))^{-\gamma}.
\end{aligned}$$

This implies that

$$\lambda(t)q^{\sigma}(t)(\Lambda^{\sigma}(t))^{-\gamma} \leqslant \frac{1}{-(\gamma-1)}\left(\Lambda^{-\gamma+1}(t)\right)^{\Delta}. \qquad (2.10)$$

Hence

$$\begin{aligned}
-u(t) &= \int_t^{\infty}\lambda(s)q^{\sigma}(s)(\Lambda^{\sigma}(s))^{-\gamma}\Delta s \\
&\leqslant \frac{1}{-(\gamma-1)}\int_t^{\infty}\left(\Lambda^{-\gamma+1}(s)\right)^{\Delta}\Delta s \leqslant \frac{1}{\gamma-1}(\Lambda(t))^{-\gamma+1}. \qquad (2.11)
\end{aligned}$$

Combining (2.9) and (2.11), we get that

$$\int_a^{\infty}\lambda(t)(\Lambda^{\sigma}(t))^{-\gamma}(\Phi^{\sigma}(t))^{p}\Delta t \leqslant \frac{1}{\gamma-1}\int_a^{\infty}(\Lambda(t))^{-\gamma+1}\left(\frac{\Phi^{p}(t)}{q(t)}\right)^{\Delta}\Delta t. \qquad (2.12)$$

Using the quotient rule (2.1), we see that

$$\left(\frac{\Phi^{p}(t)}{q(t)}\right)^{\Delta} = \frac{q(t)(\Phi^{p}(t))^{\Delta} - \Phi^{p}(t)q^{\Delta}(t)}{q(t)q^{\sigma}(t)}. \qquad (2.13)$$

Applying the chain rule

$$f^{\Delta}(g(t)) = f'(g(c))\, f^{\Delta}(t),\ \text{for}\quad c\in[t,\sigma(t)], \tag{2.14}$$

we see that

$$(\Phi^{p}(t))^{\Delta} = p\Phi^{p-1}(c)\Phi^{\Delta}(t),\ \text{for } c\in[t,\sigma(t)]. \tag{2.15}$$

Since $\Phi^{\Delta}(t)=\lambda(t)q(t)f(t)\geqslant 0$ and $\sigma(t)\geqslant c,$ we have

$$(\Phi^{p}(t))^{\Delta} \leqslant p\,(\Phi^{\sigma}(t))^{p-1}\lambda(t)q(t)f(t). \tag{2.16}$$

From (2.16) and (2.13), we have that

$$\left(\frac{\Phi^{p}(t)}{q(t)}\right)^{\Delta} \leqslant \frac{p\lambda(t)q(t)f(t)\,(\Phi^{\sigma}(t))^{p-1}}{q^{\sigma}(t)} - \frac{\Phi^{p}(t)q^{\Delta}(t)}{q(t)q^{\sigma}(t)}. \tag{2.17}$$

Substituting (2.17) into (2.12), we get that

$$\begin{aligned}\int_{a}^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{\gamma}}(\Phi^{\sigma}(t))^{p}\Delta t \leqslant{}& \frac{p}{\gamma-1}\int_{a}^{\infty}\frac{\lambda(t)q(t)f(t)\,(\Phi^{\sigma}(t))^{p-1}}{\Lambda^{\gamma-1}(t)q^{\sigma}(t)}\Delta t\\ &-\frac{1}{\gamma-1}\int_{a}^{\infty}\frac{q^{\Delta}(t)\Phi^{p}(t)}{\Lambda^{\gamma-1}(t)q(t)q^{\sigma}(t)}.\end{aligned} \tag{2.18}$$

Using the facts that $q(t)\leqslant q^{\sigma}(t)$ and $\Lambda^{\sigma}(t)\geqslant\Lambda(t)$ (since $q(t)$ and $\Lambda(t)$ are increasing rd-continuous functions and $\sigma(t)\geqslant t$), we get that

$$\begin{aligned}\int_{a}^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{\gamma}}(\Phi^{\sigma}(t))^{p}\Delta t \leqslant{}& \frac{p}{\gamma-1}\int_{a}^{\infty}\lambda(t)f(t)\,(\Lambda(t))^{-\gamma+1}(\Phi^{\sigma}(t))^{p-1}\Delta t\\ &-\frac{1}{\gamma-1}\int_{a}^{\infty}\frac{q^{\Delta}(t)\Phi^{p}(t)}{(\Lambda^{\sigma}(t))^{\gamma-1}(q^{\sigma}(t))^{2}}\Delta t.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{a}^{\infty}\lambda(t)\,(\Lambda^{\sigma}(t))^{-\gamma}(\Phi^{\sigma}(t))^{p}\left[\gamma-1+\frac{q^{\Delta}(t)\Lambda^{\sigma}(t)\Phi^{p}(t)}{\lambda(t)\,(q^{\sigma}(t))^{2}\,(\Phi^{\sigma}(t))^{p}}\right]\Delta t\\ &\qquad\leqslant p\int_{a}^{\infty}\lambda(t)\Lambda^{-\gamma+1}(t)\,(\Phi^{\sigma}(t))^{p-1}f(t)\Delta t.\end{aligned}$$

Applying (2.6) and the Hölder's inequality (2.4) with indices p and $p/(p-1),$ we see that

$$\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\Phi^\sigma(t)\right)^p \Delta t \leqslant K\int_a^\infty \lambda(t)\Lambda^{-\gamma+1}(t)\left(\Phi^\sigma(t)\right)^{p-1} f(t)\Delta t$$
$$= K\int_a^\infty \left\{\lambda^{\frac{p-1}{p}}(t)\left(\Lambda^\sigma(t)\right)^{\frac{-\gamma(p-1)}{p}}\left(\Phi^\sigma(t)\right)^{p-1}\right\}$$
$$\times\left\{\left(\Lambda^\sigma(t)\right)^{\frac{\gamma(p-1)}{p}}\Lambda^{-\gamma+1}(t)\lambda^{\frac{1}{p}}(t)f(t)\right\}\Delta t$$
$$\leqslant K\left\{\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\Phi^\sigma(t)\right)^p \Delta t\right\}^{\frac{p-1}{p}}$$
$$\times\left\{\int_a^\infty \frac{\left(\Lambda^\sigma(t)\right)^{\gamma(p-1)}}{\Lambda^{(\gamma-1)p}(t)}\lambda(t)f^p(t)\Delta t\right\}^{\frac{1}{p}}.$$

This gives us that

$$\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\Phi^\sigma(t)\right)^p \Delta t \leqslant K^p\int_a^\infty \frac{\left(\Lambda^\sigma(t)\right)^{\gamma(p-1)}}{\Lambda^{(\gamma-1)p}(t)}\lambda(t)f^p(t)\Delta t,$$

which is the desired inequality (2.7). The proof is complete.

REMARK 2.1. As a special case of Theorem 2.1, when $q(t)=1$, we see that the inequality (2.7) reduces to the inequality (1.20).

REMARK 2.2. As a special case of Theorem 2.1 when $\lambda(t)=q(t)=1$, we see that the inequality (2.7) becomes the inequality (1.19).

REMARK 2.3. If we assume that $q(t)=1$, $\gamma=p$ and there exists a constant $\beta>1$ such that

$$\frac{\Lambda(t)}{\Lambda^\sigma(t)} \geqslant \frac{1}{\beta}, \quad \text{for } t\in[a,\infty)_{\mathbb{T}}, \tag{2.19}$$

then the inequality (2.7) becomes

$$\int_a^\infty \lambda(t)\left(\frac{\Phi^\sigma(t)}{\Lambda^\sigma(t)}\right)^p \Delta t \leqslant \left(\frac{p\beta^{p-1}}{p-1}\right)^p\int_a^\infty \lambda(t)f^p(t)\Delta t, \tag{2.20}$$

which is an inequality of Copson's type.

REMARK 2.4. As a special case of inequality (2.20) when $\lambda(t)=1$, we have the following inequality of Hardy's type

$$\int_a^\infty \left(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)} f(s)\Delta s\right)^p \Delta t \leqslant \left(\frac{p\beta^{p-1}}{p-1}\right)^p\int_a^\infty f^p(t)\Delta t, \tag{2.21}$$

on a time scale $\mathbb{T}$ where $\sigma(t)$ satisfies $\sigma(t)\leqslant\beta t$ for $\beta>1$.

When $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and Theorem 2.1 gives us the following result.

COROLLARY 2.1. *Let $1<\gamma\leqslant p$ and $q(t)$ be an increasing function on $[a,\infty)$. Furthermore, assume that there exists a constant $K>0$ such that*

$$\gamma-1+\frac{q^{'}(t)\Lambda(t)}{\lambda(t)q^{2}(t)}\geqslant\frac{p}{K},\quad for\ t\geqslant a. \tag{2.22}$$

Then

$$\int_{a}^{\infty}\frac{\lambda(t)}{\Lambda^{\gamma}(t)}\Phi^{p}(t)dt\leqslant K^{p}\int_{a}^{\infty}\Lambda^{p-\gamma}(t)\lambda(t)f^{p}(t)dt, \tag{2.23}$$

where

$$\Phi(t)=\int_{a}^{t}\lambda(s)q(s)f(s)ds,\ and\ \Lambda(t)=\int_{a}^{t}\lambda(s)q(s)ds. \tag{2.24}$$

The inequality (2.23) reduces to the inequality (1.12) due to Levinson, the inequality (1.15) due to Yang and Hwang, the inequality (1.13) due to Copson and the inequality (1.9) due to Hardy by using different substitutions of the constants and the functions as listed in the following.

REMARK 2.5. When $\gamma=p>1$ and $a=0$, we get the inequality (1.15) due to Yang and Hwang.

REMARK 2.6. When $\gamma=p>1$, $a=0$ and $\lambda(t)=1$, we get the special form of the inequality (1.12) due to Levinson for $\varphi(x)=x^{p}$.

REMARK 2.7. When $q(t)=1$, we get the integral inequality (1.13) due to Copson.

REMARK 2.8. When $\lambda(t)=q(t)=1$ and $a=0$, we get the inequality (1.9) due to Hardy.

REMARK 2.9. When $\gamma=p$ and $\lambda(t)=q(t)=1$, we get the classical Hardy inequality (1.2).

REMARK 2.10. As a special case of Theorem 2.1 when $\mathbb{T}=\mathbb{N}$, $q(t)=1$ and $a=1$, we get an inequality of Copson's type (1.4).

In the following, we prove a new inequality with different operators $\bar{\Phi}(t)$ and $\Lambda(t)$ which are defined by

$$\bar{\Phi}(t):=\int_{t}^{\infty}\lambda(s)q(s)f(s)\Delta s,\ \Lambda(t):=\int_{a}^{t}\lambda(s)q(s)\Delta s,\ t\in[a,\infty)_{\mathbb{T}}. \tag{2.25}$$

THEOREM 2.2. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_{\mathbb{T}}$, $p > 1$, $0 \leqslant \gamma < 1$ and $q(t)$ be an increasing function on $[a,\infty)_{\mathbb{T}}$. Furthermore, assume that there exists a constant $K > 0$ such that*

$$1-\gamma-\frac{q^{\Delta}(t)\Lambda^{\sigma}(t)}{\lambda(t)q^{2}(t)} \geqslant \frac{p}{K}, \quad for\, t \in [a,\infty)_{\mathbb{T}}. \tag{2.26}$$

Then

$$\int_{a}^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{\gamma}}\left(\bar{\Phi}(t)\right)^{p}\Delta t \leqslant K^{p}\int_{a}^{\infty}(\Lambda^{\sigma}(t))^{p-\gamma}\lambda(t)f^{p}(t)\Delta t, \tag{2.27}$$

Proof. Integrating the left hand side of (2.27) by parts formula (2.3) with $u(t) = \left(\bar{\Phi}(t)\right)^{p}/q(t)$, and $\upsilon^{\Delta}(t) = \lambda(t)q(t)\left(\Lambda^{\sigma}(t)\right)^{-\gamma}$, we obtain

$$\begin{aligned}
&\int_{a}^{\infty}\lambda(t)\left(\Lambda^{\sigma}(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^{p}\Delta t \\
&= \int_{a}^{\infty}\lambda(t)q(t)\left(\Lambda^{\sigma}(t)\right)^{-\gamma}\left(\frac{\left(\bar{\Phi}(t)\right)^{p}}{q(t)}\right)\Delta t \\
&= \upsilon(t)\left.\frac{\left(\bar{\Phi}(t)\right)^{p}}{q(t)}\right|_{a}^{\infty} + \int_{a}^{\infty}\upsilon^{\sigma}(t)\left(-\frac{\left(\bar{\Phi}(t)\right)^{p}}{q(t)}\right)^{\Delta}\Delta t,
\end{aligned} \tag{2.28}$$

where $\upsilon(t) = \int_{a}^{t}\lambda(s)q(s)\left(\Lambda^{\sigma}(s)\right)^{-\gamma}\Delta s$. Using the facts that $\bar{\Phi}(\infty) = 0$ and $\upsilon(a) = 0$, we get from (2.28) that

$$\int_{a}^{\infty}\lambda(t)\left(\Lambda^{\sigma}(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^{p}\Delta t = \int_{a}^{\infty}\upsilon^{\sigma}(t)\left(-\frac{\left(\bar{\Phi}(t)\right)^{p}}{q(t)}\right)^{\Delta}\Delta t. \tag{2.29}$$

Applying the chain rule (2.2), we see that

$$\begin{aligned}
\left(\Lambda^{1-\gamma}(t)\right)^{\Delta} &= (1-\gamma)\int_{0}^{1}\left[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)\right]^{-\gamma}dh\Lambda^{\Delta}(t) \\
&= (1-\gamma)\int_{0}^{1}\frac{dh}{\left[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)\right]^{\gamma}}\lambda(t)q(t) \\
&\geqslant (1-\gamma)\int_{0}^{1}\frac{dh}{\left[h\Lambda^{\sigma}(t)+(1-h)\Lambda^{\sigma}(t)\right]^{\gamma}}\lambda(t)q(t) \\
&= (1-\gamma)\lambda(t)q(t)\left(\Lambda^{\sigma}(t)\right)^{-\gamma}.
\end{aligned}$$

This implies that

$$\lambda(t)q(t)\left(\Lambda^{\sigma}(t)\right)^{-\gamma} \leqslant \frac{1}{1-\gamma}\left(\Lambda^{1-\gamma}(t)\right)^{\Delta}. \tag{2.30}$$

Hence

$$\begin{aligned} \upsilon^{\sigma}(t) &= \int_a^{\sigma(t)} \lambda(s) q(s) \left(\Lambda^{\sigma}(s)\right)^{-\gamma} \Delta s \\ &\leqslant \frac{1}{1-\gamma} \int_a^{\sigma(t)} \left(\Lambda^{1-\gamma}(s)\right)^{\Delta} \Delta s = \frac{1}{1-\gamma} \left(\Lambda^{\sigma}(t)\right)^{1-\gamma}. \end{aligned} \tag{2.31}$$

Combining (2.29) and (2.31), we get that

$$\int_a^{\infty} \lambda(t) \left(\Lambda^{\sigma}(t)\right)^{-\gamma} \left(\bar{\Phi}(t)\right)^p \Delta t \leqslant \frac{1}{1-\gamma} \int_a^{\infty} \left(\Lambda^{\sigma}(t)\right)^{1-\gamma} \left(-\frac{\left(\bar{\Phi}(t)\right)^p}{q(t)}\right)^{\Delta} \Delta t. \tag{2.32}$$

Using the quotient rule (2.1), we see that

$$-\left(\frac{\left(\bar{\Phi}(t)\right)^p}{q(t)}\right)^{\Delta} = \frac{-q(t)\left(\left(\bar{\Phi}(t)\right)^p\right)^{\Delta} + \left(\bar{\Phi}(t)\right)^p q^{\Delta}(t)}{q(t) q^{\sigma}(t)}. \tag{2.33}$$

Applying the chain rule (2.14), we see that

$$-\left(\left(\bar{\Phi}(t)\right)^p\right)^{\Delta} = -p\bar{\Phi}^{p-1}(c)\left(\bar{\Phi}(t)\right)^{\Delta}, \text{ where } c \in [t, \sigma(t)]. \tag{2.34}$$

Since $\left(\bar{\Phi}(t)\right)^{\Delta} = -\lambda(t) q(t) f(t) \leqslant 0$ and $c \geqslant t,$ we have that

$$-\left(\left(\bar{\Phi}(t)\right)^p\right)^{\Delta} \leqslant p\bar{\Phi}^{p-1}(t) \lambda(t) q(t) f(t). \tag{2.35}$$

From (2.35) and (2.33), we have that

$$-\left(\frac{\left(\bar{\Phi}(t)\right)^p}{q(t)}\right)^{\Delta} \leqslant \frac{p\lambda(t) q(t) f(t) \left(\bar{\Phi}(t)\right)^{p-1}}{q^{\sigma}(t)} + \frac{\left(\bar{\Phi}(t)\right)^p q^{\Delta}(t)}{q(t) q^{\sigma}(t)}. \tag{2.36}$$

Substituting (2.36) into (2.32), we get that

$$\begin{aligned} &\int_a^{\infty} \lambda(t) \left(\Lambda^{\sigma}(t)\right)^{-\gamma} \left(\bar{\Phi}(t)\right)^p \Delta t \\ &\leqslant \frac{p}{1-\gamma} \int_a^{\infty} \frac{\lambda(t) q(t) f(t) \left(\Lambda^{\sigma}(t)\right)^{1-\gamma} \left(\bar{\Phi}(t)\right)^{p-1}}{q^{\sigma}(t)} \Delta t \\ &\quad + \frac{1}{1-\gamma} \int_a^{\infty} \frac{q^{\Delta}(t) \left(\Lambda^{\sigma}(t)\right)^{1-\gamma} \left(\bar{\Phi}(t)\right)^p}{q(t) q^{\sigma}(t)}. \end{aligned} \tag{2.37}$$

Since $q(t)$ is an increasing rd-continuous function, we have that

$$\begin{aligned}&\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^p\Delta t\\ &\leqslant \frac{p}{1-\gamma}\int_a^\infty \lambda(t)f(t)\left(\Lambda^\sigma(t)\right)^{1-\gamma}\left(\bar{\Phi}(t)\right)^{p-1}\Delta t\\ &+\frac{1}{1-\gamma}\int_a^\infty \frac{q^\Delta(t)\left(\Lambda^\sigma(t)\right)^{1-\gamma}\left(\bar{\Phi}(t)\right)^p}{q^2(t)}\Delta t.\end{aligned}$$

Hence

$$\begin{aligned}&\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^p\left[1-\gamma-\frac{q^\Delta(t)\Lambda^\sigma(t)}{\lambda(t)q^2(t)}\right]\Delta t\\ &\leqslant p\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{1-\gamma}\left(\bar{\Phi}(t)\right)^{p-1}f(t)\Delta t.\end{aligned}$$

Applying (2.26) and the Hölder's inequality (2.4) with indices p and $p/(p-1)$, we see that

$$\begin{aligned}\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^p\Delta t &\leqslant K\int_a^\infty \left(\Lambda^\sigma(t)\right)^{1-\gamma}\left(\bar{\Phi}(t)\right)^{p-1}\lambda(t)f(t)\Delta t\\ &= K\int_a^\infty \left\{\lambda^{\frac{p-1}{p}}(t)\left(\Lambda^\sigma(t)\right)^{-\frac{\gamma(p-1)}{p}}\left(\bar{\Phi}(t)\right)^{p-1}\right\}\\ &\quad\times\left\{\left(\Lambda^\sigma(t)\right)^{\frac{\gamma(p-1)}{p}}\left(\Lambda^\sigma(t)\right)^{1-\gamma}\lambda^{\frac{1}{p}}(t)f(t)\right\}\Delta t\\ &\leqslant K\left\{\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^p\Delta t\right\}^{\frac{p-1}{p}}\\ &\quad\times\left(\int_a^\infty \left(\Lambda^\sigma(t)\right)^{p-\gamma}\lambda(t)f^p(t)\Delta t\right)^{\frac{1}{p}}.\end{aligned}$$

This gives us that

$$\int_a^\infty \lambda(t)\left(\Lambda^\sigma(t)\right)^{-\gamma}\left(\bar{\Phi}(t)\right)^p\Delta t\leqslant K^p\int_a^\infty \left(\Lambda^\sigma(t)\right)^{p-\gamma}\lambda(t)f^p(t)\Delta t,$$

which is the desired inequality (2.27). The proof is complete.

REMARK 2.11. If we use the fact that $\left(\bar{\Phi}(t)\right)^\Delta\leqslant 0$, we see that

$$\int_a^\infty \frac{\lambda(t)}{\left(\Lambda^\sigma(t)\right)^\gamma}\left(\bar{\Phi}^\sigma(t)\right)^p\Delta t\leqslant\int_a^\infty \frac{\lambda(t)}{\left(\Lambda^\sigma(t)\right)^\gamma}\left(\bar{\Phi}(t)\right)^p\Delta t.$$

This and (2.27) give us a new inequality of the form

$$\int_a^\infty \frac{\lambda(t)}{\left(\Lambda^\sigma(t)\right)^\gamma}\left(\bar{\Phi}^\sigma(t)\right)^p\Delta t\leqslant K^p\int_a^\infty \left(\Lambda^\sigma(t)\right)^{p-\gamma}\lambda(t)f^p(t)\Delta t. \tag{2.38}$$

REMARK 2.12. As a special case of Theorem 2.2 when $q(t)=1$, we see that the inequality (2.27) reduces to the inequality (1.21).

REMARK 2.13. As a special case of Theorem 2.2 when $\lambda(t)=q(t)=1$, we get the time scales version of the Hardy type inequality (1.10)

$$\int_a^\infty \frac{1}{(\sigma(t)-a)^\gamma}\left(\int_t^\infty f(s)\Delta s\right)^p \Delta t \leqslant \left(\frac{p}{1-\gamma}\right)^p \int_a^\infty \frac{1}{(\sigma(t)-a)^{\gamma-p}} f^p(t)\Delta t. \quad (2.39)$$

Note that When $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and Theorem 2.2 gives us the following result.

COROLLARY 2.2. *If $p>1$, $0\leqslant\gamma<1$ and there exists a constant $K>0$ such that*

$$1-\gamma-\frac{q'(t)\Lambda(t)}{\lambda(t)q^2(t)} \geqslant \frac{p}{K}, \quad for\ t\in[a,\infty). \quad (2.40)$$

Then

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^\gamma(t)}\left(\bar{\Phi}(t)\right)^p dt \leqslant K^p \int_a^\infty \Lambda^{p-\gamma}(t)\lambda(t) f^p(t)dt, \quad (2.41)$$

where $\bar{\Phi}(t):=\int_t^\infty \lambda(s)q(s)f(s)ds$ and $\Lambda(t):=\int_a^t \lambda(s)q(s)ds$, $t\in[a,\infty)$.

The inequality (2.41) reduces to the inequalities due to Copson and Hardy by using different substitutions of the functions as listed in the following.

REMARK 2.14. If $q(t)=1$, then inequality (2.41) reduces to Copson's type integral inequality (1.14).

REMARK 2.15. As a special case of Theorem 2.2 when $\mathbb{T}=\mathbb{N}$, $q(t)=1$ and $a=1$, we get the discrete inequality of Copson's type (1.5).

REMARK 2.16. It would be interesting to prove some new results by excluding the condition that has been proposed on $q(t)$.

3. Generalizations of Pachpatte's inequality

In this section, we will extend Pachpatte's inequality (1.8) on time scales and also prove its dual which is an essentially new even when $\mathbb{T}=\mathbb{R}$. To prove our next theorems, we will use Jensen's inequality (see [2, Chapter 2])

$$F\left(\frac{\int_b^\infty |h(s)| f(s)\Delta s}{\int_b^\infty |h(s)|\Delta s}\right) \leqslant \frac{\int_b^\infty |h(s)| F(f(s))\Delta s}{\int_b^\infty |h(s)|\Delta s}, \quad (3.1)$$

where $F\in C((c,d),\mathbb{R})$ is convex, $b\in[0,\infty)_\mathbb{T}$, $c,d\in\mathbb{R}$, $f\in C_{rd}([b,\infty)_\mathbb{T},(c,d))$ and $h\in C_{rd}([b,\infty)_\mathbb{T},\mathbb{R})$ with $\int_b^\infty |h(s)|\Delta s>0$. For simplicity, we set

$$\Lambda(t)=\int_a^t \lambda(s)\Delta s\ ,\ A(t)=\int_a^t \lambda(s)f(s)\Delta s,$$

$$\psi(t)=\int_a^t \lambda(s)\varphi(f(s))\Delta s, \quad \text{and} \quad \alpha(t)=\frac{\psi(t)}{\Lambda(t)}. \tag{3.2}$$

THEOREM 3.1. *Let $\mathbb{T}$ be a time scale with $a\in[0,\infty)_{\mathbb{T}}$, and further let $\varphi(u)$ be a nondecreasing positive convex function defined for $u>0$. If $p>1$, then*

$$\begin{aligned}\int_a^{\infty} \lambda(t)\varphi^p\left(\frac{A^{\sigma}(t)}{\Lambda^{\sigma}(t)}\right)\Delta t &\leqslant \frac{p}{p-1}\int_a^x \lambda(t)\varphi(f(t))\alpha^{p-1}(\sigma(t))\Delta t \\ &\leqslant \left(\frac{p}{p-1}\right)^p \int_a^{\infty} \lambda(t)\varphi^p(f(t))\Delta t.\end{aligned} \tag{3.3}$$

Proof. From (3.2), we can write that

$$\lambda(t)\varphi(f(t))=[\Lambda(t)\alpha(t)]^{\Delta},$$

which leads directly to the following

$$\lambda(t)\varphi(f(t))=\left[\Lambda(t)\alpha^{\Delta}(t)+\alpha(\sigma(t))\lambda(t)\right]. \tag{3.4}$$

Now by using (3.4), we have that

$$\begin{aligned}&\lambda(t)\alpha^p(\sigma(t))-\frac{p}{p-1}\lambda(t)\varphi(f(t))\alpha^{p-1}(\sigma(t))\\ &=\lambda(t)\alpha^p(\sigma(t))-\frac{p}{p-1}\alpha^{p-1}(\sigma(t))\left[\Lambda(t)\alpha^{\Delta}(t)+\alpha(\sigma(t))\lambda(t)\right]\\ &=\lambda(t)\alpha^p(\sigma(t))-\frac{p}{p-1}\lambda(t)\alpha^p(\sigma(t))-\frac{p}{p-1}\Lambda(t)\alpha^{p-1}(\sigma(t))\alpha^{\Delta}(t)\\ &=-\frac{1}{p-1}\lambda(t)\alpha^p(\sigma(t))-\frac{p}{p-1}\Lambda(t)\alpha^{p-1}(\sigma(t))\alpha^{\Delta}(t).\end{aligned} \tag{3.5}$$

Using the quotient rule (2.1), we have that

$$(\alpha(t))^{\triangle}=\left(\frac{\psi(t)}{\Lambda(t)}\right)^{\triangle}=\frac{\Lambda(t)\psi^{\triangle}(t)-\psi(t)\Lambda^{\triangle}(t)}{\Lambda(t)\Lambda^{\sigma}(t)}.$$

From this, we get that $sgn\ \alpha^{\Delta}(t)=sgn\ \left[\Lambda(t)\psi^{\triangle}(t)-\psi(t)\Lambda^{\triangle}(t)\right]$. Since

$$\begin{aligned}&\Lambda(t)\psi^{\triangle}(t)-\psi(t)\Lambda^{\triangle}(t)\\ &=[\lambda(t)\varphi(f(t))]\int_a^t \lambda(s)\Delta s-\lambda(t)\int_a^t \lambda(s)\varphi(f(s))\Delta s\\ &=\lambda(t)\left[\varphi(f(t))\int_a^t \lambda(s)\Delta s-\int_a^t \lambda(s)\varphi(f(s))\Delta s\right],\end{aligned} \tag{3.6}$$

and φ is nondecreasing, we see that

$$\int_a^t \lambda(s)\varphi(f(s))\Delta s\leqslant \varphi(f(t))\int_a^t \lambda(s)\Delta s,$$

which asserts the positivity of the right-hand side of (3.6). This implies that $\alpha^{\Delta}(t) > 0$. Applying the time scales chain rule (2.2) to $(\alpha^p(t))^{\Delta}$ we see (notice that $\alpha^{\Delta}(t) > 0$) that

$$\begin{aligned}
(\alpha^p(t))^{\Delta} &= p\int_0^1 [h\alpha^{\sigma}(t) + (1-h)\alpha(t)]^{p-1} dh\alpha^{\Delta}(t) \\
&\leqslant p\int_0^1 [h\alpha^{\sigma}(t) + (1-h)\alpha^{\sigma}(t)]^{p-1} dh\alpha^{\Delta}(t) \\
&= p\alpha^{p-1}(\sigma(t))\alpha^{\Delta}(t),
\end{aligned}$$

and hence

$$\alpha^{p-1}(\sigma(t))\alpha^{\Delta}(t) \geqslant \frac{1}{p}(\alpha^p(t))^{\Delta}.$$

Using this estimate in (3.5), we get that

$$\begin{aligned}
&\lambda(t)\alpha^p(\sigma(t)) - \frac{p}{p-1}\lambda(t)\varphi(f(t))\alpha^{p-1}(\sigma(t)) \\
&\leqslant -\frac{1}{p-1}\lambda(t)\alpha^p(\sigma(t)) - \frac{1}{p-1}\Lambda(t)(\alpha^p(t))^{\Delta} \\
&= -\frac{1}{p-1}\left[\Lambda^{\Delta}(t)\alpha^p(\sigma(t)) + \Lambda(t)(\alpha^p(t))^{\Delta}\right] \\
&= -\frac{1}{p-1}(\Lambda(t)(\alpha^p(t)))^{\Delta}.
\end{aligned}$$

Integrating both sides from a to $x,$ we obtain that

$$\begin{aligned}
&\int_a^x \lambda(t)\alpha^p(\sigma(t))\Delta t - \frac{p}{p-1}\int_a^x \lambda(t)\varphi(f(t))\alpha^{p-1}(\sigma(t))\Delta t \\
&\leqslant -\frac{1}{p-1}\Lambda(x)(\alpha^p(x)) \leqslant 0,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\int_a^x \lambda(t)\alpha^p(\sigma(t))\Delta t \\
&\leqslant \frac{p}{p-1}\int_a^x \lambda(t)\varphi(f(t))\alpha^{p-1}(\sigma(t))\Delta t \\
&= \frac{p}{p-1}\int_a^x \left\{\lambda^{\frac{1}{p}}(t)\varphi(f(t))\right\}\left\{\lambda^{\frac{p-1}{p}}(t)\alpha^{p-1}(\sigma(t))\right\}\Delta t.
\end{aligned}$$

Applying Hölder's inequality (2.4) on the right hand-side with indices p and $p/(p-1)$, we have that

$$\begin{aligned}
\int_a^x \lambda(t)\alpha^p(\sigma(t))\Delta t \leqslant{}& \frac{p}{p-1}\left\{\int_a^x \lambda(t)\varphi^p(f(t))\Delta t\right\}^{\frac{1}{p}} \\
&\times\left\{\int_a^x \lambda(t)\alpha^p(\sigma(t))\Delta t\right\}^{\frac{p-1}{p}}.
\end{aligned}$$

Dividing both sides by the last factor $\{\int_a^x \lambda(t)\alpha^p(\sigma(t))\Delta t\}^{\frac{p-1}{p}}$, we obtain that

$$\int_a^x \lambda(t)\alpha^p(\sigma(t))\Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^x \lambda(t)\varphi^p(f(t))\Delta t,$$

which can be written as

$$\int_a^x \lambda(t)\left(\frac{\psi^\sigma(t)}{\Lambda^\sigma(t)}\right)^p \Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^x \lambda(t)\varphi^p(f(t))\Delta t. \tag{3.7}$$

Applying Jensen's inequality (3.1) with $F=\varphi$ and $h=\lambda$ (since φ is convex), we have that

$$\begin{aligned}\varphi\left(\frac{A^\sigma(t)}{\Lambda^\sigma(t)}\right) &= \varphi\left(\frac{\int_a^{\sigma(t)}\lambda(s)f(s)\Delta s}{\int_a^{\sigma(t)}\lambda(s)\Delta s}\right)\\ &\leqslant \left(\frac{\int_a^{\sigma(t)}\lambda(s)\varphi(f(s))\Delta s}{\int_a^{\sigma(t)}\lambda(s)\Delta s}\right) = \frac{\psi^\sigma(t)}{\Lambda^\sigma(t)}.\end{aligned}$$

Using this in (3.7), we get that

$$\begin{aligned}\int_a^x \lambda(t)\varphi^p\left(\frac{A^\sigma(t)}{\Lambda^\sigma(t)}\right)\Delta t &\leqslant \int_a^x \lambda(t)\left(\frac{\psi^\sigma(t)}{\Lambda^\sigma(t)}\right)^p\Delta t\\ &\leqslant \left(\frac{p}{p-1}\right)^p \int_a^x \lambda(t)\varphi^p(f(t))\Delta t.\end{aligned}$$

By letting $x \to \infty$, we obtain that

$$\int_a^\infty \lambda(t)\varphi^p\left(\frac{A^\sigma(t)}{\Lambda^\sigma(t)}\right)\Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))\Delta t,$$

which is the required inequality (3.3). This completes the proof.

As a special case from Theorem 3.1, by taking $\varphi(u)=u$ we get the following dynamic Hardy-type inequality which improve the inequalities due to Saker et al. [28, Theorem 2.1] and [29, Theorem 2.1] for $k=c$.

COROLLARY 3.1. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_{\mathbb{T}}$. If $p>1$, then*

$$\int_a^\infty \lambda(t)\left(\frac{A^\sigma(t)}{\Lambda^\sigma(t)}\right)^p\Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t) f^p(t)\Delta t,$$

where

$$\Lambda(t) = \int_a^t \lambda(s)\Delta s, \ \text{and} \ \ A(t) = \int_a^t \lambda(s)f(s)\Delta s.$$

REMARK 3.1. In Theorem 3.1 by taking $\varphi(u) = u$ and $\lambda(t) = 1$, we obtain the inequality

$$\int_a^\infty \left(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)} f(s)\Delta s\right)^p \Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(t)\Delta t,$$

of Hardy's type due to Řehák [24]. We should mention here that the condition $\varphi^\Delta(t) > 0$ where $\varphi(t) = \int_a^t f(s)\Delta s/(t-a)$ that has been proposed by Řehák to prove his inequality (1.16) has been excluded.

As a special case of Theorem 3.1, by choosing $\lambda(t) = 1$, we get the following dynamic Hardy-type inequality which can be considered as the time scales version of the Hardy-type inequality due to Sulaiman [33, Theorem 2.7].

COROLLARY 3.2. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_\mathbb{T}$, and φ be a positive nondecreasing convex function. If $p > 1$, then*

$$\int_0^\infty \varphi^p\left(\frac{1}{\sigma(t)}\int_0^{\sigma(t)} f(s)\Delta s\right)\Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty \varphi^p(f(t))\Delta t. \tag{3.8}$$

REMARK 3.2. If we take $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, and Theorem 3.1 gives us the following extension of the continuous inequality of Hardy's type

$$\int_a^\infty \lambda(t)\varphi^p\left(\frac{A(t)}{\Lambda(t)}\right)dt \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))dt,$$

where

$$\Lambda(t) = \int_a^t \lambda(s)ds, \quad \text{and} \quad A(t) = \int_a^t \lambda(s)f(s)ds.$$

REMARK 3.3. If $\mathbb{T} = \mathbb{N}$, then the inequality (3.3) reduces to the discrete inequality (1.8) due to Pachpatte.

To prove the next theorem, for simplicity, we set

$$\Lambda^*(t) = \int_t^\infty \lambda(s)\Delta s\ , \ A^*(t) = \int_t^\infty \lambda(s)f(s)\Delta s,$$

$$\psi^*(t) = \int_t^\infty \lambda(s)\varphi(f(s))\Delta s, \quad \text{and} \quad \alpha^*(t) = \frac{\psi^*(t)}{\Lambda^*(t)}. \tag{3.9}$$

THEOREM 3.2. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_\mathbb{T}$, and further let $\varphi(u)$ be a nonincreasing positive convex function defined for $u > 0$. If $p > 1$, then*

$$\begin{aligned}\int_a^\infty \lambda(t)\varphi^p\left(\frac{A^*(t)}{\Lambda^*(t)}\right)\Delta t &\leqslant \frac{p}{p-1}\int_a^\infty \lambda(t)\varphi(f(t))\alpha^{p-1}(t)\Delta t\\ &\leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))\Delta t.\end{aligned} \tag{3.10}$$

Proof. From (3.9), we can write that

$$\lambda(t)\varphi(f(t)) = -\left[\Lambda^*(t)\alpha^*(t)\right]^\Delta,$$

which leads directly to

$$\begin{aligned}\lambda(t)\varphi(f(t)) &= -\left[-\lambda(t)\alpha^*(t) + \Lambda^*(\sigma(t))(\alpha^*(t))^\Delta\right] \\ &= \lambda(t)\alpha^*(t) - \Lambda^*(\sigma(t))(\alpha^*(t))^\Delta.\end{aligned} \tag{3.11}$$

Now using (3.11), we have that

$$\begin{aligned}&\lambda(t)(\alpha^*(t))^p - \frac{p}{p-1}\lambda(t)\varphi(f(t))(\alpha^*(t))^{p-1} \\ &= \lambda(t)(\alpha^*(t))^p - \frac{p}{p-1}(\alpha^*(t))^{p-1}\left[\lambda(t)\alpha^*(t) - \Lambda^*(\sigma(t))(\alpha^*(t))^\Delta\right] \\ &= \lambda(t)(\alpha^*(t))^p - \frac{p}{p-1}\lambda(t)(\alpha^*(t))^p + \frac{p}{p-1}\Lambda^*(\sigma(t))(\alpha^*(t))^{p-1}(\alpha^*(t))^\Delta \\ &= -\frac{1}{p-1}\lambda(t)(\alpha^*(t))^p - \frac{p}{p-1}\Lambda^*(\sigma(t))(\alpha^*(t))^{p-1}(-\alpha^*(t))^\Delta.\end{aligned} \tag{3.12}$$

Using the quotient rule (2.1), we have that

$$(\alpha^*(t))^\Delta = \left(\frac{\psi^*(t)}{\Lambda^*(t)}\right)^\Delta = \frac{\Lambda^*(t)(\psi^*(t))^\Delta - \psi^*(t)(\Lambda^*(t))^\Delta}{\Lambda^*(t)\Lambda^*(\sigma(t))},$$

which leads us directly to $sgn\ (\alpha^*(t))^\Delta = sgn\ \left[\Lambda^*(t)(\psi^*(t))^\Delta - \psi^*(t)(\Lambda^*(t))^\Delta\right]$. Since

$$\begin{aligned}&\Lambda^*(t)(\psi^*(t))^\Delta - \psi^*(t)(\Lambda^*(t))^\Delta \\ &= -[\lambda(t)\varphi(f(t))]\int_t^\infty \lambda(s)\Delta s + \lambda(t)\int_t^\infty \lambda(s)\varphi(f(s))\Delta s \\ &= -\lambda(t)\left[\varphi(f(t))\int_t^\infty \lambda(s)\Delta s - \int_t^\infty \lambda(s)\varphi(f(s))\Delta s\right],\end{aligned} \tag{3.13}$$

and φ is nonincreasing, we get that

$$\int_t^\infty \lambda(s)\varphi(f(s))\Delta s \leqslant \varphi(f(t))\int_t^\infty \lambda(s)\Delta s,$$

which asserts the negativity of the right-hand side of (3.13) and hence $(\alpha^*(t))^\Delta < 0$. Applying the time scales chain rule (2.2) we have (notice that $(\alpha^*(t))^\Delta < 0$) that

$$\begin{aligned}(-(\alpha^*(t))^p)^\Delta &= p\int_0^1 [h\alpha^*(\sigma(t)) + (1-h)\alpha^*(t)]^{p-1}\,dh\,(-\alpha^*(t))^\Delta \\ &\leqslant p\int_0^1 [h\alpha^*(t) + (1-h)\alpha^*(t)]^{p-1}\,dh\,(-\alpha^*(t))^\Delta \\ &= p(\alpha^*(t))^{p-1}(-\alpha^*(t))^\Delta.\end{aligned}$$

This implies that

$$-(\alpha^*(t))^{p-1}(-\alpha^*(t))^{\Delta} \leqslant \frac{-1}{p}(-(\alpha^*(t))^{p})^{\Delta}.$$

Using this estimate in (3.13), we get that

$$\begin{aligned}
&\lambda(t)(\alpha^*(t))^p - \frac{p}{p-1}\lambda(t)\varphi(f(t))(\alpha^*(t))^{p-1} \\
&\leqslant \frac{1}{p-1}(-\lambda(t)(\alpha^*(t))^p) - \frac{1}{p-1}\Lambda^*(\sigma(t))(-(\alpha^*(t))^p)^{\Delta} \\
&= \frac{1}{p-1}\left[(\Lambda^*(t))^{\Delta}(\alpha^*(t))^p) + \Lambda^*(\sigma(t))((\alpha^*(t))^p)^{\Delta}\right] \\
&= \frac{1}{p-1}(\Lambda^*(t)(\alpha^*(t))^p)^{\Delta}.
\end{aligned}$$

Integrating both sides from a to x, and using the fact that $\Lambda^*(t)$ is decreasing, we get that

$$\begin{aligned}
&\int_a^x \lambda(t)(\alpha^*(t))^p\,\Delta t - \frac{p}{p-1}\int_a^x \lambda(t)\varphi(f(t))(\alpha^*(t))^{p-1}\,\Delta t \\
&\leqslant \frac{1}{p-1}[\Lambda^*(x)(\alpha^*(x))^p - \Lambda^*(a)(\alpha^*(a))^p] \\
&\leqslant \frac{\Lambda^*(a)}{p-1}[(\alpha^*(x))^p - (\alpha^*(a))^p].
\end{aligned}$$

Also, since $(\alpha^*(t))^{\Delta} < 0$, we have that $[(\alpha^*(x))^p - (\alpha^*(a))^p] \leqslant 0$, and hence

$$\int_a^x \lambda(t)(\alpha^*(t))^p\,\Delta t - \frac{p}{p-1}\int_a^x \lambda(t)\varphi(f(t))(\alpha^*(t))^{p-1}\,\Delta t \leqslant 0.$$

Letting $x \longrightarrow \infty$, we get

$$\begin{aligned}
&\int_a^{\infty} \lambda(t)(\alpha^*(t))^p\,\Delta t \\
&\leqslant \frac{p}{p-1}\int_a^{\infty} \lambda(t)\varphi(f(t))(\alpha^*(t))^{p-1}\,\Delta t \\
&= \frac{p}{p-1}\int_a^{\infty}\left\{\lambda^{\frac{1}{p}}(t)\varphi(f(t))\right\}\left\{\lambda^{\frac{p-1}{p}}(t)(\alpha^*(t))^{p-1}\right\}\Delta t.
\end{aligned}$$

Applying Hölder's inequality on the right hand-side with indices p and $p/(p-1)$, we have that

$$\int_a^{\infty} \lambda(t)(\alpha^*(t))^p\,\Delta t \leqslant \frac{p}{p-1}\left\{\int_a^{\infty} \lambda(t)\varphi^p(f(t))\Delta t\right\}^{\frac{1}{p}}\left\{\int_a^{\infty} \lambda(t)(\alpha^*(t))^p\,\Delta t\right\}^{\frac{p-1}{p}}.$$

Dividing both sides by the last factor $\{\int_a^\infty \lambda(t)(\alpha^*(t))^p \Delta t\}^{\frac{p-1}{p}}$, we obtain that

$$\int_a^\infty \lambda(t)(\alpha^*(t))^p \Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))\Delta t,$$

which can be written as

$$\int_a^\infty \lambda(t)\left(\frac{\psi^*(t)}{\Lambda^*(t)}\right)^p \Delta t \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))\Delta t. \tag{3.14}$$

Applying Jensen's inequality (3.1), we see that

$$\varphi\left(\frac{A^*(t)}{\Lambda^*(t)}\right) = \varphi\left(\frac{\int_t^\infty \lambda(s)f(s)\Delta s}{\int_t^\infty \lambda(s)\Delta s}\right)$$
$$\leqslant \left(\frac{\int_t^\infty \lambda(s)\varphi(f(s))\Delta s}{\int_t^\infty \lambda(s)\Delta s}\right) = \frac{\psi^*(t)}{\Lambda^*(t)}.$$

Using this in (3.14), we get that

$$\int_a^\infty \lambda(t)\varphi^p\left(\frac{A^*(t)}{\Lambda^*(t)}\right)\Delta t \leqslant \int_a^\infty \lambda(t)\left(\frac{\psi^*(t)}{\Lambda^*(t)}\right)^p \Delta t$$
$$\leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))\Delta t,$$

which is the required inequality (3.10). This completes the proof.

REMARK 3.4. If we take $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, and Theorem 3.2 gives us the following continuous inequality of Pachpatte's type with tails

$$\int_a^\infty \lambda(t)\varphi^p\left(\frac{A^*(t)}{\Lambda^*(t)}\right)dt \leqslant \frac{p}{p-1}\int_a^\infty \lambda(t)\varphi(f(t))\alpha^{p-1}(t)\,dt$$
$$\leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty \lambda(t)\varphi^p(f(t))dt,$$

where

$$\Lambda^*(t) = \int_t^\infty \lambda(s)ds \quad \text{and} \quad A^*(t) = \int_t^\infty \lambda(s)f(s)ds.$$

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