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ESSENTIAL NORM OF THE DIFFERENCES OF COMPOSITION OPERATORS ON THE BLOCH SPACE

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Abstract. We provide some estimates for the essential norm of the differences of composition operators $C_\varphi - C_\psi$ acting on the Bloch space by using pseudo-hyperbolic distance, Möbius transformation and $\varphi^n - \psi^n$.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the unit circle and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. We denote by $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. Let $H^\infty = H^\infty(\mathbb{D})$ denote the set of all bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathscr{B} = \mathscr{B}(\mathbb{D})$, if (see [15])

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

It is well known that $\mathscr{B}$ is a Banach space under the norm $\|\cdot\|_{\mathscr{B}}$. Note that $H^\infty \subset \mathscr{B}$ and $\|f\|_{\mathscr{B}} \leqslant 2\|f\|_\infty$ if $f \in H^\infty$. For $\varphi \in S(\mathbb{D})$, $\|\varphi\|_{\mathscr{B}} \leqslant 2\|\varphi\|_\infty \leqslant 2$. The little Bloch space, denoted by $\mathscr{B}_0 = \mathscr{B}_0(\mathbb{D})$, is a subspace of $\mathscr{B}$ consisting of all $f \in H(\mathbb{D})$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$.

For $a \in \mathbb{D}$, let σ_a be the Möbius transformation of $\mathbb{D}$ defined by

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}.$$

For z, $w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\sigma_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

It is well known that $\rho(z,w) \leqslant 1$. For $\varphi \in S(\mathbb{D})$, the Schwarz-Pick type derivative $\varphi^\#$ of φ is defined by

$$\varphi^\#(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z).$$

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By the Schwarz-Pick lemma, we have $|\varphi^{\#}(z)| \leqslant 1$ on $\mathbb{D}$.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It is well known that the composition operator is bounded on the Bloch space since $|\varphi^{\#}(z)| \leqslant 1$. The compactness of the composition operator on the Bloch space was firstly studied in [6]. They proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if $\lim_{|\varphi(z)|\to 1} |\varphi^{\#}(z)| = 0$. Tjani proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if $\lim_{|a|\to 1} \|C_\varphi \sigma_a\|_{\mathscr{B}} = 0$ in [10, 11]. In [12], Wulan, Zheng and Zhu obtained a new characterization for the compactness of the composition operator acting on the Bloch space, i.e., they proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_{\mathscr{B}} = 0$.

The essential norm of the composition operator acting on the Bloch space has been studied by several authors. In [7], Montes-Rodríeguez proved that the essential norm of the operator $C_\varphi : \mathscr{B} \to \mathscr{B}$ is

$$\|C_\varphi\|_{e,\mathscr{B}} = \limsup_{|\varphi(z)|\to 1} |\varphi^{\#}(z)|.$$

In [14], Zhao obtained another characterization for the essential norm of the operator $C_\varphi : \mathscr{B} \to \mathscr{B}$, i.e., he showed that $\|C_\varphi\|_{e,\mathscr{B}} = \frac{e}{2} \limsup_{n\to\infty} \|\varphi^n\|_{\mathscr{B}}$. Recall that the essential norm of a bounded linear operator $T : X \to X$ is its distance to the set of compact operators K mapping X into X, that is,

$$\|T\|_{e,X} = \inf\{\|T - K\|_X : K \text{ is compact }\},$$

where X is a Banach space and $\|\cdot\|_X$ is the operator norm.

Recently, one of the most interesting problems in the theory of composition operator is to characterize the boundedness, compactness and essential norm of the differences of two composition operators, more generally, the linear combination of composition operators. The study of the differences of composition operators was started on the Hardy space H^2. The main purpose for the study of the differences is to understand the topological structure of the set of composition operators $\mathscr{C}(H^2)$, see [1, 2, 9]. It is easy to see that the differences of two composition operators is also bounded on the Bloch space for any analytic self-map. In [3], the authors obtained some characterizations for the compactness of $C_\varphi - C_\psi$, among others, they obtain the following result.

THEOREM A. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Suppose that neither C_φ nor C_ψ is compact on $\mathscr{B}$. Then $C_\varphi - C_\psi$ is compact on $\mathscr{B}$ if and only if*

$$\lim_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} = 0 \quad \textit{and} \quad \lim_{|a|\to 1} \|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} = 0.$$

See [3, 4, 5, 8, 13] for more information of the compactness of the differences of composition operators on the Bloch space. Based on the idea of [12, 14] and THEOREM A, it is natural to ask whether $C_\varphi - C_\psi$ is compact on $\mathscr{B}$ if and only if

$$\lim_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}} = 0?$$

In this paper, we give a positive answer. In fact, motivated by [3, 12, 14], we obtain some estimates for the essential norm of the differences of composition operators on the Bloch space. Our main result is stated as follows.

THEOREM 1. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then,*

$$\begin{aligned} &\|C_\varphi - C_\psi\|_{e,\mathscr{B}} \\ \approx &\lim_{s\to 1}\sup_{|\varphi(z)|>s} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)| + \lim_{s\to 1}\sup_{|\psi(z)|>s} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)| \\ &\qquad + \lim_{s\to 1}\sup_{\substack{|\varphi(z)|>s\\ |\psi(z)|>s}} |\varphi^{\#}(z)-\psi^{\#}(z)| \\ \approx &\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + \limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} \\ \approx &\limsup_{n\to\infty}\|\varphi^n - \psi^n\|_{\mathscr{B}}. \end{aligned}$$

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Preliminary

In this section, we give some auxiliary results. For $z,w \in \mathbb{D}$, we define

$$b(z,w) = \sup_{\|f\|_{\mathscr{B}}\leqslant 1} |(1-|z|)^2 f'(z) - (1-|w|)^2 f'(w)|.$$

LEMMA 1. [3, Proposition 2.2] *For all $z,w \in \mathbb{D}$, we have*

$$\rho(z,w)^2 \leqslant b(z,w) \leqslant 18\rho(z,w).$$

For $r \in (0,1)$, let $K_r f(z) = f(rz)$. Then K_r is a compact operator on the space $\mathscr{B}$ or $\mathscr{B}_0$, with $\|K_r\|_{\mathscr{B}} \leqslant 1$.

LEMMA 2. [14, Lemma 4.2] *There is a sequence $\{r_k\}$, with $0 < r_k < 1$ tending to 1, such that the compact operator $L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$ acting on $\mathscr{B}_0$ satisfies*

(i) For any $t \in [0,1)$, $\lim_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{|z|\leqslant t}|((I-L_n)f)'(z)| = 0$.

(iia) $\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{|z|>s}|(I-L_n)f(z)|(\log\frac{1}{1-|z|^2})^{-1} \leqslant 1$, for s sufficiently close to 1, and

(iib) $\lim_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{|z|\leqslant s}|(I-L_n)f(z)| = 0$, for the above s.

(iii) $\limsup_{n\to\infty}\|I-L_n\| \leqslant 1$.

Furthermore, the same is true for the sequence of biadjoints L_n^{**} on $\mathscr{B}$.

LEMMA 3. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then*

(i)

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}\rho(\varphi(z),\psi(z))|\varphi^{\#}(z)|$$
$$\leqslant \limsup_{|a|\to 1}\|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + \frac{1}{2}\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}.$$

(ii)

$$\lim_{s\to 1}\sup_{|\psi(z)|>s}\rho(\varphi(z),\psi(z))|\psi^{\#}(z)|$$
$$\leqslant \limsup_{|a|\to 1}\|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + \frac{1}{2}\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}.$$

(iii)

$$\lim_{s\to 1}\sup_{\substack{|\varphi(z)|>s\\|\psi(z)|>s}}|\varphi^{\#}(z)-\psi^{\#}(z)|$$
$$\leqslant 19\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + 9\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}.$$

Proof. For any $z\in\mathbb{D}$, we have

$$\begin{aligned}\|(C_\varphi - C_\psi)\sigma_{\varphi(z)}\|_{\mathscr{B}} &\geqslant (1-|z|^2)|((C_\varphi - C_\psi)\sigma_{\varphi(z)})'(z)|\\ &= (1-|z|^2)\left|\frac{\varphi'(z)}{1-|\varphi(z)|^2} - \frac{\psi'(z)(1-|\varphi(z)|^2)}{(1-\overline{\varphi(z)}\psi(z))^2}\right|\\ &= \left|\varphi^{\#}(z) - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)}{(1-\overline{\varphi(z)}\psi(z))^2}\psi^{\#}(z)\right|\\ &\geqslant |\varphi^{\#}(z)| - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)}{|1-\overline{\varphi(z)}\psi(z))^2|}|\psi^{\#}(z)|\\ &= |\varphi^{\#}(z)| - (1-(\rho(\varphi(z),\psi(z)))^2)|\psi^{\#}(z)|\end{aligned}$$

and

$$\begin{aligned}\|(C_\varphi - C_\psi)(\sigma_{\varphi(z)})^2\|_{\mathscr{B}} &\geqslant (1-|z|^2)|((C_\varphi - C_\psi)(\sigma_{\varphi(z)})^2)'(z)|\\ &\geqslant 2(1-(\rho(\varphi(z),\psi(z)))^2)|\psi^{\#}(z)|\rho(\varphi(z),\psi(z)).\end{aligned}$$

Thus

$$\begin{aligned}&\|(C_\varphi - C_\psi)\sigma_{\varphi(z)}\|_{\mathscr{B}} + \frac{1}{2}\|(C_\varphi - C_\psi)(\sigma_{\varphi(z)})^2\|_{\mathscr{B}}\\ &\geqslant \rho(\varphi(z),\psi(z))\|(C_\varphi - C_\psi)\sigma_{\varphi(z)}\|_{\mathscr{B}} + \frac{1}{2}\|(C_\varphi - C_\psi)(\sigma_{\varphi(z)})^2\|_{\mathscr{B}}\\ &\geqslant \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)|.\end{aligned}$$

Similarly,

$$\|(C_\varphi - C_\psi)\sigma_{\psi(z)}\|_{\mathscr{B}} + \frac{1}{2}\|(C_\varphi - C_\psi)(\sigma_{\psi(z)})^2\|_{\mathscr{B}} \geqslant \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|.$$

Then we have

$$\begin{aligned}
&\lim_{s\to 1}\sup_{|\varphi(z)|>s} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)| \\
\leqslant &\lim_{s\to 1}\sup_{|\varphi(z)|>s} \|(C_\varphi - C_\psi)\sigma_{\varphi(z)}\|_{\mathscr{B}} + \frac{1}{2}\lim_{s\to 1}\sup_{|\varphi(z)|>s} \|(C_\varphi - C_\psi)(\sigma_{\varphi(z)})^2\|_{\mathscr{B}} \\
\leqslant &\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + \frac{1}{2}\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{s\to 1}\sup_{|\psi(z)|>s} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)| \\
\leqslant &\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + \frac{1}{2}\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}.
\end{aligned}$$

Moreover, by applying Lemma 1, we have

$$\begin{aligned}
&\|(C_\varphi - C_\psi)\sigma_{\varphi(z)}\|_{\mathscr{B}} \\
\geqslant &|\varphi^{\#}(z) - \psi^{\#}(z)| - |\psi^{\#}(z)|\left|1 - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)}{(1-\overline{\varphi(z)}\psi(z))^2}\right| \\
\geqslant &|\varphi^{\#}(z) - \psi^{\#}(z)| - |\psi^{\#}(z)||(1-|\varphi(z)|^2)\sigma'_{\varphi(z)}(\varphi(z)) \\
&-(1-|\psi(z)|^2)\sigma'_{\varphi(z)}(\psi(z))| \\
\geqslant &|\varphi^{\#}(z) - \psi^{\#}(z)| - |\psi^{\#}(z)|b(\varphi(z),\psi(z)) \\
\geqslant &|\varphi^{\#}(z) - \psi^{\#}(z)| - 18|\psi^{\#}(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&|\varphi^{\#}(z) - \psi^{\#}(z)| \\
\leqslant &\|(C_\varphi - C_\psi)\sigma_{\varphi(z)}\|_{\mathscr{B}} + 18\|(C_\varphi - C_\psi)\sigma_{\psi(z)}\|_{\mathscr{B}} + 9\|(C_\varphi - C_\psi)(\sigma_{\psi(z)})^2\|_{\mathscr{B}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{s\to 1}\sup_{\substack{|\varphi(z)|>s\\|\psi(z)|>s}} |\varphi^{\#}(z) - \psi^{\#}(z)| \\
\leqslant &\ 19\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + 9\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}.
\end{aligned}$$

The proof is completed. □

LEMMA 4. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then*
(i)
$$\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} \leqslant 2 \limsup_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}}.$$
(ii)
$$\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} \leqslant 8 \limsup_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

Proof. Note that the Maclaurin expansion of Möbius map σ_a is given by
$$\sigma_a(z) = a - (1-|a|^2) \sum_{k=0}^{\infty} \bar{a}^k z^{k+1}.$$
For any fix positive integer $n \geqslant 2$, it follows from the triangle inequality that
$$\begin{aligned}
&\|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} \\
&\leqslant |(C_\varphi - C_\psi)\sigma_a(0)| + (1-|a|^2) \sum_{k=0}^{\infty} |a|^k \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} \\
&= (1-|a|^2) \sum_{k=0}^{n-2} |a|^k \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} + (1-|a|^2) \sum_{k=n-1}^{\infty} |a|^k \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} \\
&\leqslant 4(n-1)(1-|a|^2) + (1-|a|^2) \sum_{k=n-1}^{\infty} |a|^k \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} \\
&\leqslant 4(n-1)(1-|a|^2) + 2 \sup_{k\geqslant n} \|\varphi^k - \psi^k\|_{\mathscr{B}},
\end{aligned}$$
where we used $\|\varphi^j - \psi^j\|_{\mathscr{B}} \leqslant 2\|\varphi^j\|_\infty + 2\|\psi^j\|_\infty \leqslant 4$ for $j = 1, 2, \ldots, n-1$ in the third inequality. Letting $|a| \to 1$ in the above inequality leads to
$$\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} \leqslant 2 \sup_{k\geqslant n} \|\varphi^k - \psi^k\|_{\mathscr{B}},$$
for any positive integer $n \geqslant 2$. Thus,
$$\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} \leqslant 2 \limsup_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

Also using Maclaurin expansion of Möbius map σ_a^2, we have
$$\begin{aligned}
\sigma_a^2(z) &= \Big(a - (1-|a|^2) \sum_{k=0}^{\infty} \bar{a}^k z^{k+1}\Big)^2 \\
&= a^2 - 2a(1-|a|^2) \sum_{k=0}^{\infty} \bar{a}^k z^{k+1} + (1-|a|^2)^2 \Big(\sum_{k=0}^{\infty} \bar{a}^k z^{k+1}\Big)^2 \\
&= a^2 - 2a(1-|a|^2) \sum_{k=0}^{\infty} \bar{a}^k z^{k+1} + (1-|a|^2)^2 \sum_{k=2}^{\infty} (k-1)\bar{a}^{k-2} z^k.
\end{aligned}$$

Therefore,

$$\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} \leqslant |(C_\varphi - C_\psi)(\sigma_a)^2(0)| + 2|a|(1-|a|^2)\sum_{k=0}^{\infty}|a|^k\|\varphi^{k+1}-\psi^{k+1}\|_{\mathscr{B}}$$
$$+(1-|a|^2)^2\sum_{k=2}^{\infty}(k-1)|a|^{k-2}\|\varphi^k-\psi^k\|_{\mathscr{B}}.$$

For each $n \geqslant 2$, we have

$$\begin{aligned}
&2|a|(1-|a|^2)\sum_{k=n-1}^{\infty}|a|^k\|\varphi^{k+1}-\psi^{k+1}\|_{\mathscr{B}}\\
&\leqslant 2|a|(1-|a|^2)\sum_{k=n-1}^{\infty}|a|^k\sup_{j\geqslant n}\|\varphi^j-\psi^j\|_{\mathscr{B}}\\
&\leqslant 4\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}}.
\end{aligned}$$

By elementary calculations, the function $h(x) = nx^{n-1}(1-x)+x^n,\ \ 0 \leqslant x \leqslant 1,$ attains its maximum value 1, at the point 1. Therefore,

$$\begin{aligned}
&(1-|a|^2)^2\sum_{k=n}^{\infty}(k-1)|a|^{k-2}\|\varphi^k-\psi^k\|_{\mathscr{B}}\\
&\leqslant (1-|a|^2)^2\sum_{k=n}^{\infty}(k-1)|a|^{k-2}\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}}\\
&\leqslant (1-|a|^2)^2\frac{n|a|^{n-1}(1-|a|)+|a|^n}{(1-|a|)^2}\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}}\\
&\leqslant 4\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}\\
&\leqslant 2|a|(1-|a|^2)\sum_{k=0}^{n-2}|a|^k\|\varphi^{k+1}-\psi^{k+1}\|_{\mathscr{B}}\\
&\quad+(1-|a|^2)^2\sum_{k=2}^{n-1}(k-1)|a|^{k-2}\|\varphi^k-\psi^k\|_{\mathscr{B}}+8\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}}\\
&\leqslant 8(n-1)|a|(1-|a|^2)+2(n-1)(n-2)(1-|a|^2)^2+8\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}}.
\end{aligned}$$

Letting $|a|\to 1$ in the above inequality leads to

$$\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} \leqslant 8\sup_{k\geqslant n}\|\varphi^k-\psi^k\|_{\mathscr{B}},$$

for any positive integer $n \geqslant 2$. Therefore we get

$$\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} \leqslant 8\limsup_{n\to\infty}\|\varphi^n-\psi^n\|_{\mathscr{B}}. \quad \square$$

3. Proof of main result

Now we are in a position to give the proof for the main result in this paper.

Proof of Theorem 1. First, we prove that

$$\|C_\varphi - C_\psi\|_{e,\mathscr{B}} \geqslant \frac{e}{2}\limsup_{n\to\infty}\|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

Let n be any positive integer. Consider the function z^n. We have

$$\|z^n\|_{\mathscr{B}} = \sup_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2) = \frac{2n}{n+1}\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$$

and hence $\lim_{n\to\infty}\|z^n\|_{\mathscr{B}} = \frac{2}{e}$. Let $f_n(z) = z^n/\|z^n\|_{\mathscr{B}}$. Then $\|f_n\|_{\mathscr{B}} = 1$ and $f_n \to 0$ weakly in $\mathscr{B}$. Thus, if K is any compact operator on $\mathscr{B}$, then $\lim_{n\to\infty}\|Kf_n\|_{\mathscr{B}} = 0$. Hence,

$$\|C_\varphi - C_\psi - K\| \geqslant \limsup_{n\to\infty}\|(C_\varphi - C_\psi - K)f_n\|_{\mathscr{B}} \geqslant \limsup_{n\to\infty}\|(C_\varphi - C_\psi)f_n\|_{\mathscr{B}}.$$

Therefore

$$\begin{aligned}
\|C_\varphi - C_\psi\|_{e,\mathscr{B}} &\geqslant \limsup_{n\to\infty}\|(C_\varphi - C_\psi)f_n\|_{\mathscr{B}} \\
&= \limsup_{n\to\infty}\frac{1}{\|z^n\|_{\mathscr{B}}}\|(C_\varphi - C_\psi)z^n\|_{\mathscr{B}} \\
&= \frac{e}{2}\limsup_{n\to\infty}\|\varphi^n - \psi^n\|_{\mathscr{B}}. \qquad (1)
\end{aligned}$$

Let $\{L_n\}$ be the sequence of operator given in Lemma 2. Since each L_n^{**} is compact on $\mathscr{B}$, $C_\varphi - C_\psi$ is bounded on $\mathscr{B}$, $(C_\varphi - C_\psi)L_n^{**}$ is also compact on $\mathscr{B}$ and we have

$$\begin{aligned}
\|C_\varphi - C_\psi\|_{e,\mathscr{B}} &\leqslant \limsup_{n\to\infty}\|C_\varphi - C_\psi - (C_\varphi - C_\psi)L_n^{**}\| \\
&= \limsup_{n\to\infty}\|(C_\varphi - C_\psi)(I - L_n^{**})\| \\
&\leqslant \limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\|(C_\varphi - C_\psi)(I - L_n^{**})f\|_{\mathscr{B}},
\end{aligned}$$

which is bounded by

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}|(I - L_n^{**})(f(\varphi(0)) - f(\psi(0)))| \\
&+\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{z\in\mathbb{D}}|((I - L_n^{**})f)'(\varphi(z))\varphi'(z) - ((I - L_n^{**})f)'(\psi(z))\psi'(z)|(1-|z|^2).
\end{aligned}$$

Lemma 2 (iib) guarantees that

$$\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}|(I - L_n^{**})(f(\varphi(0)) - f(\psi(0)))| = 0.$$

Now we consider the term

$$J = \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}} |((I-L_n^{**})f)'(\varphi(z))\varphi'(z) - ((I-L_n^{**})f)'(\psi(z))\psi'(z)|(1-|z|^2).$$

Let $f \in \mathscr{B}$ with $\|f\|_{\mathscr{B}} \leqslant 1$ and fix an arbitrary $r \in (0,1)$. For the sake of simplicity, we define

$$H_n^f(z) := |((I-L_n^{**})f)'(\varphi(z))\varphi'(z) - ((I-L_n^{**})f)'(\psi(z))\psi'(z)|(1-|z|^2)$$

and set

$$\mathbb{D}_1 := \{z\in\mathbb{D} : |\varphi(z)| \leqslant r, |\psi(z)| \leqslant r\}, \quad \mathbb{D}_2 := \{z\in\mathbb{D} : |\varphi(z)| \leqslant r, |\psi(z)| > r\},$$

$$\mathbb{D}_3 := \{z\in\mathbb{D} : |\varphi(z)| > r, |\psi(z)| \leqslant r\}, \quad \mathbb{D}_4 := \{z\in\mathbb{D} : |\varphi(z)| > r, |\psi(z)| > r\}.$$

Then

$$J = \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}} H_n^f = \max_{1\leqslant i\leqslant 4} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_i} H_n^f = \max\{J_1, J_2, J_3, J_4\}.$$

By (i) of Lemma 2,

$$\begin{aligned}
\limsup_{n\to\infty} J_1 &= \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_1} H_n^f \\
&\leqslant \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{|\varphi(z)|\leqslant r} |((I-L_n^{**})f)'(\varphi(z))||\varphi'(z)|(1-|z|^2) \\
&\quad + \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{|\psi(z)|\leqslant r} |((I-L_n^{**})f)'(\psi(z))||\psi'(z)|(1-|z|^2) \\
&= 0.
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
&H_n^f(z) = |((I-L_n^{**})f)'(\varphi(z))\varphi'(z) - ((I-L_n^{**})f)'(\psi(z))\psi'(z)|(1-|z|^2) \\
&= |((I-L_n^{**})f)'(\psi(z))|(1-|\psi(z)|^2)|\varphi^{\#}(z) - \psi^{\#}(z)| \\
&\quad + |((I-L_n^{**})f)'(\varphi(z))(1-|\varphi(z)|^2) - ((I-L_n^{**})f)'(\psi(z))(1-|\psi(z)|^2)||\varphi^{\#}(z)| \\
&\leqslant |((I-L_n^{**})f)'(\psi(z))|(1-|\psi(z)|^2)|\varphi^{\#}(z) - \psi^{\#}(z)| + b(\varphi(z),\psi(z))|\varphi^{\#}(z)| \\
&\leqslant |((I-L_n^{**})f)'(\psi(z))|(1-|\psi(z)|^2)|\varphi^{\#}(z) - \psi^{\#}(z)| + 18\rho(\varphi(z),\psi(z))|\varphi^{\#}(z)|.
\end{aligned}$$

Similarly,

$$H_n^f(z) \leqslant |((I-L_n^{**})f)'(\varphi(z))|(1-|\varphi(z)|^2)|\varphi^{\#}(z) - \psi^{\#}(z)| + 18\rho(\varphi(z),\psi(z))|\psi^{\#}(z)|.$$

Then, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} J_2\\
&\leqslant \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_2} \Big(|((I-L_n^{**})f)'(\varphi(z))|(1-|\varphi(z)|^2)|\varphi^{\#}(z)-\psi^{\#}(z)|\\
&\quad +18\rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\Big)\\
&\leqslant \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{|\varphi(z)|\leqslant r} |((I-L_n^{**})f)'(\varphi(z))|(1-|\varphi(z)|^2)|\varphi^{\#}(z)-\psi^{\#}(z)|\\
&\quad +18 \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\\
&= 18 \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|,
\end{aligned}$$

where we used (i) of Lemma 2 again in the last inequality. Since r is arbitrary, we have

$$\limsup_{n\to\infty} J_2 \leqslant 18 \lim_{r\to 1} \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|.$$

Similarly, we can prove that

$$\limsup_{n\to\infty} J_3 \leqslant 18 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)|.$$

Also,

$$\begin{aligned}
&\limsup_{n\to\infty} J_4\\
&\leqslant \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_4} \Big(|((I-L_n^{**})f)'(\varphi(z))|(1-|\varphi(z)|^2)|\varphi^{\#}(z)-\psi^{\#}(z)|\\
&\quad +18\rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\Big)\\
&\leqslant \limsup_{n\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} \|(I-L_n^{**})f\|_{\mathscr{B}}|\varphi^{\#}(z)-\psi^{\#}(z)|\\
&\quad +18 \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\\
&\leqslant \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\varphi^{\#}(z)-\psi^{\#}(z)| + 18 \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|,
\end{aligned}$$

where we used the fact

$$\limsup_{n\to\infty} \|(I-L_n^{**})f\|_{\mathscr{B}} \leqslant \limsup_{n\to\infty} \|I-L_n^{**}\|\|f\|_{\mathscr{B}} \leqslant 1$$

in the last inequality. Thus,

$$\limsup_{n\to\infty} J_4 \leqslant \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\varphi^{\#}(z)-\psi^{\#}(z)| + 18 \lim_{r\to 1} \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|.$$

Then we have

$$\begin{aligned}
&\limsup_{n\to\infty} J = \limsup_{n\to\infty} \max\{J_1, J_2, J_3, J_4\}\\
\leqslant\ & 18 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)| + 18 \lim_{r\to 1} \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\\
&+ \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\varphi^{\#}(z) - \psi^{\#}(z)|.
\end{aligned}$$

From Lemmas 3 and 4, we have

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,\mathscr{B}}\\
\leqslant\ & 18 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)| + 18 \lim_{r\to 1} \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\\
&+ \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\varphi^{\#}(z) - \psi^{\#}(z)|\\
\leqslant\ & 55 \limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + 27 \limsup_{|a|\to 1} \|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}\\
\leqslant\ & 326 \limsup_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}}.
\end{aligned} \tag{2}$$

Combining (1) with (2), we immediately obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}}\\
\lesssim\ & \|C_\varphi - C_\psi\|_{e,\mathscr{B}}\\
\lesssim\ & \lim_{r\to 1} \sup_{|\varphi(z)|>r} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)| + \lim_{r\to 1} \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|\\
&+ \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\varphi^{\#}(z) - \psi^{\#}(z)|\\
\lesssim\ & \limsup_{|a|\to 1} \|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} + \limsup_{|a|\to 1} \|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}}\\
\lesssim\ & \limsup_{n\to\infty} \|\varphi^n - \psi^n\|_{\mathscr{B}}.
\end{aligned}$$

The proof is completed. □

REMARK 1. In [5], the authors showed that

$$\begin{aligned}
\|C_\varphi - C_\psi\|_{e,\mathscr{B}} \approx \max\Big\{ &\lim_{r\to 1} \sup_{|\varphi(z)|>r} \rho(\varphi(z),\psi(z))|\varphi^{\#}(z)|,\\
&\lim_{r\to 1} \sup_{|\psi(z)|>r} \rho(\varphi(z),\psi(z))|\psi^{\#}(z)|, \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\varphi^{\#}(z) - \psi^{\#}(z)| \Big\}.
\end{aligned}$$

In [13], Yang and Zhou obtained another estimate for the essential norm of the differences of composition operators on the Bloch space.

From Theorem 1, we immediately get the following corollary.

COROLLARY 1. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then the following statements are equivalent:*

(i) $C_\varphi - C_\psi$ *is compact on* $\mathscr{B}$.

(ii)

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s}\rho(\varphi(z),\psi(z))|\varphi^{\#}(z)| = \lim_{s\to 1}\sup_{|\psi(z)|>s}\rho(\varphi(z),\psi(z))|\psi^{\#}(z)| = \lim_{s\to 1}\sup_{\substack{|\varphi(z)|>s\\ |\psi(z)|>s}}|\varphi^{\#}(z)-\psi^{\#}(z)| = 0.$$

(iii) $\lim_{|a|\to 1}\|(C_\varphi - C_\psi)\sigma_a\|_{\mathscr{B}} = \lim_{|a|\to 1}\|(C_\varphi - C_\psi)(\sigma_a)^2\|_{\mathscr{B}} = 0.$

(iv) $\lim_{n\to\infty}\|\varphi^n - \psi^n\|_{\mathscr{B}} = 0.$

REMARK 2. By results of [6, 10, 12], Corollary 1 was obviously true in case of either C_φ or C_ψ is compact on $\mathscr{B}$. The equivalence of (i), (ii) and (iii) was given in [3] for the case where neither C_φ nor C_ψ is compact on $\mathscr{B}$. The condition (iv) is a new and simple compactness criteria.

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