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SEPARATION THEOREMS FOR NONCONVEX SETS IN SPACES WITH NON–SYMMETRIC SEMINORM

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Abstract. The theory of weakly convex sets in Banach spaces with non-symmetric seminorm is developed. The separation theorem with sphere or (in a general case) with the boundary of a shifted quasiball for two closed disjoint subsets of a Banach space, one of which is prox-regular or weakly convex, and the other is the summand of a ball or quasiball is proven.

1. Introduction

The Hahn–Banach theorem on the separation of two convex disjoint sets with a hyperplane is well-known in functional analysis. The duality theory, based on this theorem, has many applications in optimization and other branches of mathematics. It is easy to see that two non-convex disjoint sets in a general case can't be separated by a hyperplane. But if one of the sets is prox-regular, and the other is the summand of a ball of a sufficiently small radius, these sets can be separated with a sphere (see Theorem 1.1 below).

Let E be a real normed vector space. For a set $A \subset E$ by $\operatorname{int} A$, $\overline{A}$, and ∂A we denote the interior, the closure, and the boundary of $A \subset E$, respectively. We use $\langle p, x\rangle$ to denote the value of the functional $p \in E^*$ at the vector $x \in E$. For $r \geqslant 0$ and $a \in E$ we define the *ball* with center a and radius r as $\mathfrak{B}_r(a) = \{x \in E : \|x-a\| \leqslant r\}$. By $\mathfrak{B}_r$ denote the ball with centre at the zero element of the space E: $\mathfrak{B}_r = \mathfrak{B}_r(0)$. *The Minkowski sum* of the sets $A \subset E$ and $B \subset E$ is called the set

$$A + B = \{a + b : \ a \in A, \ b \in B\}.$$

The set $C \subset E$ is called the *summand* of the set $M \subset E$, if there exists a set $C_1 \subset E$ such that $C + C_1 = M$.

The distance from $x \in E$ *to* $A \subset E$ is determined by the equality

$$\rho(x, A) = \inf_{a \in A} \|x - a\|. \tag{1.1}$$

The metric projection of $x \in E$ on $A \subset E$ is called the set

$$P(x, A) = \{a \in A : \ \|x - a\| = \rho(x, A)\}. \tag{1.2}$$

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By $\Omega_R(A)$ denote the R-*tube* around $A \subset E$:

$$\Omega_R(A) = \{x \in E \,:\, 0 < \rho(x,A) < R\}.$$

The principle notions of our study are prox-regular sets and weakly convex sets. They date back to the notion of sets with positive reach, introduced by H. Federer [7]. The set $A \subset \mathbb{R}^n$ is called *set with positive reach*, if $\operatorname{reach}(A) > 0$, where

$$\operatorname{reach}(A) = \sup\{R > 0 \,|\, P(x,A) \text{ is a singleton } \forall x \in \Omega_R(A)\}.$$

Clark, Stern and Wolenski [4], extending the notion of positive reach sets on Hilbert spaces, introduced the notion of *proximally smooth sets* as the sets $A \subset E$ for which the distance function $\rho(\cdot,A)$ is continuously differentiable on the set $\Omega_R(A)$ for some $R > 0$. R. Poliquin and R.-T. Rockafellar [12] introduced the notion of prox-regularity in finite dimensional Euclidean space, and then this notion was considered for sets in a Banach spaces by F. Bernard, L. Thibault and N. Zlateva [2]. A set $A \subset E$ is called *uniformly R-prox-regular sets* if

$$P(a+Rz,A) = \{a\} \qquad \forall a \in A, \quad \forall z \in N(a,A) :\ \|z\| < 1,$$

where

$$N(a,A) = \{z \in E \,:\, \exists t > 0 :\ a \in P(a+tz,A)\}$$

is the *cone of proximal normals* to the set A at the point a.

The *convexity modulus* of space E is determined by the equality

$$\delta(t) = \inf\left\{1 - \frac{\|x+y\|}{2} \,:\, x,y \in \mathfrak{B}_1,\ \|x-y\| \geqslant t\right\}, \quad t \in [0,2]. \tag{1.3}$$

The notion of convexity modulus was introduced by J. Clarkson [5]. The space E is called *uniformly convex*, if $\delta(t) > 0$ for all $t \in (0,2]$.

The *smoothness modulus* of the space E is called (see [6])

$$\beta(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 \,:\, x,y \in E \,:\ \|x\| = 1,\ \|y\| \leqslant t\right\}, \qquad t \geqslant 0. \tag{1.4}$$

The space E is called *uniformly smooth*, if $\lim_{t\to+0} \frac{\beta(t)}{t} = 0$.

As shown in [2, Theorem 6.2] a closed set $A \subset E$ is uniformly R-prox-regular iff the distance function $\rho(\cdot,A)$ is continuously differentiable on the set $\Omega_R(A)$ provided that the space E is uniformly convex and uniformly smooth with the moduli of power type. So, in such spaces the notions of the uniform prox-regularity and the proximal smoothness are equivalent.

THEOREM 1.1. (On separation with a sphere) *Let E be a uniformly convex and uniformly smooth Banach space, $0 < r < R$, the set $A \subset E$ be closed and uniformly R-prox-regular, the set $C \subset E$ be convex, closed and the summand of the ball $\mathfrak{B}_r$,*

$\operatorname{int} C \neq \emptyset$, $A \cap \operatorname{int} C = \emptyset$. *Then there exist points* $a, c \in E$ *such that* $\|a-c\| < R-r$ *and for any* $x \in A, y \in \operatorname{int} C$ *the inequalities*

$$\|c-y\| < r < R < \|a-x\|$$

hold. Geometrically it means that (see Fig. 1)

$$\operatorname{int} C \subset \operatorname{int} \mathfrak{B}_r(c) \subset \operatorname{int} \mathfrak{B}_R(a) \subset E \setminus A.$$

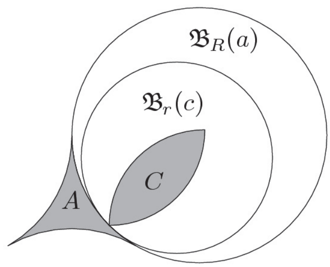

Figure 1: *to Theorem* 1.1.

Theorem 1.1 generalizes Theorem 1.18.2, obtained in monograph [8] for Hilbert space and correlates with [1, Theorem 2.6] for Banach space with generating unit ball.

In the next section we shall consider the class of weakly convex sets with respect to a non-symmetric seminorm or (in other terms) to a quasiball. In particular, if the seminorm is a norm (and the quasiball is the unit ball), the class of weakly convex sets coincides with the class of uniformly R-prox-regular sets with $R = 1$. The case of unbounded quasiball is important in investigations of weakly convex functions, whose epigraphs are weakly convex w.r.t. the epigraph of some convex function. In terms of such concept of weakly convex functions one can characterize well-posedness of the infimal convolution problem (see [9]). We shall prove Theorem 2.1 on separation with the boundary of a shifted quasiball and as a corollary we shall obtain Theorem 1.1 (see Remark 2.14).

2. Separation Theorems for weakly convex sets w.r.t. a quasiball

A closed convex set $M \subset E$, $M \neq E$ such that $0 \in \operatorname{int} M$ is called *quasiball*.

The *Minkowski function* of the quasiball $M \subset E$ is the function $\mu_M : E \to [0; +\infty)$ given by the equality

$$\mu_M(x) = \inf\{t > 0 : x \in tM\} \qquad \forall x \in E.$$

The function $\mu : E \to \mathbb{R}$ is called *sublinear*, if it is *positive homogeneous*:

$$\mu(\lambda x) = \lambda \mu(x) \qquad \forall x \in E \quad \forall \lambda \geqslant 0$$

and *subadditive*:

$$\mu(x+y) \leqslant \mu(x) + \mu(y) \qquad \forall x, y \in E.$$

A sublinear non-negative function is called a *non-symmetric seminorm*.

REMARK 2.1. The continuous function $\mu : E \to [0; +\infty)$ is a non-symmetric seminorm iff it is the Minkowski function of a quasiball.

REMARK 2.2. For any quasiball $M \subset E$ and for any vector $x \in E$ the inequality $\mu_M(x) \leqslant 1$ is equivalent to the inclusion $x \in M$, and the equality $\mu_M(x) = 1$ is equivalent to the inclusion $x \in \partial M$.

Let $M \subset E$ be a quasiball. The *M-distance* from set $C \subset E$ to set $A \subset E$ is called

$$\rho_M(C, A) = \inf_{\substack{a \in A \\ c \in C}} \mu_M(c - a). \tag{2.1}$$

In particular, the M-distance from point $x \in E$ to set $A \subset E$ is determined by the equality

$$\rho_M(x, A) = \inf_{a \in A} \mu_M(x - a). \tag{2.2}$$

The *M-projection* of $x \in E$ on $A \subset E$ is called the set

$$P_M(x, A) = A \bigcap (x - \rho_M(x, A) M). \tag{2.3}$$

The *cone of proximal normals* to set $A \subset E$ at the point $a \in A$ w.r.t. the quasiball $M \subset E$ is called the cone

$$N_M(a, A) = \{z \in E \,:\, \exists t > 0 :\, a \in P_M(a + tz, A)\}. \tag{2.4}$$

Denote

$$N_M^1(a, A) = \{z \in N_M(a, A) \,:\, \mu_M(z) = 1\}. \tag{2.5}$$

The set $A \subset E$ is called *weakly convex w.r.t. the quasiball $M \subset E$*, if

$$a + z \notin A + \operatorname{int} M \qquad \forall a \in A, \quad \forall z \in N_M^1(a, A). \tag{2.6}$$

REMARK 2.3. If $R > 0$, $M = \mathfrak{B}_R$, then for any $x \in E$, $A \subset E$ the equalities

$$\mu_M(x) = \frac{\|x\|}{R}, \quad \rho_M(x, A) = \frac{\rho(x, A)}{R}, \quad P_M(x, A) = P(x, A), \quad N_M(x, A) = N(x, A)$$

hold.

The set $M \subset E$ is called *strictly convex*, if for any distinct points $x, y \in M$ the inclusion $\frac{x+y}{2} \in \operatorname{int} M$ holds.

REMARK 2.4. If $R > 0$ and $M = \mathfrak{B}_R$ is a strongly convex ball, then the class of sets, weakly convex w.r.t. the quasiball M coincides with the class of uniformly R-prox-regular sets.

The set $M \subset E$ is called *boundedly uniformly convex*, if for any positive numbers R and ε the inequality $\delta_M(\varepsilon,R) > 0$ holds, where

$$\delta_M(\varepsilon,R) = \sup\left\{\delta \in \left[0,\frac{\varepsilon}{2}\right] : \|x-y\| \geqslant \varepsilon \Rightarrow \mathfrak{B}_\delta\left(\frac{x+y}{2}\right) \subset M \;\forall x,y \in M \cap \mathfrak{B}_R\right\}. \tag{2.7}$$

REMARK 2.5. If the set $M \subset E$ is boundedly uniformly convex then it is strictly convex.

REMARK 2.6. If $M = \mathfrak{B}_r$, then for $R \geqslant r > 0$, $\varepsilon > 0$ the equality $\delta_M(\varepsilon,R) = \delta(\varepsilon)$ holds. Therefore the ball in the uniformly convex space E is boundedly uniformly convex.

REMARK 2.7. A boundedly uniformly convex set may not be bounded. For example, if E is a uniformly convex space and the set $\{(x,y) \in E \times \mathbb{R} : y \geqslant \|x\|^2\}$ is boundedly uniformly convex in the space $E \times \mathbb{R}$ with the norm $\|(x,y)\| = \|x\| + |y|$.

For any quasiball $M \subset E$ denote

$$\sigma_M = \inf_{x \in \partial M} \|x\|. \tag{2.8}$$

REMARK 2.8. For any quasiball $M \subset E$ the inequality $\sigma_M > 0$ and the inclusion $\mathfrak{B}_{\sigma_M} \subset M$ hold.

REMARK 2.9. The Minkowski function of any quasiball $M \subset E$ satisfies the Lipschitz condition with constant $\frac{1}{\sigma_M}$ on E. For any set $A \subset E$ the function $\rho_M(\cdot,A)$ satisfies the Lipschitz condition on E with the same constant.

The quasiball $M \subset E$ is called *boundedly uniformly smooth*, if

$$\lim_{t \to +0} \frac{\beta_M(t,R)}{t} = 0 \qquad \forall R > \sigma_M, \tag{2.9}$$

where σ_M is defined by the equality (2.8) and for any $t \geqslant 0$ and $R > \sigma_M$

$$\beta_M(t,R) = \sup\left\{\frac{\mu_M(x+ty) + \mu_M(x-ty)}{2} - 1 : x \in \partial M \cap \mathfrak{B}_R,\ y \in \mathfrak{B}_1\right\}. \tag{2.10}$$

REMARK 2.10. If $M = \mathfrak{B}_r$, then for $R > r$, $t \geqslant 0$ the equality $\beta_M(t,R) = \beta(t)$ holds. Therefore the ball in the uniformly smooth space E is boundedly uniformly smooth.

The set $M \subset E$ is called *parabolic*, if for any vector $b \in E$ the set $\left(b + \frac{1}{2}M\right) \setminus M$ is bounded (see [10]).

REMARK 2.11. If a set $M \subset E$ is bounded, it is parabolic.

The term “parabolic” is due to the observation that the epigraph of the parabola $y = x^2$ is parabolic while the epigraph of the hyperbola $y = \frac{1}{x}$, $x > 0$ is not parabolic.

The set $A \subset E$ is called *closed w.r.t. the quasiball M* or *M-closed*, if for any $x \in E \setminus A$ the inequality $\rho_M(x,A) > 0$ holds.

The set $A \subset E$ is called *M-quasibounded*, if it is M-closed and for any $R \geqslant 0$ the inequality $\kappa(R) < +\infty$ holds, where

$$\kappa(R) = \sup\left\{\|z\| \,:\, z \in N^1_M(a,A),\ a \in A \cap \mathfrak{B}_R\right\}. \tag{2.11}$$

REMARK 2.12. If the quasiball M is bounded, then any closed set $A \subset E$ is M-quasibounded.

Let $M \subset E$ be a quasiball. By $\mathscr{WC}(M)$ denote the class of closed subsets of the space E which are weakly convex w.r.t. the quasiball M, and by $\mathscr{SC}(M)$ the class of convex closed subsets of the space E which are summands of the quasiball M.

THEOREM 2.1. (On separation with the boundary of a shifted quasiball) *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Let $r \in (0,1)$, $A \in \mathscr{WC}(M)$, $C \in \mathscr{SC}(-rM)$. In addition, let at least one of the following conditions hold:*

1) $\rho_M(C,A) > 0$ or

2) $\operatorname{int} C \neq \emptyset$, $A \cap \operatorname{int} C = \emptyset$, the quasiball M is boundedly uniformly smooth, the set A is M-quasibounded.

Then there exist $a, c \in E$ such that $\mu_M(c-a) \leqslant 1-r$ and for any $x \in A$, $y \in \operatorname{int} C$

$$\mu_M(c-y) < r < 1 \leqslant \mu_M(a-x).$$

According to Remark 2.2 geometrically it means that (see Fig. 2)

$$\operatorname{int} C \subset c - \operatorname{int} rM \subset a - \operatorname{int} M \subset E \setminus A.$$

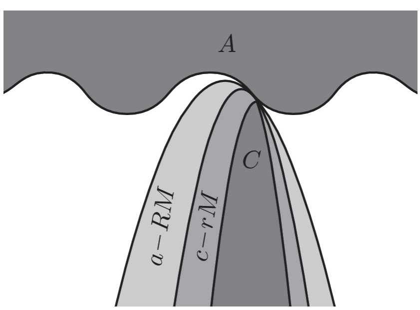

Figure 2: *to Theorem 2.1.*

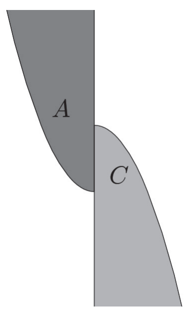

Figure 3: *to Remark 2.13.*

REMARK 2.13. The assumption of M-quasiboundedness of the set A in item 2) of Theorem 2.1 is essential. Indeed, let us consider in Euclidean space $E=\mathbb{R}^2$ the quasiball $M=\{(x,y)\in\mathbb{R}^2 : y\geqslant x^2-1\}$ and the sets $A=\{(x,y)\in\mathbb{R}^2 : y\geqslant 2x^2-1,\ x\leqslant 0\}$, $C=\{(x,y)\in\mathbb{R}^2 : y\leqslant 1-2x^2,\ x\geqslant 0\}$. Then for $r=\frac{1}{2}$ the assumptions of item 2 of Theorem 2.1 hold except for the M-quasiboundedness of the set A. And the sets A and C cannot be separated with the boundary of the shifted quasiball M (see Fig. 3).

REMARK 2.14. Due to Remarks 2.4, 2.6, 2.10–2.12 Theorem 1.1 follows from Theorems 2.1.

3. Auxiliary results

The *support function* of the set $M\subset E$ is called the function

$$s(p,M)=\sup_{x\in M}\langle p,x\rangle, \qquad p\in E^*. \tag{3.1}$$

LEMMA 3.1. *Let $M\subset E$ be a quasiball, $p\in E^*$, $s(p,M)<+\infty$. Then*

$$\langle p,x\rangle\leqslant\mu_M(x)s(p,M) \qquad \forall x\in E,\quad \forall p\in E^*.$$

Proof. Fix an arbitrary vector $x\in E$ and the functional $p\in E^*$ such that $s(p,M)<+\infty$. If $\mu_M(x)=0$, then for any $t>0$ due to Remark 2.2 the inclusion $\frac{x}{t}\in M$ holds. As $\sup_{t>0}\langle p,\frac{x}{t}\rangle\leqslant s(p,M)<+\infty$, then $\langle p,x\rangle\leqslant 0$, and the desired inequality holds. Let $\mu_M(x)>0$. Then the vector $x_1=\frac{x}{\mu_M(x)}$ satisfies the inclusion $x_1\in M$ and, therefore, $\frac{\langle p,x\rangle}{\mu_M(x)}=\langle p,x_1\rangle\leqslant s(p,M)$. □

LEMMA 3.2. ([11, Lemma 3.1]) *Let the set $C\subset E$ be the summand of the strongly convex quasiball $M\subset E$. Then*

$$C-c\subset M-z \qquad \forall c\in C,\quad \forall z\in N_M^1(c,C).$$

Further the following properties of weakly convex sets will be needed.

LEMMA 3.3. ([9, Lemma 5.4]) *Let $M\subset E$ be a parabolic and a boundedly uniformly convex quasiball. Let $A\in\mathscr{WC}(M)$ and let there exist $x\in E$ such that $\rho_M(x,A)>0$. Then $A+\operatorname{int}M\neq E$.*

THEOREM 3.1. ([9, Theorem 2.5]) *Let $M\subset E$ be a parabolic and boundedly uniformly convex quasiball. Let $A\in\mathscr{WC}(RM)$, $R>0$. Let $x\in E$ be such that $0<\rho_M(x,A)<R$. Then $P_M(x,A)$ is a singleton.*

Given a function $f:A\to\mathbb{R}\cup\{+\infty\}$, let us consider the problem

$$\min_{x\in E}f(x). \tag{3.2}$$

A sequence $\{x_k\} \subset E$ is said to be a *minimizing sequence* if

$$\lim_{k\to\infty} f(x_k) = \inf_{x\in E} f(x).$$

The problem (3.2) is called *well posed*, if every minimizing sequence of this problem converges.

Due to the continuity of the Minkowski function, if the problem

$$\min_{a\in A,\ c\in C} \mu_M(c-a) \tag{3.3}$$

is well-posed, then the minimizing sequences converge to its solution and the minimum is attained at a single pair (a,c).

THEOREM 3.2. ([11, Theorem 4.1]) *Let the quasiball M in a Banach space E be parabolic and boundedly uniformly convex. Let the set $A \subset E$ be closed an weakly convex with respect to the quasiball M. Let the set $C \subset E$ be the summand of the quasiball $-rM$, with $0 < r < 1$. Let $0 < \rho_M(C,A) < 1-r$. Then the problem (3.3) is well posed.*

LEMMA 3.4. *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Let the set $C \subset E$ be the summand of the quasiball $-rM$ and let there exist a vector $c_1 \in E$ such that $c_1 + C \subset -rM$. Let the set $A \in \mathscr{W}\mathscr{C}(RM)$ be M-quasibounded, where $0 < r < R$. Let $\rho_M(C,A) = 0$, $A \bigcap \operatorname{int} C = \emptyset$ and $\operatorname{int} C \neq \emptyset$. Then the set $A \bigcap C$ is a singleton.*

Lemma 3.4 is implied by Theorem 4.2 from [11].

LEMMA 3.5. ([10, Lemma 5.1]) *Let the set $M \subset E$ be convex and parabolic, $0 < \lambda_1 < \lambda_2$, $x_1, x_2 \in E$, then the set $(\lambda_1 M + x_1) \setminus (\lambda_2 \operatorname{int} M + x_2)$ is bounded.*

LEMMA 3.6. *Let $M \subset E$ be a quasiball, $A \in \mathscr{W}\mathscr{C}(M)$, $a_0 \in \partial A$, $z \in N_M^1(a_0,A) \cap \mathfrak{B}_R$, where $R > \sigma_M$. Let $p \in E^*$, $\langle p,z\rangle = s(p,M) = 1$. Then*

$$\langle p, a-a_0\rangle \leqslant 2\beta_M(\|a-a_0\|,R) \qquad \forall a \in A,$$

where β_M is the modulus of smoothness of M, determined by the equality (2.10).

Proof. Fix an arbitrary $a \in A$. As $z \in N_M^1(a_0,A) \cap \mathfrak{B}_R$, $A \in \mathscr{W}\mathscr{C}(M)$, then according to (2.6) we have $a_0 + z \notin A + \operatorname{int} M$. Therefore, $a_0 + z - a \notin \operatorname{int} M$, i.e. $\mu_M(a_0 + z - a) \geqslant 1$. Using the equality (2.10) and the inclusion $z \in N_M^1(a_0,A) \subset \partial M$, we get

$$\beta_M(\|a-a_0\|,R) \geqslant \frac{\mu_M(z+a_0-a)+\mu_M(z+a-a_0)}{2} - 1 \geqslant \frac{\mu_M(z+a-a_0)-1}{2}. \tag{3.4}$$

On the other hand, Lemma 3.1 and the equality $s(p,M) = 1$ imply that $\langle p, z+a-a_0\rangle \leqslant \mu_M(z+a-a_0)$. This together with the equality $\langle p,z\rangle = 1$ and the inequality (3.4) yields the required inequality. □

REMARK 3.1. For any $R > \sigma_M$ the function $\beta_M(\cdot,R)$ is a convex function as the supremum of convex functions. This and the equality $\beta_M(0,R) = 0$ and the inequality $\beta_M(t,R) \geqslant 0$ imply that the function $t \mapsto \frac{\beta_M(t,R)}{t}$ non-strictly increases for any $R > \sigma_M$.

LEMMA 3.7. *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasiball. Let the M-quasibounded set $A \in \mathscr{WC}(M)$ contain the points x_0, x_1 such that for any $\lambda \in (0,1)$ the inequality $\|x_1 - x_0\| \min\{\lambda, 1-\lambda\} < \sigma_M$ holds, where the number σ_M is defined by equality (2.8). Let $x_\lambda = (1-\lambda)x_0 + \lambda x_1$, $a \in P_M(x_\lambda, A)$, $R \geqslant \max\{\sigma_M + 1, \kappa(\|a\|)\}$, $\varkappa = \frac{\kappa(\|a\|)}{\sigma_M}$, where the function $\kappa(\cdot)$ is defined by equality (2.11). Then the inequality*

$$\rho_M(x_\lambda, A) \leqslant 4\lambda(1-\lambda)\beta_M((1+\varkappa)\|x_1 - x_0\|, R)$$

holds.

Proof. If $\rho_M(x_\lambda, A) = 0$, then the required inequality holds trivially. Let us suppose that $\rho_M(x_\lambda, A) > 0$. According to inequalities (2.4), (2.5) the vector $z = \frac{x_\lambda - a}{\rho_M(x_\lambda, A)}$ satisfies the inclusion $z \in N^1_M(a, A)$. According to equality (2.11) we get

$$\|z\| \leqslant \kappa(\|a\|) \leqslant R. \tag{3.5}$$

As $z \in \partial M$, then, according to the Hahn–Banach separation theorem, there exists a functional $p \in E^*$ such that $\langle p, z\rangle = s(p, M) = 1$. Lemma 3.6 implies that

$$\langle p, x_0 - a\rangle \leqslant 2\beta_M(\|x_0 - a\|, R), \qquad \langle p, x_1 - a\rangle \leqslant 2\beta_M(\|x_1 - a\|, R).$$

Therefore,

$$\begin{aligned} \langle p, x_\lambda - a\rangle &= (1-\lambda)\langle p, x_0 - a\rangle + \lambda\langle p, x_1 - a\rangle \\ &\leqslant 2(1-\lambda)\beta_M(\|x_0 - a\|, R) + 2\lambda\beta_M(\|x_1 - a\|, R). \end{aligned} \tag{3.6}$$

According to Remark 2.9, we get

$$\begin{aligned} \rho_M(x_\lambda, A) &\leqslant \min\{\mu_M(x_\lambda - x_0), \mu_M(x_\lambda - x_1)\} \\ &\leqslant \frac{\min\{\|x_\lambda - x_0\|, \|x_\lambda - x_1\|\}}{\sigma_M} = \frac{\|x_1 - x_0\|}{\sigma_M} \min\{\lambda, 1-\lambda\} < 1, \end{aligned}$$

and, using inequality (3.5), we have

$$\|x_\lambda - a\| \leqslant \kappa(\|a\|)\rho_M(x_\lambda, A) \leqslant \varkappa\|x_1 - x_0\| \min\{\lambda, 1-\lambda\}.$$

Therefore,

$$\|x_0 - a\| \leqslant \|x_0 - x_\lambda\| + \|x_\lambda - a\| \leqslant \lambda(1+\varkappa)\|x_1 - x_0\|,$$

$$\|x_1 - a\| \leqslant \|x_1 - x_\lambda\| + \|x_\lambda - a\| \leqslant (1-\lambda)(1+\varkappa)\|x_1 - x_0\|.$$

This and inequalities (3.6) imply that

$$\langle p, x_\lambda - a\rangle \leqslant 2(1-\lambda)\beta_M(\lambda(1+\varkappa)\|x_1 - x_0\|, R) + 2\lambda\beta_M((1-\lambda)(1+\varkappa)\|x_1 - x_0\|, R).$$

Using Remark 3.1, we obtain the inequality

$$\langle p, x_\lambda - a\rangle \leqslant 4\lambda(1-\lambda)\beta_M((1+\varkappa)\|x_1 - x_0\|, R). \tag{3.7}$$

As $\langle p, z\rangle = 1$, then $\langle p, x_\lambda - a\rangle = \rho_M(x_\lambda, A)$. This and inequalities (3.7) imply the required inequality. □

LEMMA 3.8. *Let $M \subset E$ be a parabolic and boundedly uniformly convex quasi-ball. Let the set $A \in \mathscr{W}\mathscr{C}(M)$, be M-closed, $a_0 \in \partial A$. Then*

$$\eta := \sup\left\{\|a\| \,:\, a \in P_M(x, A),\ x \in \mathfrak{B}_{\frac{1}{4}\sigma_M}(a_0)\right\} < +\infty. \tag{3.8}$$

Proof. Let $x \in \mathfrak{B}_{\frac{1}{4}\sigma_M}(a_0)$, $a \in P_M(x, A)$. Then

$$\begin{aligned}\mu_M(a_0 - a) &\leqslant \mu_M(x-a) + \mu_M(a_0 - x) = \rho_M(x, A) + \mu_M(a_0 - x)\\ &\leqslant \mu_M(x - a_0) + \mu_M(a_0 - x).\end{aligned}$$

The last inequality is implied by the fact that $a_0 \in \partial A$. As $\{x - a_0, a_0 - x\} \subset \mathfrak{B}_{\frac{1}{4}\sigma_M} \subset \frac{1}{4}M$, then $\mu_M(x - a_0) \leqslant \frac{1}{4}$ and $\mu_M(a_0 - x) \leqslant \frac{1}{4}$. Therefore, $\mu_M(a_0 - a) \leqslant \frac{1}{2}$, and, thus, $a \in A \cap \left(a_0 - \frac{1}{2}M\right)$. Therefore, it is sufficient to show that the set $A \cap \left(a_0 - \frac{1}{2}M\right)$ is bounded.

As the set A is M-closed and $A \neq E$, there exists $x_0 \in E$ such that $\rho_M(x_0, A) > 0$. Lemma 3.3 implies that there exists $w \in E \setminus (A + \operatorname{int} M)$. Therefore, $A \subset E \setminus (w - \operatorname{int} M)$, and, thus, $A \cap \left(a_0 - \frac{1}{2}M\right) \subset \left(a_0 - \frac{1}{2}M\right) \setminus (w - \operatorname{int} M)$. Using the parabolicity of the set M, by Lemma 3.5 we obtain the boundedness of the set $A \cap \left(a_0 - \frac{1}{2}M\right)$. □

4. Convexity of the contingent cone

The *contingent cone* to the set $A \subset E$ at $a_0 \in E$ was introduced by Bouligand [3] and is determined by the equality

$$T(a_0, A) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{t \in (0, \delta]} \left(\frac{1}{t}(A - a_0) + \mathfrak{B}_\varepsilon\right).$$

REMARK 4.1. The vector v belongs to $T(a_0, A)$ iff there exist sequences $\{v_k\} \subset E$ and $\{t_k\} \subset (0, +\infty)$ such that $\lim_{k\to\infty} v_k = v$, $\lim_{k\to\infty} t_k = 0$ and $a_0 + t_k v_k \in A$ for all $k \in \mathbb{N}$.

LEMMA 4.1. *Let $M \subset E$ be parabolic, boundedly uniformly convex and boundedly uniformly smooth, the set $A \in \mathscr{W}\mathscr{C}(M)$ is M-quasibounded, $a_0 \in \partial A$. Then the cone $T(a_0, A)$ is convex.*

Proof. Let $u,v \in T(a_0,A)$. We have to prove the inclusion

$$u+v \in T(a_0,A). \tag{4.1}$$

According to Remark 4.1, there exist sequences $\{u_k\} \subset E$, $\{v_k\} \subset E$, $\{t_k\} \subset (0,+\infty)$ and $\{\tau_k\} \subset (0,+\infty)$ such that

$$\lim_{k\to\infty} u_k = u, \qquad \lim_{k\to\infty} v_k = v, \qquad \lim_{k\to\infty} t_k = 0, \qquad \lim_{k\to\infty} \tau_k = 0, \tag{4.2}$$

$a_0 + t_k u_k \in A$ and $a_0 + \tau_k v_k \in A$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ denote

$$\xi_k = \frac{t_k \tau_k}{t_k + \tau_k}, \qquad y_k = a_0 + \xi_k(u_k + v_k), \qquad \rho_k = \rho_M(y_k, A).$$

If for any index k_0 there exists an index $k \geqslant k_0$ such that $y_k \in A$, then Remark 4.1 and relation (4.2) imply inclusion (4.1). Therefore, we will suppose that $y_k \notin A$ for all k, starting with k_0. Due to the M-quasiboundedness of the set A, we obtain the inequality $\rho_k > 0$ for all $k \geqslant k_0$.

As $0 < \xi_k < t_k \to 0$, $\tau_k \to 0$ for $k \to \infty$, and the sequences $\{u_k\}$ and $\{v_k\}$ are bounded, then $y_k \to a_0$ and $\rho_k \to 0$ as $k \to \infty$. Therefore there exists an index $k_1 \geqslant k_0$ such that

$$\rho_k < 1, \quad y_k \in \mathfrak{B}_{\frac{1}{4}\sigma_M}(a_0), \quad \|t_k u_k - \tau_k v_k\| < \sigma_M \quad \forall k \geqslant k_1.$$

Theorem 3.1 implies that for any $k \geqslant k_1$ there exists a point $a_k \in P_M(y_k, A)$. According to relation (3.8) we have $\sup_{k \geqslant k_1} \|a_k\| \leqslant \eta < +\infty$. For any fixed $k \geqslant k_1$ Lemma 3.7, applied to $x_0 = a_0 + t_k u_k$, $x_1 = a_0 + \tau_k v_k$, $\lambda = \frac{t_k}{t_k+\tau_k}$, implies that

$$\rho_k \leqslant 4\xi_k \varepsilon_k, \tag{4.3}$$

where

$$\varepsilon_k = \frac{\beta_M\Big((1+\varkappa)\|t_k u_k - \tau_k v_k\|, R\Big)}{t_k + \tau_k}, \qquad \varkappa = \frac{\kappa(\eta)}{\sigma_M}, \qquad R = \max\{\sigma_M + 1, \kappa(\eta)\}.$$

Using relation (2.9) and the boundedness of sequences $\{u_k\}$ and $\{v_k\}$, we obtain that

$$\lim_{k\to\infty} \varepsilon_k = 0. \tag{4.4}$$

For $k \geqslant k_1$ denote $z_k = \frac{y_k - a_k}{\rho_k}$. As $a_k \in P_M(y_k, A)$, then $z_k \in N_M^1(a_k, A)$. This and the M-quasiboundedness of A imply that $\sup_{k \geqslant k_1} \|z_k\| = C < +\infty$. Using inequality (4.3), for any $k \geqslant k_1$ we get

$$\|y_k - a_k\| = \|z_k\| \rho_k \leqslant C\rho_k \leqslant 4C\xi_k \varepsilon_k.$$

Denoting $w_k = \frac{a_k - a_0}{\xi_k}$, we obtain the inequality $\|u_k + v_k - w_k\| \leqslant 4C\varepsilon_k$ as $k \geqslant k_1$. Using (4.4) and (4.2), we get $w_k \to u+v$ as $k \to \infty$. As $a_k = a_0 + \xi_k w_k \in A$, then relation (4.1) holds. □

5. Other properties of the contingent cone

LEMMA 5.1. *Let the set* $A \subset E$ *and the convex set* $C \subset E$ *be such that* $0 \in A \cap C$, $\operatorname{int} C \neq \emptyset$, $A \cap \operatorname{int} C = \emptyset$. *Then* $T(0,A) \cap \operatorname{int} C = \emptyset$.

Proof. Suppose the contrary: there exists a vector $v \in T(0,A) \cap \operatorname{int} C$. As $v \in \operatorname{int} C$, there exists $\delta > 0$ such that $\mathfrak{B}_\delta(v) \subset \operatorname{int} C$. As $v \in T(0,A)$, there exists $u \in \mathfrak{B}_\delta(v)$ and $t \in (0,1)$ such that $tu \in A$. On the other hand, the inclusions $u \in \mathfrak{B}_\delta(v) \subset \operatorname{int} C$, $0 \in C$ and the convexity of C imply that $tu \in \operatorname{int} C$. This contradicts the assumption $A \cap \operatorname{int} C = \emptyset$. $\square$

LEMMA 5.2. *Let* $M \subset E$ *be a parabolic, boundedly uniformly convex and boundedly uniformly smooth quasiball. Let the set* $A \in \mathscr{W}\mathscr{C}(M)$ *be* M*-quasibounded and* $0 \in \partial A$. *Then there exist positive numbers* δ, C_1, C_2 *and number* $R > \sigma_M$ *such that for any non-zero vector* $a \in A \cap \mathfrak{B}_\delta$ *there exists a non-zero vector* $v \in T(0,A)$ *that satisfies the inequality*

$$\|v - a\| \leqslant C_1 \beta_M\Big(C_2\|a\|, R\Big), \tag{5.1}$$

where the function $\beta_M(\cdot)$ *is defined by equality* (2.10).

Proof. Using the function $\kappa(\cdot)$, defined by formula (2.11) and the number η, given by the equality (3.8), denote

$$\varkappa = \frac{\kappa(\eta)}{\sigma_M}, \qquad R = \max\{\sigma_M + 1, \kappa(\eta)\}. \tag{5.2}$$

For any $t > 0$ define $\gamma_M(t) = \frac{\beta_M((1+\varkappa)t,R)}{t}$. Remark 3.1 implies that function $\gamma_M(\cdot)$ non-strictly increases. Relation (2.9) implies that $\lim\limits_{t\to+0} \gamma_M(t) = 0$.

Therefore there exists a positive number $\delta \leqslant \min\left\{\frac{\sigma_M}{2}, \eta\right\}$ such that

$$8R\gamma_M(\delta) \leqslant 1. \tag{5.3}$$

Fix an arbitrary non-zero vector $a \in A \cap \mathfrak{B}_\delta$. Suppose that $a_0 = a$, $\delta_0 = \|a_0\|$. As $\lim\limits_{t\to+0} \gamma_M(t) = 0$, then for any $k \in \mathbb{N}$ there exists a number $\delta_k \in (0,\delta]$ such that

$$\gamma_M(\delta_k) \leqslant \frac{\gamma_M(\delta_0)}{2^k}. \tag{5.4}$$

Let for some $k \in \mathbb{N} \cup \{0\}$ be given a vector $a_k \in A$ such that

$$0 < \|a_k\| \leqslant \delta_k. \tag{5.5}$$

Fix any $\lambda_k \in (0,1)$ such that

$$2\lambda_k\|a_k\| < \delta_{k+1}. \tag{5.6}$$

As $\|\lambda_k a_k\| < \frac{\delta_{k+1}}{2} \leqslant \frac{\delta}{2} \leqslant \frac{\sigma_M}{4}$, Remark 2.9 implies the inequalities $\rho_M(\lambda_k a_k, A) \leqslant \frac{\|\lambda_k a_k\|}{\sigma_M} < \frac{1}{4}$. Therefore, according to Theorem 3.1, there exists $a_{k+1} \in P_M(\lambda_k a_k, A)$.

Inequality $\|\lambda_k a_k\| < \frac{\sigma_M}{4}$ and equality (3.8) imply the inequality $\|a_{k+1}\| \leqslant \eta$. From Lemma 3.7, applied to $x_0 = 0$, $x_1 = a_k$, $\lambda = \lambda_k$, $a = a_{k+1}$, and equality $\gamma_M(t) = \frac{\beta_M((1+\varkappa)t,R)}{t}$, we obtain

$$\rho_M(\lambda_k a_k, A) \leqslant 4\lambda_k \beta_M((1+\varkappa)\|a_k\|, R) = 4\lambda_k \|a_k\| \gamma_M(\|a_k\|). \tag{5.7}$$

Equality (2.11) implies that $\|\lambda_k a_k - a_{k+1}\| \leqslant \kappa(\eta)\mu_M(\lambda_k a_k - a_{k+1}) \leqslant R\mu_M(\lambda_k a_k - a_{k+1})$. This, inequality (5.7), and the equality $\mu_M(\lambda_k a_k - a_{k+1}) = \rho_M(\lambda_k a_k, A)$ imply that

$$\|\lambda_k a_k - a_{k+1}\| \leqslant R\mu_M(\lambda_k a_k - a_{k+1}) \leqslant 4R\lambda_k \|a_k\| \gamma_M(\|a_k\|). \tag{5.8}$$

As, according to Remark 3.1, the inequality $\gamma_M(\|a_k\|) \leqslant \gamma_M(\delta)$ holds, inequality (5.3) implies that $\|\lambda_k a_k - a_{k+1}\| \leqslant \frac{\lambda_k \|a_k\|}{2}$. Therefore, using inequality (5.6), we obtain the chain of inequalities

$$0 < \frac{1}{2}\lambda_k \|a_k\| \leqslant \|a_{k+1}\| \leqslant \frac{3}{2}\lambda_k \|a_k\| < \delta_{k+1}.$$

Therefore, $0 < \|a_{k+1}\| \leqslant \delta_{k+1}$ and we may continue the process of constructing the sequence a_k. Thus, the recursively constructed sequence $\{a_k\}_{k=0}^{\infty} \subset A$ is such that for any $k \in \mathbb{N} \cup \{0\}$ inequalities (5.5) and (5.8) hold.

For any $k \in \mathbb{N} \cup \{0\}$ the chain of inequalities

$$\left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_k}{\|a_k\|} \right\| \leqslant \left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_{k+1}}{\lambda_k \|a_k\|} \right\| + \left\| \frac{a_{k+1}}{\lambda_k \|a_k\|} - \frac{a_k}{\|a_k\|} \right\| \leqslant \frac{2\|\lambda_k a_k - a_{k+1}\|}{\lambda_k \|a_k\|},$$

relations (5.4), (5.5), (5.8) and Remark 3.1 imply that

$$\left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_k}{\|a_k\|} \right\| \leqslant 8R\gamma_M(\|a_k\|) \leqslant 8R\gamma_M(\delta_k) \leqslant \frac{8R\gamma_M(\delta_0)}{2^k}.$$

Therefore,

$$\left\| \frac{a_m}{\|a_m\|} - \frac{a_n}{\|a_n\|} \right\| \leqslant \sum_{k=n}^{m-1} \left\| \frac{a_{k+1}}{\|a_{k+1}\|} - \frac{a_k}{\|a_k\|} \right\| \leqslant \frac{R\gamma_M(\delta_0)}{2^{n-4}} \qquad \forall n \in \mathbb{N} \cup \{0\}, \quad \forall m > n.$$

Thus, the sequence $\left\{ \frac{a_k}{\|a_k\|} \right\}$ is a Cauchy sequence and, therefore, converges to some $v_0 \in E$. Moreover,

$$\left\| v_0 - \frac{a}{\delta_0} \right\| \leqslant 16R\gamma_M(\delta_0) = \frac{16R\beta_M\Big((1+\varkappa)\delta_0, R\Big)}{\delta_0}.$$

Therefore, vector $v = \delta_0 v_0$ satisfies inequality (5.1) as $C_1 = 16R$, $C_2 = 1+\varkappa$. As $a_k \in A$, $\frac{a_k}{\|a_k\|} \to v_0$, $\|a_k\| \to 0$ when $k \to \infty$, we have that $v_0 \in T(0,A)$ and, therefore, $v \in T(0,A)$. □

Remind that the *effective domain* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is

$$\operatorname{dom} f = \{x \in E \mid f(x) \in \mathbb{R}\}.$$

LEMMA 5.3. ([10, Lemma 5.2]) *Let the set $M \subset E$ be parabolic. Then*

1) for any functional $p \in \operatorname{dom} s(\cdot, M) \setminus \{0\}$ the set $\{x \in M : \langle p, x\rangle \geqslant -1\}$ is bounded;

2) the set $\operatorname{dom} s(\cdot, M) \setminus \{0\}$ is open.

LEMMA 5.4. ([9, Lemma 4.5(ii)]) *If in the space E there exists a parabolic boundedly uniformly convex quasiball, then the space E is reflexive.*

LEMMA 5.5. ([9, Lemma 7.2]) *Let $M \subset E$ be a boundedly uniformly convex and parabolic quasiball. Let the functional $p \in E^* \setminus \{0\}$ and the bounded sequence $\{w_k\} \subset \partial M$ be such that $\lim_{k\to\infty}\langle p, w_k\rangle = s(p, M) < +\infty$. Then the sequence $\{w_k\}$ converges.*

LEMMA 5.6. *Let $M \subset E$ be a parabolic, boundedly uniformly convex, and boundedly uniformly smooth quasiball, let the set $A \in \mathscr{W}\mathscr{C}(M)$ be M-quasibounded, $0 \in \partial A$. Let the functional $p \in E^*$ satisfy the equality $s(p, T(0, A)) = 0$. Then*

1) $s(p, M) < +\infty$;

2) if $z \in \partial M$ satisfies the equalities $\langle p, z\rangle = s(p, M) = 1$, then $0 \in P_M(z, A)$.

Proof. Fix an arbitrary $z \in M$ such that $\langle p, z\rangle > 0$. If for any k_0 there exists a number $k \geqslant k_0$ such that $\frac{z}{k} \in A$, then $z \in T(0, A)$. Therefore, $0 < \langle p, z\rangle \leqslant s(p, T(0, A)) = 0$. The contradiction obtained proves the existence of a number k_0 such that $\frac{z}{k} \notin A$ for all $k \geqslant k_0$. This and the M-quasiboundedness of set A imply that $\rho_M\left(\frac{z}{k}, A\right) > 0$ as $k \geqslant k_0$. As $0 \in A$, we get

$$\rho_M\left(\frac{z}{k}, A\right) \leqslant \mu_M\left(\frac{z}{k}\right) \leqslant \frac{1}{k} \qquad \forall k \in \mathbb{N}. \tag{5.9}$$

Theorem 3.1 implies that for any $k > k_0$ there exists $a_k \in P_M\left(\frac{z}{k}, A\right)$. For any $k > k_0$ denote

$$w_k = \frac{\frac{z}{k} - a_k}{\mu_M\left(\frac{z}{k} - a_k\right)}. \tag{5.10}$$

Choose an index $k_1 > k_0$ such that $\frac{\|z\|}{k_1} < \frac{\sigma_M}{4}$. Then, according to equality (3.8), we get $\sup_{k \geqslant k_1}\|a_k\| \leqslant \eta$. Equality (2.11) and the M-quasiboundedness of set A imply that

$$\|w_k\| \leqslant \kappa(\eta) < +\infty \quad \forall k \geqslant k_1.$$

This and relations (5.9), (5.10), and equality $\mu_M\left(\frac{z}{k} - a_k\right) = \rho_M\left(\frac{z}{k}, A\right)$ imply that

$$\left\|\frac{z}{k} - a_k\right\| \leqslant \frac{\kappa(\eta)}{k} \qquad \forall k \geqslant k_1. \tag{5.11}$$

Therefore,

$$\|a_k\| \leqslant \left\|\frac{z}{k} - a_k\right\| + \frac{\|z\|}{k} \leqslant \frac{\kappa(\eta) + \|z\|}{k} \to 0, \qquad k \to \infty. \tag{5.12}$$

Lemma 5.2 implies that there exist an index $k_2 \geqslant k_1$, positive numbers C_1, C_2 and number $R > \sigma_M$ such that, for any $k \geqslant k_2$ there exists a vector $v_k \in T(0,A)$ that satisfies the inequality

$$\|a_k - v_k\| \leqslant \varepsilon_k := C_1 \beta_M(C_2\|a_k\|, R). \tag{5.13}$$

Relations (2.9), (5.12) imply that

$$k\varepsilon_k = C_1 k \beta_M\Big(C_2\|a_k\|, R\Big) \leqslant C_1 k \beta_M\left(\frac{C_2(\kappa(\eta)+\|z\|)}{k}, R\right) \to 0, \qquad k \to \infty. \tag{5.14}$$

As $v_k \in T(0,A)$, we obtain that $\langle p, v_k\rangle \leqslant s(p, T(0,A)) = 0$. Therefore, according to inequality (5.13), we have

$$\langle p, a_k\rangle \leqslant \langle p, a_k - v_k\rangle \leqslant \|p\| \cdot \|a_k - v_k\| \leqslant \varepsilon_k\|p\| \qquad \forall k \geqslant k_2.$$

Thus,

$$\left\langle p, \frac{z}{k}\right\rangle \leqslant \left\langle p, \frac{z}{k} - a_k\right\rangle + \varepsilon_k\|p\| \qquad \forall k \geqslant k_2. \tag{5.15}$$

Using inequality (5.11), we obtain the inequalities

$$\langle p, z\rangle \leqslant \left(\left\|\frac{z}{k} - a_k\right\| + \varepsilon_k\right) k\|p\| \leqslant (\kappa(\eta) + k\varepsilon_k)\|p\| \qquad \forall k \geqslant k_2.$$

Passing to the limit as $k \to \infty$, taking into account relations (5.14), we obtain that

$$\langle p, z\rangle \leqslant \kappa(\eta)\|p\|, \tag{5.16}$$

where $\kappa(\cdot)$ and the number η are defined by equalities (2.11) and (3.8) accordingly, and, therefore, $\kappa(\eta)$ does not depend on point z. If $\langle p, z\rangle \leqslant 0$, then inequality (5.16) also holds. Thus, inequality (5.16) holds for any $z \in M$. Therefore, $s(p,M) \leqslant \kappa(\eta)\|p\| < +\infty$. This proves the first assertion of the Lemma.

Let us prove the second assertion. Let $z \in \partial M$ satisfy the equalities $\langle p, z\rangle = s(p,M) = 1$. Using inequality (5.15), we have

$$\left\langle p, \frac{z}{k} - a_k\right\rangle \geqslant \left\langle p, \frac{z}{k}\right\rangle - \varepsilon_k\|p\| = \frac{1 - k\varepsilon_k\|p\|}{k} \qquad \forall k \geqslant k_2.$$

Relation (5.14) implies that there exists an index $k_3 \geqslant k_2$ such that $k\varepsilon_k\|p\| < 1$ for all $k \geqslant k_3$. Using the equality $\mu_M\left(\frac{z}{k} - a_k\right) = \rho_M\left(\frac{z}{k}, A\right)$ and the relations (5.9), (5.10), we obtain that

$$s(p,M) \geqslant \langle p, w_k\rangle = \frac{\left\langle p, \frac{z}{k} - a_k\right\rangle}{\mu_M\left(\frac{z}{k} - a_k\right)} \geqslant \frac{1 - k\varepsilon_k\|p\|}{k\,\mu_M\left(\frac{z}{k} - a_k\right)} \geqslant 1 - k\varepsilon_k\|p\| \qquad \forall k \geqslant k_2.$$

As $w_k \in M$ then, according to (5.14), we have

$$s(p,M) \geqslant \langle p, w_k\rangle \geqslant 1 - k\varepsilon_k\|p\| \to 1 = s(p,M), \qquad k \to \infty.$$

According to Lemma 5.3(1), the sequence $\{w_k\}$ is bounded. Therefore, in view of Lemma 5.5, this sequence converges to some vector $w \in E$. In addition, $w \in M$ and

$\langle p,w\rangle = s(p,M) = \langle p,z\rangle$. Taking into consideration the strict convexity of M, we obtain the equality $w = z$. Thus,

$$\lim_{k\to\infty} w_k = z. \tag{5.17}$$

As $a_k \in P_M\left(\frac{z}{k},A\right)$, according to equalities (2.4), (2.5) we have $w_k \in N_M^1(a_k,A)$. Thus, taking into consideration the inclusion $A \in \mathscr{W}\mathscr{C}(M)$ and formula (2.6), we have $a_k + w_k \notin A + \operatorname{int} M$, i.e. $\rho_M(a_k + w_k, A) = 1$. Using (5.12), (5.17), and the continuity of function $\rho_M(\cdot,A)$, we obtain the equality $\rho_M(z,A) = 1$, i.e. $0 \in P_M(z,A)$. $\square$

For any set $X \subset E$ and any functional $p \in E^*$ consider the set

$$\operatorname{Exp}(p,X) = \{x \in X : \langle p,x\rangle = s(p,X)\}. \tag{5.18}$$

REMARK 5.1. Definition (5.18) and formula $s(p,X_1+X_2) = s(p,X_1)+s(p,X_2)$ imply that

$$\operatorname{Exp}(p,X_1+X_2) = \operatorname{Exp}(p,X_1)+\operatorname{Exp}(p,X_2) \qquad \forall X_1,X_2 \subset E, \quad \forall p \in E^*.$$

6. Proof of Theorem 2.1

1) Let $\rho_M(C,A) > 0$.

a) First, suppose that $\rho_M(C,A) < 1-r$. By Theorem 3.2 the minimum (3.3) is attained at a single pair of points $a_0 \in A, c_0 \in C$. Therefore,

$$a_0 \in P_M(c_0,A), \quad c_0 \in P_{-M}(a_0,C).$$

Denote $w = \frac{c_0-a_0}{\mu_M(c_0-a_0)}$. As $A \in \mathscr{W}\mathscr{C}(M)$, then, according to (2.6), we have

$$a_0 + w \notin A + \operatorname{int} M.$$

Lemma 3.2 implies that

$$C \subset c_0 - r(M-w).$$

Let $c = c_0 + rw, \quad a = a_0 + w$. Then $\mu_M(a-x) \geqslant 1$ for all $x \in A$ and $\mu_M(c-y) \leqslant r$ for all $y \in C$. It remains to prove that $\mu_M(a-c) < (1-r)$. The definition of vector w implies that

$$\begin{aligned}\mu_M(a-c) &= \mu_M(a_0-c_0+(1-r)w) = \mu_M\left((1-r-\mu_M(c_0-a_0))w\right)\\ &= 1-r-\mu_M(c_0-a_0) < 1-r.\end{aligned}$$

b) Let now $\rho_M(C,A) \geqslant 1-r$. Inclusion $C \in \mathscr{S}\mathscr{C}(-rM)$ implies that there exists a set $C' \subset E$ such that $C + C' = -rM$. Fix $c_0 \in C'$ and for any $t \in [0,1]$ consider the set $C_t = C + t(C'-c_0)$. Suppose that $\rho_M(C_1,A) \geqslant 1-r$. As $C_1 = C_t|_{t=1} = -c_0 - rM$, then $\rho_M(-c_0-rM,A) \geqslant 1-r$. Therefore, $(-c_0-rM) \cap (A+(1-r)\operatorname{int} M) = \emptyset$, and, thus, $-c_0 \notin A + \operatorname{int} M$. Denoting $a = c = -c_0$, we obtain the inclusions we needed to prove. Let, at last, $\rho_M(C_1,A) < 1-r$. As function $t \mapsto \rho_M(C_t,A)$ is continuous and

$\rho_M(C_0,A) = \rho_M(C,A) \geqslant 1-r > 0$, there exists $\tau \in (0,1]$ such that $0 < \rho_M(C_\tau,A) < 1-r$. Note that $C_\tau \in \mathscr{SC}(-rM)$. In view of the assertion, proven in part a), there exist $a,c \in E$ such that

$$\operatorname{int} C_\tau \subset c - \operatorname{int} rM \subset a - \operatorname{int} M \subset E \setminus A.$$

As $C \subset C_\tau$, we obtain the inclusions to be proven.

2) Let the assumptions of item 2) of Theorem 2.1 hold. If $\rho_M(C,A) > 0$, then the desired result was obtained in the previous part. Therefore, let us suppose that $\rho_M(C,A) = 0$. Lemma 3.4 implies that $A \cap C$ consists of a single element. Without loss of generality, let us consider that $A \cap C = \{0\}$. Lemma 5.1 implies that $T(0,A) \cap \operatorname{int} C = \emptyset$. According to Lemma 4.1, the cone $T(0,A)$ is convex. The Hahn–Banach separation theorem yields a non-zero functional $p \in E^*$ such that $s(p,T(0,A)) \leqslant -s(-p,C)$. Using equality $A \cap C = \{0\}$ we get $s(p,T(0,A)) = 0$ and $s(-p,C) = 0$. According to Lemma 5.6(1) the inequality $s(p,M) < +\infty$ holds, i.e. $p \in \operatorname{dom} s(\cdot,M)$. Lemma 5.3(2) implies the inclusion $p \in \operatorname{int}\operatorname{dom} s(\cdot,M)$. Using Lemma 5.4 we obtain the reflexivity of space E. Thus the function $s(\cdot,M)$ has a non-empty subdifferential at p. This means that there exists a vector $z \in \partial M$ such that $\langle p,z\rangle = s(p,M)$. According to Lemma 5.6(2) we have $0 \in P_M(z,A)$. Therefore

$$z - \operatorname{int} M \subset E \setminus A. \tag{6.1}$$

As $C \in \mathscr{SC}(-rM)$, there exists a set $C_1 \subset E$ such that $C + C_1 = -rM$. As $z \in M$, $\langle p,z\rangle = s(p,M)$, and the set M is strictly convex then, according to equality (5.18), we have $\operatorname{Exp}(p,M) = \{z\}$, i.e. $\operatorname{Exp}(-p,-rM) = \{-rz\}$. Thus, by Remark 5.1 there exists a single pair of points $c_0 \in \operatorname{Exp}(-p,C)$, $c_1 \in \operatorname{Exp}(-p,C_1)$ such that $c_0 + c_1 = -rz$. As $0 \in C$ and $s(-p,C) = 0$, then $0 \in \operatorname{Exp}(-p,C)$. Therefore, $c_0 = 0$. Thus, $-rz = c_0 + c_1 = c_1 \in C_1$, and, therefore, $C - rz \subset C + C_1 = -rM$. Thus, using inclusions (6.1) and the fact that $z \in M$, we obtain that

$$\operatorname{int} C \subset rz - \operatorname{int} rM \subset z - \operatorname{int} M \subset E \setminus A.$$

Denoting $c = rz$, $a = z$ we obtain the desired statement.

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