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CONCAVE FUNCTIONS OF PARTITIONED MATRICES WITH NUMERICAL RANGES IN A SECTOR

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Abstract. We prove two inequalities for concave functions and partitioned matrices whose numerical ranges in a sector. These complement some results of Zhang in [Linear Multilinear Algebra 63 (2015) 2511–2517].

1. Introduction

Let $\mathbb{M}_n$, $\mathbb{M}_n^+$ denote the set of $n\times n$ complex matrices and the set of $n\times n$ positive semi-definite matrices, respectively. For $A\in\mathbb{M}_n$, we denote by $|A|=(AA^*)^{\frac{1}{2}}$, $\|A\|$, $\|A\|_k$, A^*, $s_j(A)$ and $\lambda_j(A)$ the modulus, the unitarily invariant norm, the Ky Fan k-norms, the conjugate transpose, the singular values and eigenvalues of A, respectively, $j=1,\dots,n$. The singular values are always arranged in nonincreasing order: $s_1(A)\geqslant s_2(A)\cdots\geqslant s_n(A)$. If A is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$. Note that $s_j(A)=\lambda_j(|A|)$, $j=1,\dots,n$. For two Hermitian matrices A, $B\in\mathbb{M}_n$, we write $A\leqslant B$ to mean $B-A\in\mathbb{M}_n^+$. For $A\in\mathbb{M}_n$, recall the Cartesian decomposition $A=\Re A+i\Im A$, where,

$$\Re A=\frac{1}{2}(A+A^*),\ \ \Im A=\frac{1}{2i}(A-A^*).$$

There are many interesting properties for such a decomposition. A celebrated result due to Fan and Hoffman (see, e.g. [2, p. 73]) states that,

$$\lambda_j(\Re A)\leqslant s_j(A),\ \ j=1,2,\cdots,n.$$

This says that,

$$\Re A\leqslant U|A|U^* \tag{1}$$

for some unitary matrix $U\in\mathbb{M}_n$.

We say that $A\in\mathbb{M}_n$ is an accretive-dissipative matrix if $\Re A\in\mathbb{M}_n^+$ and $\Im A\in\mathbb{M}_n^+$. This class of matrices has been recently considered in George [8], Ikramov [9, 10], Lin [13, 14], Lin and Zhou [16].

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The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} | \Re z \geqslant 0, |\Im z| \leqslant (\Re z)\tan(\alpha)\} = \{re^{i\theta} | r \geqslant 0, |\theta| \leqslant \alpha\}$$

and let

$$S'_\alpha = \{z \in \mathbb{C} | \Re z \geqslant 0, \Im z \geqslant 0, \Im z \leqslant (\Re z)\tan(\alpha)\} = \{re^{i\theta} | r \geqslant 0, 0 \leqslant \theta \leqslant \alpha\}$$

Relevant studies on matrices with numerical ranges in a sector can be found in Drury and Lin [6], Fu [7], Li [12], Lin [15], Zhang [18] and Zhang [20].

Let H be a Hermitian matrix and let f be a real-valued function defined on an interval containing all the eigenvalues of H. Then, $f(H)$ is well defined through spectral decomposition. Consider a partitioned matrix $A \in \mathbb{M}_n$ in the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are square.} \tag{2}$$

Lee [11, Theorem 2.1] proved the following result which is considered as an extension of the classical Rotfel'd theorem.

THEOREM 1. [11, Theorem 2.1] *Let $A \in \mathbb{M}_n^+$ be partitioned as in (2), and let $f : [0,\infty) \longmapsto [0,\infty)$ be a concave function. Then,*

$$\|f(A)\| \leqslant \|f(A_{11})\| + \|f(A_{22})\|.$$

Here, and in the sequel, the symbol $\|\cdot\|$ stands for an arbitrary unitarily invariant norm on $\mathbb{M}_n$. Recall that this also induces a norm on $\mathbb{M}_k$, $k \leqslant n$.

As a further extension of the classic Rotfel'd theorem, Zhang extended Theorem 1 to matrices with $W(A) \subseteq S_\alpha$, for $\alpha \in [0, \frac{\pi}{2})$ as follows:

THEOREM 2. [19, Theorem 3.4] *Let $f : [0,\infty) \longmapsto [0,\infty)$ be a concave function and let A with $W(A) \subseteq S_\alpha$ for $\alpha \in [0, \pi/2)$ be partitioned as in (2). Then,*

$$\|f(|A|)\| \leqslant \|f(|A_{11}|)\| + \|f(|A_{22}|)\| + 2(\|f(\tan(\alpha)|A_{11}|)\| + \|f(\tan(\alpha)|A_{22}|)\|).$$

And Zhang left an open problem whether the constant 2 in Theorem 2 can be replaced by 1.

In this paper, we partially answer the open problem of Zhang and we improve the consequence of [19, Theorem 3.1] to some extent, when $W(A) \subseteq S'_\alpha$. Our approach is quite parallel to P. Zhang's one; however, we also use a simple eigenvalue inequality (Lemma 1 below) which sharpens an estimate of Bhatia and Kittaneh.

2. Some lemmas

The main new observation allowing us to improve P. Zhang's estimates consists in the following lemma.

LEMMA 1. *Let* $A,B \in \mathbb{M}_n^+$ *and* $W(A+iB) \subseteq S'_\alpha$. *Then*

$$s_j(A+B) \leqslant a s_j(A+iB), \quad j=1,2,\dots,n,$$

where $a = \min\{1+\tan(\alpha), \sqrt{2}\}$.

Proof. Let e_j be eigenvectors of $A+B$ belonging to its eigenvalues $\lambda_j(A+B)$. For $W(A+iB) \subseteq S'_\alpha$, we get

$$B \leqslant A\tan(\alpha), \;\text{ where }\; A,\, B \in \mathbb{M}_n^+. \tag{3}$$

So

$$\begin{aligned} \lambda_j(A+B) &= \langle e_j, (A+B)e_j\rangle \\ &= \langle e_j, Ae_j\rangle + \langle e_j, Be_j\rangle \\ &\leqslant \langle e_j, Ae_j\rangle + \langle e_j, A\tan(\alpha)e_j\rangle \text{ (by (3))} \\ &= (1+\tan(\alpha))\langle e_j, Ae_j\rangle \\ &\leqslant (1+\tan(\alpha))|\langle e_j, Ae_j\rangle + i\langle e_j, Be_j\rangle| \\ &= (1+\tan(\alpha))|\langle e_j, (A+iB)e_j\rangle|. \end{aligned}$$

Since $s_j(A) = \max\limits_{dim(\mathbb{M})=j} \min\limits_{\substack{x\in\mathbb{M}\\ \|x\|=1}} \|Ax\|$ (see, e.g. [2, p. 75]), where $\mathbb{M}$ represent a subspace of $\mathbb{C}^n$ for $A \in \mathbb{M}_n$ and since, by a result of Bhatia and Kittaneh,

$$s_j(A+B) \leqslant \sqrt{2} s_j(A+iB), \quad j=1,2,\dots,n,$$

in [3, Theorem 1.1, (1.8)]. □

Lemma 1 means that

$$U(A+B)U^* \leqslant a|A+iB|, \tag{4}$$

where $a = \min\{1+\tan(\alpha), \sqrt{2}\}$ for some unitary matrix $U \in \mathbb{M}_n$.

We will also use, likewise in P. Zhang's paper, the following series of three well-known results, due to Bourin-Lee, Aujla-Bourin, and Thompson.

LEMMA 2. [5, Lemma 3.4] *Let* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{m+n}^+$, *where* $A \in \mathbb{M}_m^+$ *and* $B \in \mathbb{M}_n^+$. *Then there exist unitary matrices* U, $V \in \mathbb{M}_{m+n}$ *such that*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}U^* + V\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}V^*.$$

LEMMA 3. [1, Theorem 2.1] *Let* $f : [0,\infty) \longmapsto [0,\infty)$ *be a concave function and let* $R,\ S \in \mathbb{M}_n^+$. *Then there exist unitary matrices* U *and* V *such that*

$$f(R+S) \leqslant Uf(R)U^* + Vf(S)V^*.$$

LEMMA 4. [17, Theorem 2] *Let* $A,\ B \in \mathbb{M}_n$. *Then there exist unitary matrices* $U,\ V$ *such that*

$$|A+B| \leqslant U|A|U^* + V|B|V^*.$$

3. Main results

In this section, we present our main results. Firstly, according to [19, Theorem 3.1], we get a new upper bound under S'_α.

THEOREM 3. *Let* $f : [0,\infty) \mapsto [0,\infty)$ *be concave and let* $A \in \mathbb{M}_n$, $W(A) \subseteq S'_\alpha$, *partitioned as* (2). *Then,*

$$\|f(|A|)\| \leqslant 2\left(\left\|f\left(\frac{a}{2}|A_{11}|\right)\right\| + \left\|f\left(\frac{a}{2}|A_{22}|\right)\right\|\right), \tag{5}$$

where $a = \min\{1+\tan(\alpha), \sqrt{2}\}$.

Proof. Arguing as in Lee's paper, we may assume that $f(0) = 0$. Consider the Cartesian decomposition $A = R+iS$, where $R,\ S$ are positive semidefinite. First, by Bourin and Ricard [4, (2.8)], we have, for some unitary matrix U_1,

$$|\, R+iS \,| \leqslant \frac{1}{2}\{(R+S) + U_1(R+S)U_1^*\}.$$

From this, we have, for some unitary matrices U_j, V_j, $j = 2,3,4$,

$$\begin{aligned}
|\, A \,| &\leqslant \frac{1}{2}\{(R+S) + U_1(R+S)U_1^*\} \\
&= \frac{1}{2}\left\{U_2\begin{bmatrix} R_{11}+S_{11} & 0 \\ 0 & 0\end{bmatrix}U_2^* + V_2\begin{bmatrix} 0 & 0 \\ 0 & R_{22}+S_{22}\end{bmatrix}V_2^*\right\} \quad \text{(by Lemma 2)} \\
&\quad + \frac{1}{2}U_1\left\{U_2\begin{bmatrix} R_{11}+S_{11} & 0 \\ 0 & 0\end{bmatrix}U_2^* + V_2\begin{bmatrix} 0 & 0 \\ 0 & R_{22}+S_{22}\end{bmatrix}V_2^*\right\}U_1^* \\
&\leqslant \frac{a}{2}\left\{U_3\begin{bmatrix} |R_{11}+iS_{11}| & 0 \\ 0 & 0\end{bmatrix}U_3^* + V_3\begin{bmatrix} 0 & 0 \\ 0 & |R_{22}+iS_{22}|\end{bmatrix}V_3^*\right\} \quad \text{(by (4))} \\
&\quad + \frac{a}{2}U_1\left\{U_3\begin{bmatrix} |R_{11}+iS_{11}| & 0 \\ 0 & 0\end{bmatrix}U_3^* + V_3\begin{bmatrix} 0 & 0 \\ 0 & |R_{22}+iS_{22}|\end{bmatrix}V_3^*\right\}U_1^* \\
&= \frac{a}{2}\left\{U_3\begin{bmatrix} |A_{11}| & 0 \\ 0 & 0\end{bmatrix}U_3^* + V_3\begin{bmatrix} 0 & 0 \\ 0 & |A_{22}|\end{bmatrix}V_3^*\right\} \\
&\quad + \frac{a}{2}\left\{U_4\begin{bmatrix} |A_{11}| & 0 \\ 0 & 0\end{bmatrix}U_4^* + V_4\begin{bmatrix} 0 & 0 \\ 0 & |A_{22}|\end{bmatrix}V_4^*\right\}.
\end{aligned}$$

Since f is nondecreasing, for some unitary matrices U_j, V_j, $j=5,6$,

$$\begin{aligned}
\|f(|A|)\| &\leqslant \left\|f\left(\left\{U_3\begin{bmatrix}\frac{a}{2}|A_{11}| & 0\\ 0 & 0\end{bmatrix}U_3^*+V_3\begin{bmatrix}0 & 0\\ 0 & \frac{a}{2}|A_{22}|\end{bmatrix}V_3^*\right\}\right.\right.\\
&\quad \left.\left.+\left\{U_4\begin{bmatrix}\frac{a}{2}|A_{11}| & 0\\ 0 & 0\end{bmatrix}U_4^*+V_4\begin{bmatrix}0 & 0\\ 0 & \frac{a}{2}|A_{22}|\end{bmatrix}V_4^*\right\}\right)\right\|\\
&\leqslant \left\|U_5 f\left(\begin{bmatrix}\frac{a}{2}|A_{11}| & 0\\ 0 & 0\end{bmatrix}\right)U_5^*+V_5 f\left(\begin{bmatrix}0 & 0\\ 0 & \frac{a}{2}|A_{22}|\end{bmatrix}\right)V_5^*\right. \quad \text{(by Lemma 3)}\\
&\quad \left.+U_6 f\left(\begin{bmatrix}\frac{a}{2}|A_{11}| & 0\\ 0 & 0\end{bmatrix}\right)U_6^*+V_6 f\left(\begin{bmatrix}0 & 0\\ 0 & \frac{a}{2}|A_{22}|\end{bmatrix}\right)V_6^*\right\|\\
&\leqslant 2\left\|f\left(\frac{a}{2}|A_{11}|\right)\right\|+2\left\|f\left(\frac{a}{2}|A_{22}|\right)\right\|,
\end{aligned}$$

which leads to the desired result. □

REMARK 1. In the sector S'_α, we get a apparently better result than in [19, Theorem 3.1], when $1+\tan(\alpha)<\sqrt{2}$.

For the open problem proposed in section 4 of [19], we give a partial answer as follows:

THEOREM 4. *Let* $f:[0,\infty)\longmapsto[0,\infty)$ *be a concave function and let* A *with* $W(A)\subseteq S'_\alpha$ *for* $\alpha\in[0,\pi/2)$ *be partitioned as in* (2). *Then,*

$$\|f(|A|)\|\leqslant\|f(|A_{11}|)\|+\|f(|A_{22}|)\|+\|f(\tan(\alpha)|A_{11}|)\|+\|f(\tan(\alpha)|A_{22}|)\|.$$

Proof. Here, we also suppose that $f(0)=0$. Consider the Cartesian decomposition $A=R+iS$, where R and S are positive semi-definite. The condition $W(A)\subseteq S'_\alpha$ tells that

$$S\leqslant\tan(\alpha)R. \tag{6}$$

By Lemma 4, we obtain, for unitary matrices U_j, V_j, $j=1,2,3$,

$$\begin{aligned}
|A| &= |R+iS|\leqslant U_1RU_1^*+V_1SV_1^*\\
&\leqslant U_1RU_1^*+V_1(\tan(\alpha)R)V_1^* \quad \text{(by (6))}\\
&= U_1U_2\begin{bmatrix}R_{11} & 0\\ 0 & 0\end{bmatrix}U_2^*U_1^*+U_1V_2\begin{bmatrix}0 & 0\\ 0 & R_{22}\end{bmatrix}V_2^*U_1^* \quad \text{(by Lemma 2)}\\
&\quad +V_1U_2\begin{bmatrix}\tan(\alpha)R_{11} & 0\\ 0 & 0\end{bmatrix}U_2^*V_1^*+V_1V_2\begin{bmatrix}0 & 0\\ 0 & \tan(\alpha)R_{22}\end{bmatrix}V_2^*V_1^*\\
&\leqslant U_1U_2U_3\begin{bmatrix}|R_{11}+iS_{11}| & 0\\ 0 & 0\end{bmatrix}U_3^*U_2^*U_1^*+U_1V_2V_3\begin{bmatrix}0 & 0\\ 0 & |R_{22}+iS_{22}|\end{bmatrix}V_3^*V_2^*U_1^*\\
&\quad +V_1U_2U_3\begin{bmatrix}\tan(\alpha)|R_{11}+iS_{11}| & 0\\ 0 & 0\end{bmatrix}U_3^*U_2^*V_1^* \quad \text{(by (1))}\\
&\quad +V_1V_2V_3\begin{bmatrix}0 & 0\\ 0 & \tan(\alpha)|R_{22}+iS_{22}|\end{bmatrix}V_3^*V_2^*V_1^*.
\end{aligned}$$

It follows from Lemma 3 and the above inequality that

$$f(|A|) \leqslant U_4U_1U_2U_3 f\Big(\begin{bmatrix} |R_{11}+iS_{11}| & 0 \\ 0 & 0 \end{bmatrix}\Big) U_3^*U_2^*U_1^*U_4^*$$
$$+V_4U_1V_2V_3 f\Big(\begin{bmatrix} 0 & 0 \\ 0 & |R_{22}+iS_{22}| \end{bmatrix}\Big) V_3^*V_2^*U_1^*V_4^*$$
$$+U_5V_1U_2U_3 f\Big(\begin{bmatrix} \tan(\alpha)|R_{11}+iS_{11}| & 0 \\ 0 & 0 \end{bmatrix}\Big) U_3^*U_2^*V_1^*U_5^*$$
$$+V_5V_1V_2V_3 f\Big(\begin{bmatrix} 0 & 0 \\ 0 & \tan(\alpha)|R_{22}+iS_{22}| \end{bmatrix}\Big) V_3^*V_2^*V_1^*V_5^*,$$

where U_j, V_j, $j = 4,5$, are unitary matrices. Taking the unitarily invariant norm on both sides of the above inequality can easily lead to the desired result. □

If we take $f(t) = t^p$, $0 < p \leqslant 1$, in Theorems 3 and 4, respectively, the following two corollaries can easily be derived.

COROLLARY 1. *Let* $0 < p \leqslant 1$, *and let* $W(A) \subseteq S'_\alpha$ *partitioned as in* (2). *Then,*

$$\||A|^p\| \leqslant \frac{a^p}{2^{p-1}}(\||A_{11}|^p\| + \||A_{22}|^p\|) \tag{7}$$

and

$$\||A|^p\| \leqslant (1+(\tan(\alpha))^p)(\||A_{11}|^p\| + \||A_{22}|^p\|), \tag{8}$$

where $a = \min\{1+\tan(\alpha), \sqrt{2}\}$.

REMARK 2. The matrix A is accretive-dissipative if and only if $W(e^{-\frac{\pi}{4}i}A) \subseteq S_{\frac{\pi}{4}}$. If we only take the upper sector into account, i. e. $S'_{\frac{\pi}{4}}$, from Theorem 4 we can derive

$$\|f(|A|)\| = 2(\|f(|A_{11}|)\| + \|f(|A_{22}|)\|). \tag{9}$$

The bound in (9) is equal to that in *Theorem* 3.

REMARK 3. When $\alpha = 0$, Theorem 4 reduces to Theorem 1.

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