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SOME NEW ESTIMATIONS FOR THE HADAMARD PRODUCT OF A NONSINGULAR M–MATRIX AND ITS INVERSE

ZHENGGE HUANG, LIGONG WANG AND ZHONG XU

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Abstract. In this paper, some new bounds for the minimum eigenvalue of the Hadamard product of a nonsingular M-matrix and its inverse are obtained, which improve some existing results. Finally, numerical examples are given to show that these bounds are better than some existing ones.

1. Introduction

For a positive integer n, N denotes the set $N = \{1,2,\cdots,n\}$. The set of all $n\times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$, and $\mathbb{C}^{n\times n}$ denotes the set of all $n\times n$ complex matrices. For a matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$, we write $A\geqslant 0(>0)$ if $a_{ij}\geqslant 0(>0)$ for any $i,j\in N$. If $A\geqslant 0(>0)$, A is called a nonnegative (positive) matrix.

Let Z_n denote the set of $n\times n$ real matrices all of whose off-diagonal entries are nonpositive. A matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is called an M-matrix [1] if there exist a nonnegative matrix B and a nonnegative real number λ such that

$$A=\lambda I-B,\ \lambda\geqslant\rho(B),$$

where I is the identity matrix and $\rho(B)$ is the spectral radius of the matrix B. If $\lambda=\rho(B)$, then A is a singular M-matrix; if $\lambda>\rho(B)$, then A is called a nonsingular M-matrix. Denote by M_n the set of all $n\times n$ nonsingular M-matrices. Let us denote

$$\tau(A)=\min\{\mathrm{Re}(\lambda):\lambda\in\sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A. It is known that [2] $\tau(A)=\frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A\in M_n$. If $A\in M_n$, we write $C_A=D_A-A$, where $D_A=\mathrm{diag}(a_{11},a_{22},\cdots,a_{nn})$. Note that $a_{ii}>0$ for all $i\in N$ if $A\in M_n$. Thus we define the Jacobi iterative matrix of A by $J_A=D_A^{-1}C_A$. It is easy to check that J_A is nonnegative and $\rho(J_A)<1$ [12].

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Recall that $A=(a_{ij})\in\mathbb{C}^{n\times n}$ is called row diagonally dominant if $|a_{ii}|\geqslant\sum\limits_{k\neq i}|a_{ik}|$ for all $i\in N$. If $|a_{ii}|>\sum\limits_{k\neq i}|a_{ik}|$, we say that A is strictly row diagonally dominant. For a matrix $A\in\mathbb{R}^{n\times n}$, if $A\geqslant 0$ and there exists $e=(1,\cdots,1)^T\in\mathbb{R}^n$ such that $Ae=e$ and $A^Te=e$, we call A is a doubly stochastic matrix.

The Hadamard product of two matrices $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ is the matrix $A\circ B=(a_{ij}b_{ij})_{n\times n}$. In [3] it is shown that if A and B are nonsingular M-matrices, then $A\circ B^{-1}$ is also a nonsingular M-matrix.

A matrix A is irreducible if there does not exist a permutation matrix P such that

$$PAP^T=\begin{pmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix},$$

where A_{11} and A_{22} are square matrices.

For $\alpha\subseteq N$, denote by $|\alpha|$ the cardinality of α and $\alpha'=N-\alpha$ the complement of α in N. If $\alpha,\beta\subseteq N$, we write $A(\alpha,\beta)$ to mean the submatrix of A lying in the rows indicated by α and the columns indicated by β. In particular, $A(\alpha,\alpha)$ is abbreviated to $A(\alpha)$. Assume that $A(\alpha)$ is nonsingular. Then

$$A/\alpha=A/A(\alpha)=A(\alpha')-A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha')$$

is called the Schur complement of A respect to $A(\alpha)$ [4].

Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant matrix. For any $i\in N$, let us denote

$$R_i=\sum_{k\neq i}|a_{ik}|,\quad C_i=\sum_{k\neq i}|a_{ki}|,\quad d_i=\frac{R_i}{|a_{ii}|},\quad c_i=\frac{C_i}{|a_{ii}|},\ i\in N;$$

$$s_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|d_k}{|a_{jj}|},\ j\neq i,\ j\in N;\quad s_i=\max_{j\neq i}\{s_{ij}\},\ i\in N;$$

$$m_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|s_{ki}}{|a_{jj}|},\ j\neq i,\ j\in N;\quad m_i=\max_{j\neq i}\{m_{ij}\},\ i\in N;$$

$$p_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|m_{ki}}{|a_{jj}|},\ j\neq i,\ j\in N;\quad p_i=\max_{j\neq i}\{p_{ij}\},\ i\in N;$$

$$h_{ji}=\frac{|a_{ji}|}{|a_{jj}|p_{ji}-\sum\limits_{k\neq j,i}|a_{jk}|p_{ki}},\ j\neq i,\ j\in N;\quad h_i=\max_{j\neq i}\{h_{ji}\},\ i\in N;$$

$$g_{ji}=\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|p_{ki}h_i}{|a_{jj}|},\ j\neq i,\ j\in N;\quad g_i=\max_{j\neq i}\{g_{ij}\},\ i\in N;$$

$$r_{ji}=\frac{|a_{ji}|}{|a_{jj}|-\sum\limits_{k\neq j,i}|a_{jk}|},\ j\neq i,\ j\in N;\quad r_i=\max_{j\neq i}\{r_{ji}\},\ i\in N;$$

$$t_{ji} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \ j \neq i, \ j \in N; \quad t_i = \max_{j\neq i}\{t_{ij}\}, \ i \in N;$$

$$u_{ji} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| t_{ki}}{|a_{jj}|}, \ j \neq i, \ j \in N; \quad u_i = \max_{j\neq i}\{u_{ij}\}, \ i \in N;$$

$$v_{ji} = \frac{|a_{ji}|}{|a_{jj}| t_{ji} - \sum\limits_{k\neq j,i} |a_{jk}| t_{ki}}, \ j \neq i, \ j \in N; \quad v_i = \max_{j\neq i}\{v_{ji}\}, \ i \in N;$$

$$w_{ji} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| t_{ki} v_i}{|a_{jj}|}, \ j \neq i, \ j \in N; \quad w_i = \max_{j\neq i}\{w_{ij}\}, \ i \in N.$$

Let $A \in M_n$ and $A^{-1} = (b_{ij})_{n\times n}$. It was proved in [5] that

$$0 < \tau(A \circ A^{-1}) \leqslant 1.$$

Fiedler and Markham [3] gave a lower bound for $\tau(A \circ A^{-1})$ and showed that $\tau(A \circ A^{-1}) \geqslant \frac{1}{n}$. They also conjectured that $\tau(A \circ A^{-1}) \geqslant \frac{2}{n}$. Chen [6], Song [7] and Yong [8] have independently proved this conjecture.

In [9], Li *et al.* obtained the following result:

$$\tau(A \circ A^{-1}) \geqslant \min_{i\in N} \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum\limits_{j\neq i} s_{ji}} \right\}. \tag{1}$$

In [10], Li *et al.* derived the following result:

$$\tau(A \circ A^{-1}) \geqslant \min_{i\in N} \left\{ \frac{a_{ii} - t_i R_i}{1 + \sum\limits_{j\neq i} t_{ji}} \right\}. \tag{2}$$

In [11], Cheng *et al.* improved the results in [9] and [10], showing that

$$\tau(A \circ A^{-1}) \geqslant \min_{i\in N} \left\{ \frac{a_{ii} - u_i R_i}{1 + \sum\limits_{j\neq i} u_{ji}} \right\}. \tag{3}$$

In [12], Zhou *et al.* presented

$$\tau(A \circ A^{-1}) \geqslant 1 - \rho^2(J_A). \tag{4}$$

In [13], Li *et al.* arrived at

$$\tau(A \circ A^{-1}) \geqslant \min_{i\in N} \left\{ \frac{a_{ii} - w_i \sum\limits_{j\neq i} |a_{ji}|}{1 + \sum\limits_{j\neq i} w_{ji}} \right\}. \tag{5}$$

Recently, Chen in [14] improved the results in [9] and gave the following result:

$$\tau(A\circ A^{-1})\geqslant \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 +4\Big(m_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(m_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\}. \tag{6}$$

In this paper, we exhibit some new lower bounds for $\tau(A\circ A^{-1})$. These bounds improve the ones in [9, 10, 11, 12, 13, 14].

The rest of this paper is organized as follows. In Section 2, we recollect and derive some notations and lemmas which are utilized in the next sections. We focus on the bounds of $\tau(A\circ A^{-1})$ and establish some new lower bounds for $\tau(A\circ A^{-1})$ in Section 3. Section 4 is devoted to some numerical experiments to show the advantages of the new lower bounds for $\tau(A\circ A^{-1})$. Finally, the paper is ended with some conclusions in Section 5.

2. Some preliminaries and notations

In this section, we start with some notations and lemmas for the entries of A^{-1} and the strictly diagonally dominant matrix. They will be useful in the proofs.

LEMMA 1. [8] *Let $A\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant matrix, that is, $|a_{ii}|>\sum_{j\neq i}|a_{ij}|$, $\forall i\in N$. Then $A^{-1}=(b_{ij})_{n\times n}$ exists, and*

$$|b_{ji}|\leqslant d_j|b_{ii}|,\ j\neq i,\forall j\in N.$$

LEMMA 2. [15] *Let $A=(a_{ij})_{n\times n}$ be a nonsingular M-matrix, $C=(c_{ij})_{n\times n}\in Z_n$ and $A\leqslant C$, i.e., $a_{ij}\leqslant c_{ij}$ for all $i,j\in N$. Then C is a nonsingular M-matrix and $A^{-1}\geqslant C^{-1}\geqslant 0$.*

LEMMA 3. [15] *Let $A=(a_{ij})_{n\times n}$, $\emptyset\neq\alpha\subseteq N$, and assume that $A(\alpha)$ is nonsingular. Then*

$$\det A=\det A(\alpha)\det A/\alpha.$$

LEMMA 4. *Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1}=(b_{ij})_{n\times n}$, we have*

$$b_{ji}\leqslant\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|p_{ki}h_i}{a_{jj}}b_{ii}\leqslant g_jb_{ii},\ j\neq i,\forall j\in N.$$

Proof. For $i \in N$ and $j \neq i$, let $d_j(\varepsilon) = \frac{\sum_{k \neq j} |a_{jk}| + \varepsilon}{a_{jj}}$,

$$s_{ji}(\varepsilon) = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k(\varepsilon) + \varepsilon}{a_{jj}}, \quad m_{ji}(\varepsilon) = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| s_{ki}(\varepsilon) + \varepsilon}{a_{jj}}$$

and

$$p_{ji}(\varepsilon) = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| m_{ki}(\varepsilon) + \varepsilon m_{ji}(\varepsilon)}{a_{jj}}, \quad h_{ji}(\varepsilon) = \frac{|a_{ji}| + \varepsilon m_{ji}(\varepsilon)}{|a_{jj}| p_{ji}(\varepsilon) - \sum_{k \neq j,i} |a_{jk}| p_{ki}(\varepsilon)}.$$

Since A is a strictly row diagonally dominant matrix, it holds that $0 \leqslant d_j < 1$, $0 \leqslant s_{ji} < 1$ and $0 \leqslant m_{ji} < 1$ $(i, j \in N, i \neq j)$. By making use of Theorem 3.3 in [14], we have $d_j \geqslant s_{ji}$, $j \neq i$. Thus, we have the results

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{|a_{jj}|} \geqslant \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| s_{ki}}{|a_{jj}|} = m_{ji}, \; j \neq i$$

and

$$m_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| s_{ki}}{|a_{jj}|} \geqslant \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| m_{ki}}{|a_{jj}|} = p_{ji},$$

$$h_{ji} = \frac{|a_{ji}|}{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| (m_{ki} - p_{ki})} \leqslant 1, \; j \neq i.$$

Thus $0 \leqslant h_i \leqslant 1$ $(i \in N)$ and for sufficiently small $\varepsilon > 0$ we have $0 < d_j(\varepsilon), s_{ji}(\varepsilon) < 1$, $0 < m_{ji}(\varepsilon), p_{ji}(\varepsilon) < 1$, $p_{ji}(\varepsilon) < m_{ji}(\varepsilon)$ $(j \neq i)$ and $0 < h_i(\varepsilon) \leqslant 1$. For any $i \in N$, let

$$G_i(\varepsilon) = \operatorname{diag}(p_{1i}(\varepsilon) h_i(\varepsilon), \cdots, p_{i-1,i}(\varepsilon) h_i(\varepsilon), 1, p_{i+1,i}(\varepsilon) h_i(\varepsilon), \cdots, p_{n,i}(\varepsilon) h_i(\varepsilon)).$$

For a given $i \in N$, one checks that the matrix $AG_i(\varepsilon)$ is a strictly row diagonally dominant M-matrix. In fact, for $j \neq i$, we have

$$h_i(\varepsilon) \geqslant h_{ji}(\varepsilon) = \frac{|a_{ji}| + \varepsilon m_{ji}(\varepsilon)}{|a_{jj}| p_{ji}(\varepsilon) - \sum_{k \neq j,i} |a_{jk}| p_{ki}(\varepsilon)} > \frac{|a_{ji}|}{|a_{jj}| p_{ji}(\varepsilon) - \sum_{k \neq j,i} |a_{jk}| p_{ki}(\varepsilon)},$$

which results in

$$|a_{jj}| p_{ji}(\varepsilon) h_i(\varepsilon) > |a_{ji}| + \sum_{k \neq j,i} |a_{jk}| p_{ki}(\varepsilon) h_i(\varepsilon). \tag{7}$$

While, for $j = i$, we have

$$|a_{ii}| > \sum_{k \neq i} |a_{ik}| \geqslant \sum_{k \neq i} |a_{ik}| p_{ki}(\varepsilon) h_i(\varepsilon). \tag{8}$$

From Inequalities (7) and (8) it follows that $AG_i(\varepsilon)$ is a strictly row diagonally dominant matrix, so it is also a nonsingular M-matrix. From Lemma 1 we derive the following inequality:

$$\frac{b_{ji}}{p_{ji}(\varepsilon)h_i(\varepsilon)} \leqslant \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| p_{ki}(\varepsilon) h_i(\varepsilon)}{a_{jj} p_{ji}(\varepsilon) h_i(\varepsilon)} b_{ii},\ j \neq i,\ j \in N,$$

i.e.,

$$b_{ji} \leqslant \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| p_{ki}(\varepsilon) h_i(\varepsilon)}{a_{jj}} b_{ii},\ j \neq i,\ j \in N.$$

Letting $\varepsilon \to 0$ yields

$$b_{ji} \leqslant \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| p_{ki} h_i}{a_{jj}} b_{ii} \leqslant g_j b_{ii},\ j \neq i,\ j \in N. \quad \square$$

REMARK 1. Since A is a strictly row diagonally dominant matrix, we obtain

$$s_{ji} \geqslant m_{ji} \geqslant p_{ji} \geqslant p_{ji} h_i,\ j \neq i,\ j \in N$$

by virtue of the proof of Lemma 4. Additionally, we see that

$$h_i \geqslant h_{ji} = \frac{|a_{ji}|}{|a_{jj}| p_{ji} - \sum\limits_{k\neq j,i} |a_{jk}| p_{ki}},\ j \neq i,\ j \in N.$$

Then, it has

$$s_{ji} \geqslant m_{ji} \geqslant p_{ji} \geqslant p_{ji} h_i \geqslant \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| p_{ki} h_i}{a_{jj}} = g_{ji},\ j \neq i,\ j \in N$$

and

$$s_i \geqslant m_i \geqslant p_i \geqslant g_i,\ i \in N.$$

This means that the result of Lemma 4 is sharper than those of Theorem 2.1 in [9] and Lemma 2.2 in [14].

LEMMA 5. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (b_{ij})_{n\times n}$, we have*

$$\frac{1}{a_{ii} - \sum\limits_{k\neq i} \frac{a_{ik} a_{ki}}{a_{kk}}} \leqslant b_{ii} \leqslant \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| g_{ji}},\ i \in N. \tag{9}$$

Proof. Let $B = A^{-1}$. Due to A is a nonsingular M-matrix, it has $B \geqslant 0$. Let A_i denote the submatrix of A which is obtained by deleting the ith row and the ith column of A. Then,

$$A_i \leqslant \operatorname{diag}(a_{11}, \cdots, a_{i-1,i-1}, a_{i+1,i+1}, \cdots, a_{nn}).$$

Thus, by Lemma 2 and since $\operatorname{diag}(a_{11}^{-1}, \cdots, a_{i-1,i-1}^{-1}, a_{i+1,i+1}^{-1}, \cdots, a_{nn}^{-1})$ is a nonsingular M-matrix, we have

$$A_i^{-1} \geqslant \operatorname{diag}(a_{11}^{-1}, \cdots, a_{i-1,i-1}^{-1}, a_{i+1,i+1}^{-1}, \cdots, a_{nn}^{-1}). \tag{10}$$

Lemma 3 implies that for $i \in N$,

$$b_{ii} = \frac{\det A_i}{\det A} = \frac{\det A_i}{\det A_i \det A/A_i} = \frac{1}{\det A/A_i}. \tag{11}$$

According to (10), we deduce that

$$\det A/A_i = a_{ii} - (a_{i1}, \cdots, a_{i,i-1}, a_{i,i+1}, \cdots, a_{in}) A_i^{-1} \begin{pmatrix} a_{1i} \\ \vdots \\ a_{i-1,i} \\ a_{i+1,i} \\ \vdots \\ a_{ni} \end{pmatrix} \leqslant a_{ii} - \sum_{k \neq i} \frac{a_{ik} a_{ki}}{a_{kk}},$$

which together with Inequality (11) gives

$$b_{ii} = \frac{\det A_i}{\det A} \geqslant \frac{1}{a_{ii} - \sum\limits_{k \neq i} \frac{a_{ik} a_{ki}}{a_{kk}}}, \ i \in N. \tag{12}$$

Combining Lemma 4 and $AB = I$ results in

$$\begin{aligned} 1 &= \sum_{j=1}^{n} a_{ij} b_{ji} = a_{ii} b_{ii} - \sum_{j \neq i} |a_{ij}| b_{ji} \\ &\geqslant a_{ii} b_{ii} - \sum_{j \neq i} |a_{ij}| \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| p_{ki} h_i}{a_{jj}} b_{ii} \\ &= \left(a_{ii} - \sum_{j \neq i} |a_{ij}| g_{ji} \right) b_{ii}, \end{aligned}$$

i.e.,

$$b_{ii} \leqslant \frac{1}{a_{ii} - \sum\limits_{j \neq i} |a_{ij}| g_{ji}}, \ i \in N. \tag{13}$$

The claim now follows by combining Inequalities (12) and (13). $\square$

REMARK 2. According to Remark 1, we have

$$s_{ji} \geqslant m_{ji} \geqslant p_{ji} \geqslant g_{ji},\ j \neq i,\ j \in N,$$

which implies that

$$\frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| g_{ji}} \leqslant \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| p_{ji}} \leqslant \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| m_{ji}} \leqslant \frac{1}{a_{ii} - \sum\limits_{j\neq i} |a_{ij}| s_{ji}}.$$

Moreover, it is not difficult to verify that

$$\frac{1}{a_{ii}} \leqslant \frac{1}{a_{ii} - \sum\limits_{k\neq i} \frac{a_{ik}a_{ki}}{a_{kk}}},\ i \in N.$$

This readily shows that the bounds in Lemma 5 are always better than those in Theorem 2.3 of [9] and Lemma 2.3 of [14].

LEMMA 6. [16] *If* A^{-1} *is a doubly stochastic matrix, then* $Ae = e$, $A^T e = e$, *where* $e = (1,1,\cdots,1)^T$.

LEMMA 7. [14] *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *be an irreducible nonsingular M-matrix, then* $0 < \tau(A) < a_{ii}$ *for all* $i \in N$.

LEMMA 8. [17] *Let* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ *and* $x_1, x_2, \cdots, x_n$ *be positive real numbers. Then all the eigenvalues of* A *lie in the region*

$$\bigcup_{i,j=1, i\neq j}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leqslant \left(x_i \sum_{k\neq i} \frac{1}{x_k} |a_{ki}| \right) \left(x_j \sum_{k\neq j} \frac{1}{x_k} |a_{kj}| \right) \right\}.$$

LEMMA 9. [15] *Let* $A \in Z_n$. A *is a nonsingular M-matrix if and only if all its leading principal minors are positive.*

LEMMA 10. [3] *If* A *is an irreducible nonsingular M-matrix, and* $Az \geqslant kz$ *for a nonnegative nonzero vector* z, *then* $\tau(A) \geqslant k$.

3. Main results

In this section, we develop some lower bounds for $\tau(A \circ A^{-1})$, which improve the ones in [9, 14].

THEOREM 1. *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *be a nonsingular M-matrix, and suppose that* $A^{-1} = (b_{ij})_{n\times n}$ *is doubly stochastic. Then*

$$b_{ii} \geqslant \frac{1}{1 + \sum\limits_{j\neq i} g_{ji}},\ i \in N.$$

Proof. Since A^{-1} is doubly stochastic and A is a nonsingular M-matrix, by Lemma 6, we have

$$b_{ii}+\sum_{j\neq i} b_{ji}=1,\ i\in N.$$

Taking into account that A is a strictly row diagonally dominant matrix and using Lemma 4, for $i\in N$, we infer that

$$\begin{aligned}1=b_{ii}+\sum_{j\neq i} b_{ji}&\leqslant b_{ii}+\sum_{j\neq i}\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|p_{ki}h_i}{a_{jj}}b_{ii}\\&=\left(1+\sum_{j\neq i}\frac{|a_{ji}|+\sum\limits_{k\neq j,i}|a_{jk}|p_{ki}h_i}{a_{jj}}\right)b_{ii}\\&=\left(1+\sum_{j\neq i} g_{ji}\right)b_{ii},\end{aligned}$$

which is equivalent to

$$b_{ii}\geqslant\frac{1}{1+\sum\limits_{j\neq i} g_{ji}},\ i\in N. \qquad \square$$

REMARK 3. According to Remark 1, it holds that

$$s_{ji}\geqslant m_{ji}\geqslant g_{ji},\ j\neq i,\ j\in N,$$

which yields that

$$\frac{1}{1+\sum\limits_{j\neq i} g_{ji}}\geqslant\frac{1}{1+\sum\limits_{j\neq i} m_{ji}}\geqslant\frac{1}{1+\sum\limits_{j\neq i} s_{ji}}.$$

This implies that the bounds in Theorem 1 are tighter than those in Lemma 3.2 of [9] and Theorem 3.1 of [14].

THEOREM 2. *Let $A=(a_{ij})_{n\times n}$ be a nonsingular M-matrix, and let $A^{-1}=(b_{ij})_{n\times n}$ be doubly stochastic. Then*

$$\begin{aligned}\tau(A\circ A^{-1})\geqslant\min_{i\neq j}\frac{1}{2}\Bigg\{&a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\&+4\left(g_i\sum_{k\neq i}|a_{ki}|b_{ii}\right)\left(g_j\sum_{k\neq j}|a_{kj}|b_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}.\end{aligned} \tag{14}$$

Proof. It is evident that Inequality (14) holds with equality for $n=1$. Below we assume that $n \geqslant 2$.

Firstly, we assume that A is irreducible. According to Lemma 6, we see that

$$a_{ii} = \sum_{k\neq i} |a_{ik}| + 1 = \sum_{k\neq i} |a_{ki}| + 1 > 1,\ i \in N.$$

Let

$$g_j = \max_{i\neq j}\{g_{ji}\} = \max_{i\neq j}\left\{\frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| p_{ki} h_i}{a_{jj}}\right\},\ j \in N.$$

Note that A is an irreducible matrix, $0 < g_j < 1$. Let $\tau(A\circ A^{-1}) = \lambda$. By making use of Lemma 7, it is easily seen that $0 < \lambda < a_{ii}b_{ii}$, $i \in N$. By combining Lemma 8 with Lemma 4, there is a pair (i,j) of positive integers with $i \neq j$ such that

$$\begin{aligned}
|\lambda - a_{ii}b_{ii}||\lambda - a_{jj}b_{jj}| &\leqslant \left(g_i \sum_{k\neq i} \frac{1}{g_k}|a_{ki}b_{ki}|\right)\left(g_j \sum_{k\neq j} \frac{1}{g_k}|a_{kj}b_{kj}|\right) \\
&\leqslant \left(g_i \sum_{k\neq i} \frac{1}{g_k}|a_{ki}|g_k b_{ii}\right)\left(g_j \sum_{k\neq j} \frac{1}{g_k}|a_{kj}|g_k b_{jj}\right) \\
&= \left(g_i \sum_{k\neq i} |a_{ki}|b_{ii}\right)\left(g_j \sum_{k\neq j} |a_{kj}|b_{jj}\right). \qquad (15)
\end{aligned}$$

It follows from Inequality (15) that

$$(\lambda - a_{ii}b_{ii})(\lambda - a_{jj}b_{jj}) \leqslant \left(g_i \sum_{k\neq i} |a_{ki}|b_{ii}\right)\left(g_j \sum_{k\neq j} |a_{kj}|b_{jj}\right). \qquad (16)$$

Solving the quadratic Inequality (16) yields

$$\begin{aligned}
\lambda \geqslant \frac{1}{2}\Bigg\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
& +4\left(g_i \sum_{k\neq i} |a_{ki}|b_{ii}\right)\left(g_j \sum_{k\neq j} |a_{kj}|b_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}, \qquad (17)
\end{aligned}$$

Inequality (17) directly leads to the following result

$$\begin{aligned}
\tau(A\circ A^{-1}) \geqslant \min_{i\neq j} \frac{1}{2}\Bigg\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
& +4\left(g_i \sum_{k\neq i} |a_{ki}|b_{ii}\right)\left(g_j \sum_{k\neq j} |a_{kj}|b_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}. \qquad (18)
\end{aligned}$$

If A is reducible, by Lemma 9 we know that all leading principal minors of A are positive. Denote by $P=(p_{ij})$ the $n\times n$ permutation matrix with $p_{12}=p_{23}=\cdots=p_{n-1,n}=p_{n,1}=1$, and the remaining p_{ij} being zero. If $\varepsilon>0$ is sufficiently small, then all the leading principal minors of $A-\varepsilon P$ are positive and $A-\varepsilon P$ is an irreducible nonsingular M-matrix. Now we substitute $A-\varepsilon P$ for A in the previous case. Letting $\varepsilon\to 0$, the result follows by continuity. □

REMARK 4. Having obtained the bound in Theorem 2, we can compare the bound in Theorem 2 with those in Theorem 3.1 of [9] and Theorem 3.2 of [14]. By Remark 1, we have

$$s_i\geqslant m_i\geqslant g_i,\ i\in N,$$

which leads to

$$\begin{aligned}\tau(A\circ A^{-1})&\geqslant \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\&\qquad+4\Big(g_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(g_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\}\\&\geqslant \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\&\qquad+4\Big(m_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(m_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\}.\end{aligned}$$

Furthermore, it follows from Theorem 3.3 in [14] that

$$\begin{aligned}&\min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(m_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(m_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\}\\&\qquad\geqslant \min_{i\in N}\left\{\frac{a_{ii}-s_iR_i}{1+\sum\limits_{j\neq i}s_{ji}}\right\}.\end{aligned}$$

Thus the bound in Theorem 2 is an improvement on those in Theorem 3.1 of [9] and Theorem 3.2 of [14].

THEOREM 3. *Let $A=(a_{ij})_{n\times n}$ be a strictly row diagonally dominant M-matrix. Then*

$$\tau(A\circ A^{-1})\geqslant \min_{i\in N}\left\{\frac{a_{ii}-\sum\limits_{j\neq i}|a_{ji}|g_{ji}}{a_{ii}-\sum\limits_{k\neq i}\frac{a_{ki}a_{ik}}{a_{kk}}}\right\}. \tag{19}$$

Proof. Firstly, we assume that A is irreducible, then $A^{-1} > 0$, and $A \circ A^{-1}$ is again irreducible. Note that

$$\tau(A \circ A^{-1}) = \tau((A \circ A^{-1})^T) = \tau(A^T \circ (A^T)^{-1}).$$

Let

$$(A^T \circ (A^T)^{-1})e = (l_1, l_2, \cdots, l_n)^T,$$

where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. Without loss of generality, we may assume that $l_1 = \min_{i \in N}\{l_i\}$. We deduce by Lemma 4 and Lemma 5 that

$$\begin{aligned} l_1 &= \sum_{j=1}^{n} a_{j1} b_{j1} = a_{11} b_{11} - \sum_{j \neq 1} |a_{j1}| b_{j1} \\ &\geqslant a_{11} b_{11} - \sum_{j \neq 1} |a_{j1}| \frac{|a_{j1}| + \sum\limits_{k \neq j,1} |a_{jk}| p_{k1} h_1}{a_{jj}} b_{11} \\ &= a_{11} b_{11} - \sum_{j \neq 1} |a_{j1}| g_{j1} b_{11} = (a_{11} - \sum_{j \neq 1} |a_{j1}| g_{j1}) b_{11} \\ &\geqslant \frac{a_{11} - \sum\limits_{j \neq 1} |a_{j1}| g_{j1}}{a_{11} - \sum\limits_{k \neq 1} \frac{a_{k1} a_{1k}}{a_{kk}}}. \end{aligned}$$

Then it follows from the above inequality and Lemma 10 that

$$\tau(A \circ A^{-1}) \geqslant \min_{i \in N} \left\{ \frac{a_{ii} - \sum\limits_{j \neq i} |a_{ji}| g_{ji}}{a_{ii} - \sum\limits_{k \neq i} \frac{a_{ki} a_{ik}}{a_{kk}}} \right\}.$$

The case when A is reducible can be treated similarly as in the proof of Theorem 2. This completes our proof. □

REMARK 5. By making using of Remark 1 and Remark 2, we have

$$s_{ji} \geqslant m_{ji} \geqslant g_{ji}, \; j \neq i, \; j \in N$$

and

$$\frac{1}{a_{ii}} \leqslant \frac{1}{a_{ii} - \sum\limits_{k \neq i} \frac{a_{ik} a_{ki}}{a_{kk}}}, \; i \in N.$$

As a result, it holds that

$$\frac{a_{ii} - \sum\limits_{j \neq i} |a_{ji}| g_{ji}}{a_{ii} - \sum\limits_{k \neq i} \frac{a_{ki} a_{ik}}{a_{kk}}} \geqslant 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \geqslant 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| s_{ji}.$$

Hence the bound in Theorem 3 always improves the corresponding ones in Theorem 3.5 of [9] and Theorem 3.4 of [14].

THEOREM 4. *Let $A=(a_{ij})_{n\times n}$ be a strictly row diagonally dominant M-matrix and $A^{-1}=(b_{ij})_{n\times n}$. Then*

$$\tau(A\circ A^{-1}) \geqslant \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 +4\Bigg(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\Bigg)\Bigg(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\}. \tag{20}$$

Proof. It is not difficult to verify that the result holds with equality for $n=1$. We next assume that $n\geqslant 2$.

Firstly, we assume that A is irreducible. Let $\tau(A\circ A^{-1})=\lambda$. Having in mind that $0<\lambda<a_{ii}b_{ii}$ for all $i\in N$. Thus, by Lemma 8 and Lemma 2.2 in [14], there exists a pair (i,j) of positive integers with $i\neq j$ such that

$$\begin{aligned}|\lambda-a_{ii}b_{ii}||\lambda-a_{jj}b_{jj}| &\leqslant \Bigg(\sum_{k\neq i}|a_{ki}b_{ki}|\Bigg)\Bigg(\sum_{k\neq j}|a_{kj}b_{kj}|\Bigg)\\ &\leqslant \Bigg(\sum_{k\neq i}|a_{ki}|b_{ii}m_{ki}\Bigg)\Bigg(\sum_{k\neq j}|a_{kj}|b_{jj}m_{kj}\Bigg).\end{aligned} \tag{21}$$

From Inequality (21), we see that

$$(\lambda-a_{ii}b_{ii})(\lambda-a_{jj}b_{jj}) \leqslant \Bigg(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\Bigg)\Bigg(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\Bigg),$$

which yields that

$$\lambda \geqslant \frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 +4\Bigg(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\Bigg)\Bigg(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\},$$

from which one may deduce the following result

$$\tau(A\circ A^{-1}) \geqslant \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 +4\Bigg(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\Bigg)\Bigg(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\Bigg)\Bigg]^{\frac{1}{2}}\Bigg\}. \tag{22}$$

With a quite similar strategy utilized in Theorem 2, the result of Theorem 4 is still valid for the case that A is reducible. □

REMARK 6. Without loss of generality, for $i \neq j$, we assume that

$$a_{ii}b_{ii} - b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki} \leqslant a_{jj}b_{jj} - b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj},$$

we can rewrite the above equation into the following equivalent form

$$b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj} \leqslant a_{jj}b_{jj} - a_{ii}b_{ii} + b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}. \tag{23}$$

By applying Inequalities (20) and (23) together with Lemma 2.3 of [14] we obtain

$$\begin{aligned}
&\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\left(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)\left(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\right)\right]^{\frac{1}{2}}\right\}\\
\geqslant{}& \frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\right.\right.\\
&\left.\left.+4\left(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)\left(a_{jj}b_{jj}-a_{ii}b_{ii}+b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)\right]^{\frac{1}{2}}\right\}\\
={}& \frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\right.\right.\\
&\left.\left.+4\left(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)^2+4\left(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)\left(a_{jj}b_{jj}-a_{ii}b_{ii}\right)\right]^{\frac{1}{2}}\right\}\\
={}& \frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[\left(a_{jj}b_{jj}-a_{ii}b_{ii}+2b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)^2\right]^{\frac{1}{2}}\right\}\\
={}& a_{ii}b_{ii}-b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}=\left(a_{ii}-\sum_{k\neq i}|a_{ki}|m_{ki}\right)b_{ii}\geqslant\frac{a_{ii}-\sum\limits_{k\neq i}|a_{ki}|m_{ki}}{a_{ii}}.
\end{aligned}$$

Then it immediately leads to the following result

$$\begin{aligned}
&\min_{i\neq j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\left(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\right)\left(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\right)\right]^{\frac{1}{2}}\right\}\\
&\qquad\geqslant\min_{i\in N}\left\{1-\frac{1}{a_{ii}}\sum_{j\neq i}|a_{ji}|m_{ji}\right\}.
\end{aligned}$$

Moreover, by Remark 3.2 in [14], we have

$$\min_{i\in N}\left\{1-\frac{1}{a_{ii}}\sum_{j\neq i}|a_{ji}|m_{ji}\right\}\geqslant\min_{i\in N}\left\{1-\frac{1}{a_{ii}}\sum_{j\neq i}|a_{ji}|s_{ji}\right\}.$$

This implies that the bound in Theorem 4 is always tighter than those in Theorem 3.5 of [9] and Theorem 3.4 of [14].

REMARK 7. If A is a nonsingular M-matrix, we know that there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant nonsingular M-matrix. So the results of Theorems 3-4 also hold for a nonsingular M-matrix.

REMARK 8. According to Theorem 2 and Theorem 4, it is clearly seen that the forms of the lower bounds in Theorem 2 and Theorem 4 are similar. For given $i,j \in N$ and $j \neq i$, the only difference between these two bounds is that the forms of

$$\left(g_i \sum_{k\neq i} |a_{ki}|\right)\left(g_j \sum_{k\neq j} |a_{kj}|\right) \text{ and } \left(\sum_{k\neq i} |a_{ki}|m_{ki}\right)\left(\sum_{k\neq j} |a_{kj}|m_{kj}\right)$$

are different. Thus, if we attempt to compare the bounds in Theorem 2 and Theorem 4, we need to get the relations between g_i and m_{ki} $(i,k \in N, k \neq i)$. By making use of the definitions of g_i and m_{ki}, we have

$$g_i = \max_{j\neq i}\left\{\frac{|a_{ij}| + \sum\limits_{k\neq i,j} |a_{ik}|p_{kj}h_j}{a_{ii}}\right\} \text{ and } m_{ki} = \frac{|a_{ki}| + \sum\limits_{t\neq k,i} |a_{kt}|s_{ti}}{a_{kk}}. \tag{24}$$

Although

$$g_i = \max_{j\neq i}\left\{\frac{|a_{ij}| + \sum\limits_{k\neq i,j} |a_{ik}|p_{kj}h_j}{a_{ii}}\right\} \leqslant m_i = \max_{j\neq i}\{m_{ij}\} = \max_{j\neq i}\left\{\frac{|a_{ij}| + \sum\limits_{k\neq i,j} |a_{ik}|s_{kj}}{a_{ii}}\right\},$$

it is not necessarily that $g_i \leqslant m_{ki}$ $(k \neq i)$. Therefore, we can not conclude that which bound in Theorem 2 and Theorem 4 is better because there is no way to compare the relations between g_i and m_{ki} $(k \neq i)$ in theory. Example 3 is also implemented to illustrate this fact.

In the following, we compare the bounds in Theorems 2 and 4 with that in Theorem 3, respectively. From Remark 5 and Remark 6, we can get

$$\min_{i\in N}\left\{\frac{a_{ii} - \sum\limits_{j\neq i} |a_{ji}|g_{ji}}{a_{ii} - \sum\limits_{k\neq i} \frac{a_{ki}a_{ik}}{a_{kk}}}\right\} \geqslant \min_{i\in N}\left\{1 - \frac{1}{a_{ii}}\sum_{j\neq i} |a_{ji}|m_{ji}\right\} \tag{25}$$

and

$$\min_{i\neq j} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\left(b_{ii}\sum_{k\neq i} |a_{ki}|m_{ki}\right)\left(b_{jj}\sum_{k\neq j} |a_{kj}|m_{kj}\right)\right]^{\frac{1}{2}}\right\}$$
$$\geqslant \min_{i\in N}\left\{1 - \frac{1}{a_{ii}}\sum_{j\neq i} |a_{ji}|m_{ji}\right\},$$

whereas we can not prove that the bound in Theorem 4 is sharper than that in Theorem 3. Furthermore, in a manner similar to that done for Remark 6 and by Lemma 5, it is not difficult to verify that

$$\min_{i\neq j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\left(g_i\sum_{k\neq i}|a_{ki}|b_{ii}\right)\left(g_j\sum_{k\neq j}|a_{kj}|b_{jj}\right)\right]^{\frac{1}{2}}\right\}$$
$$\geqslant \min_{i\in N}\left\{\frac{a_{ii}-g_i\sum\limits_{j\neq i}|a_{ji}|}{a_{ii}-\sum\limits_{k\neq i}\frac{a_{ki}a_{ik}}{a_{kk}}}\right\}.$$

However, we can not determine the relations between g_i and g_{ji} $(j\neq i)$, so we also can not see that which bound in Theorem 2 and Theorem 3 is better.

From the above discussions, it can be seen that we can not conclude that which bound in Theorems 2-4 is the best. Numerical results of Example 3 are provided to confirm this fact. Nevertheless, as observed in the results of Example 3, the bound in Theorem 2 outperforms that in Theorem 4 for many cases and the bounds in Theorem 2 and Theorem 4 are better than that in Theorem 3 for almost cases. However, we see that the bounds in Theorems 3-4 hold under weaker conditions compared with Theorem 2. What is more, when we derive the bounds in Theorems 2 and 4, we need to compute the inverse of the matrix A, but it is not necessarily for Theorem 3. This implies that the bound in Theorem 3 needs less basic arithmetic operations than those of Theorems 2 and 4 as the size of A is large. On the other hand, the bound in Theorem 3 is only depending on the entries of the matrix A. So, it is more easily computed than those in Theorems 2 and 4.

4. Numerical examples

EXAMPLE 1. Consider the following nonsingular M-matrix [14]:

$$A=\begin{pmatrix}4 & -1 & -1 & -1\\ -2 & 5 & -1 & -1\\ 0 & -2 & 4 & -1\\ -1 & -1 & -1 & 4\end{pmatrix}.$$

Since $Ae=e$ and $A^Te=e$, A^{-1} is doubly stochastic. By calculations we have

$$A^{-1}=\begin{pmatrix}0.4 & 0.2 & 0.2 & 0.2\\ 0.2333 & 0.3667 & 0.2 & 0.2\\ 0.1667 & 0.2333 & 0.4 & 0.2\\ 0.2 & 0.2 & 0.2 & 0.4\end{pmatrix}.$$

(1) Estimate the upper bounds for the entries of $A^{-1} = (b_{ij})_{n\times n}$. By Theorem 2.1 (a) of [9], we have

$$A^{-1} \leqslant \begin{pmatrix} 1 & 0.6250 & 0.6375 & 0.6375 \\ 0.7000 & 1 & 0.6500 & 0.6500 \\ 0.5875 & 0.6875 & 1 & 0.6500 \\ 0.6375 & 0.6250 & 0.6375 & 1 \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

By Lemma 2.2 of [14], we have

$$A^{-1} \leqslant \begin{pmatrix} 1 & 0.5781 & 0.5718 & 0.5750 \\ 0.6450 & 1 & 0.5825 & 0.5850 \\ 0.5093 & 0.6562 & 1 & 0.5750 \\ 0.5718 & 0.5781 & 0.5718 & 1 \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

If we apply Lemma 4, we obtain

$$A^{-1} \leqslant \begin{pmatrix} 1 & 0.5479 & 0.5021 & 0.5035 \\ 0.5934 & 1 & 0.5013 & 0.5026 \\ 0.4263 & 0.6383 & 1 & 0.5041 \\ 0.5104 & 0.5479 & 0.5021 & 1 \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

By combining the result of Lemma 4 with the results of Theorem 2.1 (a) of [9] and Lemma 2.2 of [14] we see that the result of Lemma 4 is the best.

By Theorem 2.3 and Lemma 3.2 of [9], we can get the following bounds for the diagonal entries of A^{-1}:

$$0.3419 \leqslant b_{11} \leqslant 0.5882,\ 0.3404 \leqslant b_{22} \leqslant 0.5128,$$
$$0.3419 \leqslant b_{33} \leqslant 0.6061,\ 0.3404 \leqslant b_{44} \leqslant 0.5882.$$

By some calculations with Lemma 2.3 and Theorem 3.1 of [14], we get

$$0.3668 \leqslant b_{11} \leqslant 0.4397,\ 0.3556 \leqslant b_{22} \leqslant 0.3832,$$
$$0.3668 \leqslant b_{33} \leqslant 0.4419,\ 0.3656 \leqslant b_{44} \leqslant 0.4415.$$

Now from Lemma 5 and Theorem 1, we obtain

$$0.3952 \leqslant b_{11} \leqslant 0.4049,\ 0.3658 \leqslant b_{22} \leqslant 0.3679,$$
$$0.3991 \leqslant b_{33} \leqslant 0.4007,\ 0.3984 \leqslant b_{44} \leqslant 0.4016.$$

(2) Lower bounds for $\tau(A\circ A^{-1})$. By Theorem 3.1 of [9], we have

$$\tau(A\circ A^{-1}) \geqslant \min_{i\in N}\left\{\frac{a_{ii}-s_iR_i}{1+\sum\limits_{j\neq i} s_{ji}}\right\} = 0.6624.$$

By Theorem 3.2 of [10], we have

$$\tau(A\circ A^{-1}) \geqslant \min_{i\in N}\left\{\frac{a_{ii}-t_iR_i}{1+\sum\limits_{j\neq i} t_{ji}}\right\} = 0.7999.$$

By Theorem 3.1 of [11], we have

$$\tau(A\circ A^{-1}) \geqslant \min_{i\in N}\left\{\frac{a_{ii}-u_iR_i}{1+\sum\limits_{j\neq i}u_{ji}}\right\} = 0.8250.$$

By Corollary 2.5 of [12], we have

$$\tau(A\circ A^{-1}) \geqslant 1-\rho^2(J_A) = 0.4145.$$

By Corollary 2 of [13], we have

$$\tau(A\circ A^{-1}) \geqslant \min_{i\in N}\left\{\frac{a_{ii}-w_i\sum\limits_{j\neq i}|a_{ji}|}{1+\sum\limits_{j\neq i}w_{ji}}\right\} = 0.8321.$$

By Theorem 3.2 of [14], we have

$$\begin{aligned}\tau(A\circ A^{-1}) \geqslant\ & \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\ &+4\Big(m_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(m_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\} = 0.8456.\end{aligned}$$

By Theorem 2, we obtain

$$\begin{aligned}\tau(A\circ A^{-1}) \geqslant\ & \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\ &+4\Big(g_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(g_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\} = 0.8904.\end{aligned}$$

By Theorem 4, we obtain

$$\begin{aligned}\tau(A\circ A^{-1}) \geqslant\ & \min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Bigg[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\ &+4\Big(b_{ii}\sum_{k\neq i}|a_{ki}|m_{ki}\Big)\Big(b_{jj}\sum_{k\neq j}|a_{kj}|m_{kj}\Big)\Bigg]^{\frac{1}{2}}\Bigg\} = 0.8811.\end{aligned}$$

This numerical example shows that the bounds in Theorem 2 and Theorem 4 are better than the corresponding ones in [9, 10, 11, 12, 13, 14].

EXAMPLE 2. Next we conduct an experiment to demonstrate the lower bounds referred. We randomly construct a 4×4 symmetric nonsingular M-matrix A where each entry is generated by uniform distribution $(-1,0)$, and we adjust the diagonal entries of the matrix A such that $Ae=e$ and $A^Te=e$, where $e=(1,1,1,1)^T$. Therefore A^{-1} is doubly stochastic. We compare the lower bounds in Theorems 2-4 with (1)–(6), Theorem 3.5 in [9] and Theorem 3.4 in [14]. Their differences are denoted by symbols:

- The bounds in Theorems 2–4 — the bound in (1): plus symbol in blue color;
- The bounds in Theorems 2–4 — the bound in (2): star symbol in black color;
- The bounds in Theorems 2–4 — the bound in (3): circle symbol in red color;
- The bounds in Theorems 2–4 — the bound in (4): diamond symbol in magenta color;
- The bounds in Theorems 2–4 — the bound in (5): inverse triangle symbol in green color;
- The bounds in Theorems 2–4 — the bound in (6): square symbol in cyan color;
- The bounds in Theorems 2–4 — the bound in Theorem 3.5 in [9]: hexagonal symbol in yellow color;
- The bounds in Theorems 2–4 — the bound in Theorem 3.4 in [14]: pentastar symbol in black color.

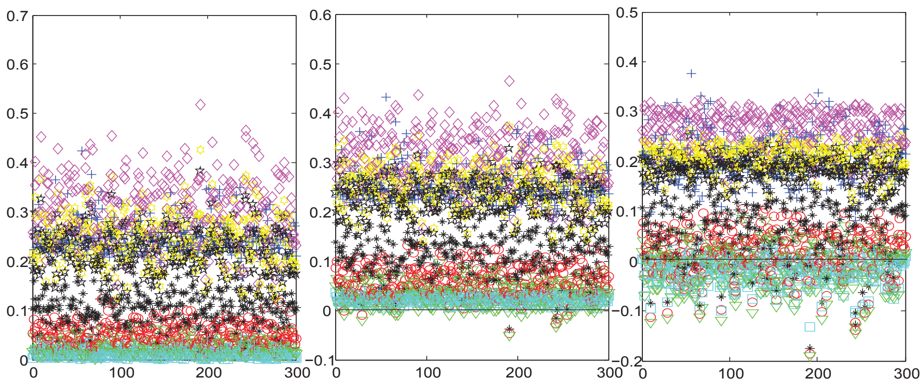

Figure 1: *The randomly generated results for Theorem 2 (left), Theorem 4 (middle) and Theorem 3 (right) compared with (1)–(6), Theorem 3.5 in* [9] *and Theorem 3.4 in* [14].

In Figure 1, we show the results of 300 generated matrices for Example 2. The x-axis refers to these 300 random generated cases. From the first figure in Figure 1, we observe that all symbols are above the x-axis, i.e., the bound in Theorem 2 are better than those in (1)–(6), Theorem 3.5 in [9] and Theorem 3.4 in [14]. In the meanwhile, the percentage of different kinds of symbols which are above the x-axis are listed in Table 1. We use ‘plus’, ‘star’, ‘circle’, ‘square’, ‘diamond’, ‘triangle’, ‘hexagonal’ and ‘pentastar’ to denote the symbols mentioned in the above. As observed in the third line of Table 1, almost all symbols are above the x-axis except the star, circle and inverse triangle symbols. This means that the bound in Theorem 4 outperforms those in (1), (4) and (6), Theorem 3.5 in [9] and Theorem 3.4 in [14]. For the star symbol in black color, there are 99.3% of cases above the x-axis, and for the circle symbol in red color and the inverse triangle symbol in green color, there are 97.3% and 91.7% of cases above the x-axis, respectively. From the observations in Table 1, we can conclude that the bound in Theorem 4 is sharper than the bounds in (2)–(3) and (5) for most cases. Additionally, from the results of the fourth line in Table 1, it can be clearly seen that the plus symbol in blue color, the diamond symbol in magenta color, the hexagonal symbol in yellow color and the pentastar symbol in black color are above the x-axis, i.e., the bound in Theorem 3 is tighter than those in (1), (4), Theorem 3.5 in [9] and Theorem 3.4 in [14]. For the star symbol in black color and the circle symbol in red color, there are 86.3% and 63% of cases above the x-axis, respectively. This implies that the bound in Theorem 3 is sharper than the bounds in (2)–(3) for many cases, whereas for the square symbol in cyan color and the inverse triangle symbol in green color, there are only 20.3% and 39% of cases above the x-axis, respectively. Hence the bounds in (5)–(6) are often better than that in Theorem 3. From all the observations above we can conclude that the proposed lower bounds for $\tau(A\circ A^{-1})$ are competitive and effective compared with some existing ones, especially the bound in Theorem 2.

Table 1: *The percentage of different kinds of symbols which are above the x-axis.*

	plus	star	circle	square	diamond	triangle	hexagonal	pentastar
left	100%	100%	100%	100%	100%	100%	100%	100%
middle	100%	99.3%	97.3%	100%	100%	91.7%	100%	100%
right	100%	86.3%	63%	20.3%	100%	39%	100%	100%

EXAMPLE 3. In the following, we establish another experiment to compare the lower bounds derived in this paper. We randomly construct a 5×5 symmetric nonsingular M-matrix A where each entry is generated by uniform distribution $(-1,0)$, and we adjust the diagonal entries of the matrix A such that $Ae=e$ and $A^Te=e$, where $e=(1,1,1,1,1)^T$. Therefore A^{-1} is doubly stochastic. We compare the lower bounds for $\tau(A\circ A^{-1})$ in Theorems 2-4. Their differences are denoted by symbols:

- The bound in Theorem 2 — the bound in Theorem 4: plus symbol in blue color;
- The bound in Theorem 2 — the bound in Theorem 3: star symbol in black color;
- The bound in Theorem 4 — the bound in Theorem 3: circle symbol in red color.

The numerical results for Example 3 are showed in Figure 2. The x-axis of Figure 2 refers to these 300 random generated cases. As seen from Figure 2, for the plus symbol in blue color, there are 71.7% of cases above the x-axis. This indicates that the bound in Theorem 2 outperforms that in Theorem 4 for many cases. Besides, for the star symbol in black color and the circle symbol in red color, there are 97% and 99.7% of cases above the x-axis, respectively. These observations imply that the bounds in

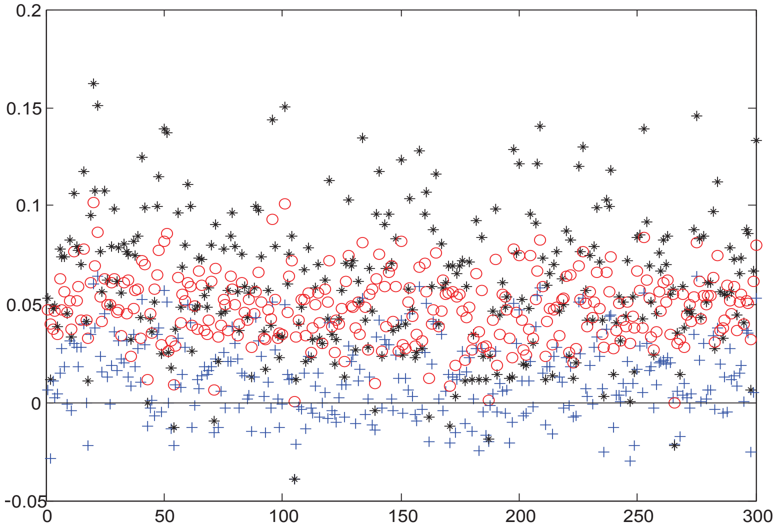

Figure 2: *The randomly generated results for Theorems 2-4 developed in this paper.*

Theorem 2 and Theorem 4 are better than that in Theorem 3 for almost cases. However, it is worthy noting that we can not determine which bound in Theorems 2-4 is the best and these numerical results are in accordance with the discussions given in Remark 8.

5. Conclusions

In this paper, by constructing new compression factors, we establish new lower bounds for $\tau(A \circ A^{-1})$ which are better than the existing ones in [9, 10, 11, 12, 13, 14]. Numerical results given in Section 4 (Table 1 and Figures 1–2) show that the results obtained by the new bounds are more sharper than those derived in [9, 10, 11, 12, 13, 14]. Numerical results also present the feasibility and effectiveness of the new bounds when they are used to estimate $\tau(A \circ A^{-1})$.

However, we do not give the error analysis, i.e., how accurately these bounds can be computed. At present, it is very difficult for us to do this. We will continue to investigate this problem in the future.

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Zhengge Huang
Department of Applied Mathematics
Northwestern Polytechnical University
Shaanxi, Xi'an, 710072, P. R. China
e-mail: ZhenggeHuang@mail.nwpu.edu.cn

Corresponding author
Ligong Wang
Department of Applied Mathematics
Northwestern Polytechnical University
Shaanxi, Xi'an, 710072, P. R. China
e-mail: lgwang@nwpu.edu.cn

Zhong Xu
Department of Applied Mathematics
Northwestern Polytechnical University
Shaanxi, Xi'an, 710072, P. R. China
e-mail: zhongxu@nwpu.edu.cn