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WEIGHTED COMPOSITION OPERATOR ON TWO–DIMENSIONAL LORENTZ SPACES

RENÉ ERLÍN CASTILLO AND HÉCTOR CAMILO CHAPARRO

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Abstract. The boundedness, compactness, closed range and invertibility of the weighted composition operator on two-dimensional Lorentz spaces are characterized.

1. Introduction

1.1. Some historical background and definitions

Let f be a complex-valued measurable function defined on a σ-finite measure space $(X,\mathscr{A},\mu)$. For $\lambda \geqslant 0$, $D_f(\lambda)$, the distribution function of f, is defined as

$$D_f(\lambda)=\mu\left(\{x\in X:|f(x)|>\lambda\}\right). \tag{1}$$

Observe that D_f depends only on the absolute value $|f|$ of the function f and D_f may take the value $+\infty$.

The distribution function D_f provides information about the size of f, but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from (1) that D_f is a decreasing function of λ (not strictly necessarily) and continuous from the right. For more details on distribution function see [10, 20, 24].

By f^* we mean the non-increasing rearrangement of f given as

$$f^*(t)=\inf\{\lambda>0:D_f(\lambda)\leqslant t\},\quad t\geqslant 0, \tag{2}$$

where we use the convention that $\inf\emptyset=\infty$. f^* is decreasing and right-continuous. Notice that

$$f^*(0)=\inf\{\lambda>0:D_f(\lambda)\leqslant 0\}=\|f\|_\infty,$$

since

$$\|f\|_\infty=\inf\{\alpha\geqslant 0:\mu(\{x\in X:|f(x)|>\alpha\})=0\}.$$

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Also observe that if D_f is strictly decreasing, then

$$f^*(D_f(t)) = \inf\{\lambda > 0 : D_f(\lambda) \leqslant D_(f)t\} = t.$$

This fact demonstrates that f^* is the inverse function of the distribution function D_f. Let $\mathscr{F}(X,\mathscr{A})$ denote the set of all $\mathscr{A}$-measurable functions on X. Let $(X,\mathscr{A}_0,\mu)$ and $(Y,\mathscr{A}_1,\nu)$ be two measure spaces.

Two functions $f \in \mathscr{F}(X,\mathscr{A}_0)$ and $g \in \mathscr{F}(X,\mathscr{A}_1)$ are said to be equimeasurable if they have the same distribution function, that is, if

$$\mu(\{x \in X : |f(x)| > \lambda\}) = \nu(\{y \in Y : |g(y)| > \lambda\}), \quad \text{for all } \lambda \geqslant 0. \tag{3}$$

So then there exists only one right-continuous decreasing function f^* equimeasurable with f. Hence the decreasing rearrangement is unique.

Decreasing rearrangements of functions were introduced by Hardy and Littlewood [26]; the authors attribute their motivation to understanding cricket averages.

One of the most important properties of f^* is that

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p} = \left(\int_0^\infty (f^*(t))^p \, dt\right)^{1/p},$$

which is obtained from the fact that f and f^* are equimeasurable. This allows us to study L_p spaces via decreasing reordering. In this way, the Lorentz spaces $\Lambda^p(w)$ are the spaces of all functions $f : \mathbb{R}^n \to \mathbb{C}$ for which

$$\|f\|_{\Lambda^p(w)} := \left(\int_0^\infty (f^*(t))^p \, w(t) \, dt\right)^{1/p}$$

is finite. Here w is a weight in $\mathbb{R}^+$ and $0 < p < \infty$.

Lorentz spaces were introduced by G. G. Lorentz in [32, 33] as a generalization of classical Lebesgue spaces L_p, and have become a standard tool in mathematical analysis, cf. [2, 10, 12, 14, 15, 18, 21, 19, 24].

The spaces $L^{p,q}$ are defined to be $\Lambda^p(w)$ with $w(t) = \frac{q}{p} t^{q/p-1}$.

In [27], Hunt did a general treatment of the $L^{p,q}$ spaces. Elementary properties, topological properties, interpolation theorems and some applications were studied there. The $L^{p,q}$ spaces play a central role in the study of Banach function spaces. Oftentimes, the methods used to investigate the $L^{p,q}$ spaces are useful for obtaining results for more generalized Banach function spaces. And results for the $L^{p,q}$ spaces often have natural analogues in the more generalized settings.

1.2. Multiplication operators

If we denote (Ω,Σ,μ) for a σ-finite and complete measure space and $L_0(\Omega)$ is the linear space of all classes of Σ-measurable functions on Ω, then every function $u : \Omega \to \mathbb{R}$ measurable on Ω allows us to define a linear transformation which assigns to every $f \in L_0(\Omega)$, the function $M_u(f) \in L_0(\Omega)$ defined by

$$M_u(f)(t) = u(t) \cdot f(t), \;\; t \in \Omega, \;\; f \in L_0(\Omega). \tag{4}$$

In the case in which normed and complete subspaces of $L_0(\Omega)$ are considered as the domain of M_u, this operator will be called multiplication operator induced by the symbol u. These operators have received considerable attention in the last years, specially in the L_p spaces and play an important role in the study of operators in Hilbert spaces.

Multiplication operators generalize the notion of operator given by a diagonal matrix. More precisely, one of the results of operator theory is a spectral theorem, which states that every self-adjoint operator on a Hilbert space is unitarily equivalent to a multiplication operator on an L_2 space.

The basic properties of the multiplication operator on spaces of measurable functions have been studied by many mathematicians. Among them we can name Abrahamese [1] (1978), Halmos [25] (1961), Axler [5] (1982), Takagi [37] (1993), Takagi and Yokouchi [38] (1999), Komal and Gupta [28] (2001), Arora, Datt and Verma [3] (2006), Castillo, León and Trousselot [18] (2009), Douglas [23] (2012), among others. Notably, Castillo, Ramos and Salas in [21] (2014), studied the properties of the multiplication operator M_u in Köthe spaces. The problems studied about the multiplication operator on those spaces are the following:

What are the properties required on the symbol u for the multiplication operator $M_u : X \to Y$, with X and Y Banach subspaces of $L_0(\Omega)$ to be continuous, compact, Fredholm, and have finite or closed range?

It is also of some interest to try to give a formula of the essential norm of M_u in terms of the symbol u.

1.3. Composition operators

Let $(X,\mathscr{A},\mu)$ be a σ-finite complete measure space and let $T : X \to X$ be a measurable transformation, that is, $T^{-1}(A) \in \mathscr{A}$ for any $A \in \mathscr{A}$.

If $\mu\left(T^{-1}(A)\right) = 0$ for all $A \in \mathscr{A}$ with $\mu(A) = 0$, then T is said to be nonsingular. This condition means that the measure $\mu \circ T^{-1}$, defined by $\mu \circ T^{-1}(A) = \mu\left(T^{-1}(A)\right)$ for $A \in \mathscr{A}$ is absolutely continuous with respect to μ (this is usually denoted $\mu \circ T^{-1} \ll \mu$). Then the Radon-Nikodym theorem ensures the existence of a non-negative locally integrable function f_T on X such that

$$\mu \circ T^{-1}(A) = \int_A f_T \, d\mu \quad \text{for } A \in \mathscr{A}.$$

Any measurable nonsingular transformation T induces a linear operator (composition operator) C_T from $\mathfrak{F}(X,\mathscr{A},\mu)$ into itself defined by

$$C_T(f)(x) = f(T(x)), \ \ x \in X, \ \ f \in \mathfrak{F}(X,\mathscr{A},\mu),$$

where $\mathfrak{F}(X,\mathscr{A},\mu)$ denotes the linear space of all equivalence classes of $\mathscr{A}$-measurable functions on X, where we identify any two functions that are equal μ-almost everywhere on X.

Here the nonsingularty of T guarantees that the operator C_T is well defined as a mapping of equivalence classes of functions into itself since $f = g$ μ-a.e. implies $C_T(f) = C_T(g)$ μ-a.e.

The first apparition of a composition transformation was in 1871 in a paper of Schrljeder, where it is asked to find a function f and a number α such that

$$(f \circ T)(z) = \alpha f(z),$$

for every z, in a suitable domain, if the function T is given. A solution was given by Koenigs in 1884. In 1925, this operators were employed in the Littlewood subordination theory. In early 1931 Koopman used the composition operators to study problems of mathematical physics, specially classical mechanics. In those days, these operators were known as substitutes operators. The systematic study of composition operators was initiated by Nordgren in 1968. After that, the study of composition operators has been extended in several directions by many mathematicians. For more details about composition operators in spaces of measurable functions, see Singh and Kumar [35] (1977), Kumar [30] (1980), Komal and Pathania [29] (1991), Takagi and Yokouchi [38] (1999), Cui, Hudzik, Kumar and Maligranda [22] (2004), Arora, Datt and Verma [4] (2007), among others.

In recent years, R. E. Castillo and other authors have done studies on spaces of functions and operator theory, as is shown in [16, 21, 17], in which they have studied some properties of multiplication and composition operator on Bloch spaces and Köthe spaces. In [19], R. E. Castillo, F. Vallejo and J.C. Ramos-Fernández did a remarkable study of the multiplication and composition operators in Weak L_p spaces. In [14] we studied the composition operator in Orlicz-Lorentz spaces. In [15] we studied the multiplication operator in Orlicz-Lorentz spaces.

1.4. Weighted composition operator

Now we talk about a more general operator which encapsulates the classical multiplication and composition operators.

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, $T : X \to X$ be a measurable transformation (i.e. $T^{-1}(A) \in \mathscr{A}$ for each $A \in \mathscr{A}$) and non-singular (i.e. $\mu\left(T^{-1}(A)\right) = 0$ for all $A \in \mathscr{A}$ with $\mu(A) = 0$, which means that μT^{-1} is absolutely continuous with respect to μ $(\mu T^{-1} \ll \mu))$ and $\mu : X \to \mathbb{C}$ be a measurable function. The linear transformation $W_{u,T}$ is defined as follows:

$$\begin{aligned} W_{u,T} : \mathscr{F}(X, \mathscr{A}) &\to \mathscr{F}(X, \mathscr{A}) \\ f &\mapsto W_{u,T}(f) = u \circ T \cdot f \circ T, \end{aligned}$$

where

$$\begin{aligned} W_{u,T} : X &\to \mathbb{C} \\ x &\mapsto (W_{u,T}(f))(x) = u(T(x)) \cdot f(T(x)). \end{aligned}$$

If the operator $W_{u,T}$ is bounded and has range in $\Lambda_2^p(w)$, then it is called the weighted composition operator on $\Lambda_2^p(w)$.

The reader may note that this operator generalize the multiplication and composition operators defined in Sections 1.2 and 1.3, as is shown below:

1. If $u = 1$, then $W_{u,T} = W_{1,T} = C_T : f \mapsto f \circ T$ is called the composition operator induced by T.
2. If $T = I_X$, identity on X, then $W_{u,T} = W_{u,I_X} = M_u : f \mapsto u \cdot f$ is called the multiplication operator induced by u.

The aim of this paper is to study the compactness, boundedness and closed range of the weighted composition operator defined on the space $\Lambda_2^p(w)$.

1.5. Two-dimensional decreasing rearrangement

Since many operations with functions defined on function spaces are iterative, C. J. Neugebauer suggested that it should be possible to obtain multivariate rearrangements by such a process. For simplicity, we are going to reduce the definitions to the two-dimensional case (the definitions for higher dimensions are analogous). Basically, the multidimensional rearrangement can be obtained as an iterative process. More precisely, if $f : \mathbb{R}^2 \to \mathbb{R}$ is a function and we take $f_x(y) = f(x,y)$, then the two-dimensional rearrangement of f may be obtained in the following way

$$\tilde{f}(s,t) = (f_x^*(\cdot,t))_y^*(s).$$

That is, we first rearrange with respect to y (keeping x fixed) to obtain a function which depends on x and t. Then, this new function is rearranged with respect to x (keeping t fixed) to finally obtain the function $\tilde{f}$. The order in which the reordering takes place is very important, because, in general, we do not get the same function if we first rearrange with respect to x and then with respect to y, we show this in Example 3. This is a huge difference with respect to the classical one-variable decreasing rearrangement defined in (2), which is unique. See [11] for some related work.

In [8] there is another way to obtain the multidimensional rearrangement. There, the authors define the decreasing rearrangement E^* of a set E and use this and the layer cake formula to define the multidimensional rearrangement of a function f as

$$f_2^*(s,t) = \int_0^\infty \chi_{\{|f|>\lambda\}^*}(s,t)\, d\lambda.$$

Note that Blozinski defined the multivariate rearrangement via an iterative process, and Barza-Persson-Soria defined the multivariate rearrangement by means of the level sets of the function (using the layer cake formula). Although, at first, these definitions look different, they lead to the same result, i.e., $f_2^* = \tilde{f}$.

The two-dimensional Lorentz space $\Lambda_2^p(w)$ is the space of all functions f for which the norm

$$\|f\|_{\Lambda_2^p(w)} := \left(\int_{\mathbb{R}_+^2} (f_2^*(x))^p\, w(x)\, dx \right)^{1/p}$$

is finite. Here w is a nonnegative, locally integrable function on $\mathbb{R}_+^2$, not identically 0.

One of the reasons to study the space $\Lambda_2^p(w)$ is that it is the standard space to consider multidimensional analogs of classical inequalities: Hardy's inequality, Chebyshev's inequality, embeddings for weighted Lorentz spaces, etc. (see [2], [6], [7],[9], [13], [34], [36].)

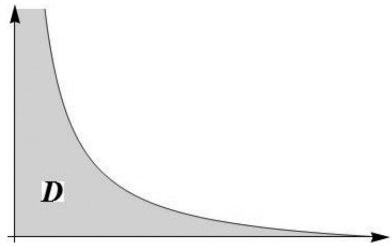

Figure 1: *The graph of a decreasing set.*

2. Two-dimensional Lorentz spaces

For the sake of completeness and convenience of the reader, we give some definitions and results that can be found in [8]. Besides, we include some graphics to illustrate some of the concepts.

DEFINITION 1. We say that a set $D \subset \mathbb{R}^2_+$ is decreasing if the function χ_D is decreasing on each variable.

Throughout this paper, $m_1(\cdot)$ will denote the Lebesgue measure on $\mathbb{R}$ and $m_2(\cdot)$ will denote the Lebesgue measure on $\mathbb{R}^2$.

DEFINITION 2. Let $E \subset \mathbb{R}^2$ and $\varphi_E(x) = m_1(E_x) = m_1(\{y \in \mathbb{R} : (x,y) \in E\})$, $x \in \mathbb{R}$. Let the function φ_E^*, defined by

$$\begin{aligned} \varphi_E^*(s) &= \inf\{\lambda : m_1(\{x \in \mathbb{R} : \varphi_E(x) > \lambda\}) \leqslant s\} \\ &= \inf\{\lambda : D_{\varphi_E}(\lambda) \leqslant s\} \quad (s \geqslant 0.) \end{aligned}$$

φ_E^* is the usual decreasing rearrangement of φ_E (see [10]). Then, the two-dimensional decreasing rearrangement of the set E is

$$E^* = \{(s,t) \in \mathbb{R}^2_+ : 0 < t < \varphi_E^*(s)\}.$$

EXAMPLE 1. Take

$$E = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant r^2\} = \{(x,y) \in \mathbb{R}^2 : -\sqrt{r^2 - x^2} \leqslant y \leqslant \sqrt{r^2 - x^2}\}.$$

Then

$$\begin{aligned} \varphi_E(x) &= m_1\left(\{y \in \mathbb{R} : -\sqrt{r^2-x^2} \leqslant y \leqslant \sqrt{r^2-x^2}\}\right) \\ &= \begin{cases} \sqrt{(\sqrt{r^2-x^2} + \sqrt{r^2-x^2})^2}, & \text{if } -r \leqslant x \leqslant r \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 2\sqrt{r^2-x^2}, & \text{if } -r \leqslant x \leqslant r \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

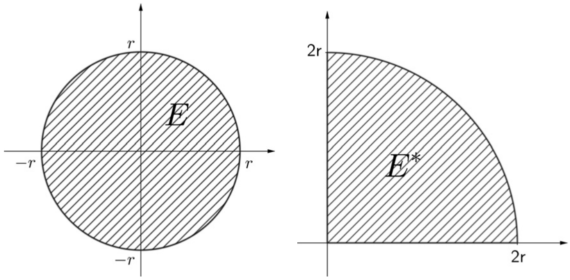

Figure 2: *The graphs of E and E^* in Example 1. Note that $m_2(E) = m_2(E^*)$.*

After some routine calculation, we see that

$$D_{\varphi_E}(\lambda) = \begin{cases} 2\sqrt{r^2 - \frac{\lambda^2}{4}}, & 0 \leqslant \lambda \leqslant 2r \\ 0, & \lambda \geqslant 2, \end{cases}$$

and

$$\varphi_E^*(s) = \begin{cases} \sqrt{4r^2 - s^2}, & 0 \leqslant s \leqslant 2r \\ 0, & s \geqslant 2. \end{cases}$$

This way

$$\begin{aligned} E^* &= \{(s,t) \in \mathbb{R}_+^2 : 0 < t < \varphi_E^*(s)\} \\ &= \{(s,t) \in \mathbb{R}_+^2 : 0 < t < \sqrt{4r^2 - s^2}, 0 < s < 2r\}. \end{aligned}$$

The following definition is inspired by the layer cake formula, see [31] for more information.

DEFINITION 3. The two-dimensional decreasing rearrangement f_2^* for a function f on $\mathbb{R}^2$ is given by

$$f_2^*(x) = \int_0^\infty \chi_{\{|f|>t\}^*}(x)\,dt, \quad x \in \mathbb{R}_+^2.$$

EXAMPLE 2. We calculate the two dimensional decreasing rearrangement for $f(x,y) = x\chi_{[0,1]}(x)\chi_{[0,1]}(y)$. In this case

$$\begin{aligned} E &= \{(x,y) \in \mathbb{R}^2 : |f(x,y)| > t\} \\ &= \{(x,y) \in \mathbb{R}^2 : |x\chi_{[0,1]}(x)\chi_{[0,1]}(y)| > t\} \\ &= \{(x,y) \in [0,1] \times [0,1] : x > t\}. \end{aligned}$$

Thus,

$$\begin{aligned} \varphi_E(x) &= m_1(\{y \in \mathbb{R} : (x,y) \in E\}) = m_1(\{y \in [0,1] : x > t\}) \\ &= (1-t)\chi_{[0,1]}(x). \end{aligned}$$

Then

$$\varphi_E^*(s) = (1-t)\chi_{[0,1)}(s).$$

So,

$$\begin{aligned} f_2^*(s,u) &= \int_0^\infty \chi_{\{|f|>t\}^*}(s,u)\,dt \\ &= \int_0^\infty \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<\varphi_E^*(s)\}}(s,u)\,dt \\ &= \int_0^\infty \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<(1-t)\chi_{[0,1)}(s)\}}(s,u)\,dt. \end{aligned} \tag{5}$$

Now,

$$\begin{aligned} \chi_{\{(s,u)\in\mathbb{R}_+^2 : 0<u<(1-t)\chi_{[0,1)}(s)\}}(s,u) &= \chi_{[0,1)\times\{u\in\mathbb{R}_+ : 0<u<1-t\}}(s,u) \\ &= \chi_{[0,1)\times(0,1-t)}(s,u). \end{aligned} \tag{6}$$

Since

$$\chi_{A\times B}(x,y) = \chi_A(x)\chi_B(y) = \chi_B(y)\chi_A(x) = \chi_{B\times A}(y,x).$$

Returning to (6) we obtain

$$\chi_{[0,1)\times(0,1-t)}(s,u) = \chi_{(0,1-t)\times[0,1)}(u,s).$$

Replacing in (5),

$$\begin{aligned} f_2^*(u,s) &= \int_0^\infty \chi_{(0,1-t)\times[0,1)}(u,s)\,dt \\ &= \int_0^\infty \chi_{(0,1-t)}(u)\chi_{[0,1)}(s)\,dt \\ &= \chi_{[0,1)}(s)\int_0^\infty \chi_{(0,1-t)}(u)\,dt. \end{aligned} \tag{7}$$

Since

$$\begin{aligned} \chi_{(0,1-t)}(u) &= \begin{cases} 1, & \text{if } 0<u<1-t \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } t<u+t<1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } t<u+t \\ 0, & \text{otherwise} \end{cases} \cdot \begin{cases} 1, & \text{if } u+t<1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } 0<u \\ 0, & \text{otherwise} \end{cases} \cdot \begin{cases} 1, & \text{if } t<1-u \\ 0, & \text{otherwise} \end{cases} \\ &= \chi_{(0,\infty)}(u)\chi_{(0,1-u)}(t). \end{aligned} \tag{8}$$

From (7) and (8), we deduce that

$$\begin{aligned} f_2^*(u,s) &= \chi_{[0,1)}(s)\int_0^\infty \chi_{(0,\infty)}(u)\chi_{(0,1-u)}(t)\,dt \\ &= \chi_{[0,1)}(s)\chi_{(0,\infty)}(u)\int_0^\infty \chi_{(0,1-u)}(t)\,dt \\ &= \chi_{[0,1)}(s)\chi_{(0,\infty)}(u)m[(0,\infty)\cap(0,1-u)] \\ &= \chi_{[0,1)}(s)\chi_{(0,\infty)}(u)(1-u)\chi_{(0,1)}(u). \end{aligned}$$

Then

$$f_2^*(u,s) = (1-u)\chi_{(0,1)}(u)\chi_{[0,1)}(s).$$

Just to keep the notation, we write

$$f_2^*(s,t) = (1-s)\chi_{(0,1)}(s)\chi_{[0,1)}(t).$$

REMARK 1. The examples above show that, in general, it is not easy to calculate f_2^*. However, in the next theorem we will show a better way to find it as an iterative rearrangement.

The definition of the two-dimensional reordering is based in a geometrical approach: we obtain the rearrangement of the function by summing its level sets (layer cake formula). In the next theorem, we present a direct way to obtain the two-dimensional rearrangement as an iterative process with respect to the usual rearrangement on each variable (see [11] for some related work).

For clarity to the notation used below, given a function $f(x,y)$ defined on $\mathbb{R}^2$, we write $R_t(x) = (f_x)^{*y}(t)$, where $f_x(y) = f(x,y)$ and $t>0$ (i.e. R_t is the usual rearrangement of the function f_x with respect to the variable y). In a similar way, we write $\tilde{f}(s,t) = (R_t)^{*x}(s)$, $s,t>0$.

THEOREM 1. *If f is a measurable function on $\mathbb{R}^2$, then*

$$f_2^*(s,t) = \tilde{f}(s,t), \quad \forall\, s,t>0.$$

EXAMPLE 3. This example, due to Blozinski (see [11]), shows us that the order in which the rearrangement takes place is fundamental. With this we mean that, in general, we get different functions if the order in which the rearrangement is calculated changes. We present it with all details.

Consider the function $f(x,y) = \sum_{i,j}^{2,3} C(i,j)\chi_{E(i,j)}(x,y)$, where $E(i,j) = [i-1,i)\times[j-1,j)$ and $C(1,1)=1$, $C(1,2)=4$, $C(1,3)=3$, $C(2,1)=5$, $C(2,2)=2$ and $C(2,3)=6$. That is,

$$\begin{aligned} f(x,y) = {} & \chi_{[0,1)\times[0,1)}(x,y) + 4\chi_{[0,1)\times[1,2)}(x,y) + 3\chi_{[0,1)\times[2,3)}(x,y) \\ & + 5\chi_{[1,2)\times[0,1)}(x,y) + 2\chi_{[1,2)\times[0,1)}(x,y) + 6\chi_{[1,2)\times[2,3)}(x,y). \end{aligned}$$

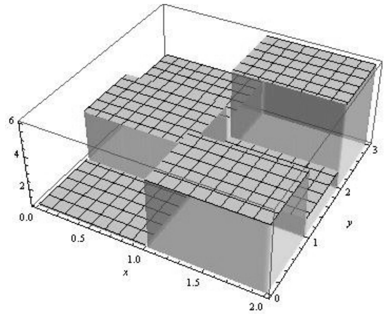

Figure 3: *The graph of the simple function f used in Example 3.*

Let us calculate $f_2^*(s,t) = \tilde{f}(s,t)$. For that purpose, we are going to write f in the following way

$$\begin{aligned} f(x,y) = {} & \chi_{[0,1)}(x)\left[4\chi_{[1,2)}(y) + 3\chi_{[2,3)}(y) + \chi_{[0,1)}(y)\right] \\ & + \chi_{[1,2)}(x)\left[6\chi_{[2,3)}(y) + 5\chi_{[0,1)}(y) + 2\chi_{[1,2)}(y)\right]. \end{aligned}$$

We have

$$R_t(x) = (f_x)^{*y}(t) = \begin{cases} 4\chi_{[0,1)}(t) + 3\chi_{[1,2)}(t) + \chi_{[2,3)}(t), & \text{if } x \in [0,1) \\ 6\chi_{[0,1)}(t) + 5\chi_{[1,2)}(t) + 2\chi_{[2,3)}(t), & \text{if } x \in [1,2). \end{cases}$$

So

$$\begin{aligned} R_t(x) = {} & \left[4\chi_{[0,1)}(t) + 3\chi_{[1,2)}(t) + \chi_{[2,3)}(t)\right]\chi_{[0,1)}(x) \\ & + \left[6\chi_{[0,1)}(t) + 5\chi_{[1,2)}(t) + 2\chi_{[2,3)}(t)\right]\chi_{[1,2)}(x). \end{aligned}$$

And then

$$R_t(x) = \begin{cases} 6\chi_{[1,2)}(x) + 4\chi_{[0,1)}(x), & \text{if } t \in [0,1) \\ 5\chi_{[1,2)}(x) + 3\chi_{[0,1)}(x), & \text{if } t \in [1,2) \\ 2\chi_{[1,2)}(x) + \chi_{[0,1)}(x), & \text{if } t \in [2,3). \end{cases}$$

Now, $\tilde{f}(s,t) = (R_t)^{*x}(s)$, therefore

$$\begin{aligned} (R_t)^{*x}(s) & = \begin{cases} 6\chi_{[0,1)}(s) + 4\chi_{[1,2)}(s), & \text{if } t \in [0,1) \\ 5\chi_{[0,1)}(s) + 3\chi_{[1,2)}(s), & \text{if } t \in [1,2) \\ 2\chi_{[0,1)}(s) + \chi_{[1,2)}(s), & \text{if } t \in [2,3) \end{cases} \\ & = \tilde{f}(s,t) = f_2^*(s,t). \end{aligned}$$

Now, we calculate the iterated rearrangement but in the reverse order. We will use the following notation

$$f_y(x) = f(x,y), \quad G_t(y) = (f_y)^{*x}(t) \quad \text{and} \quad \hat{f}(s,t) = (G_t)^{*y}(s).$$

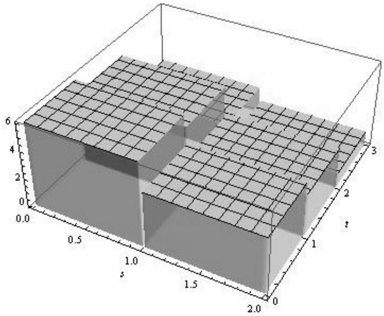

Figure 4: *The graph of* f_2^* *in Example* 3.

We write f as

$$\begin{aligned} f(x,y) = {} & \chi_{[0,1)}(y)\left[5\chi_{[1,2)}(x)+\chi_{[0,1)}(y)\right]+\chi_{[1,2)}(y)\left[4\chi_{[0,1)}(x)+2\chi_{[1,2)}(x)\right] \\ & +\chi_{[2,3)}(y)\left[6\chi_{[1,2)}(x)+3\chi_{[0,1)}(x)\right]. \end{aligned}$$

So

$$G_t(y) = (f_y)^{*x}(t) = \begin{cases} 5\chi_{[0,1)}(t)+\chi_{[1,2)}(t), & \text{if } y\in[0,1) \\ 4\chi_{[0,1)}(t)+2\chi_{[1,2)}(t), & \text{if } y\in[1,2) \\ 6\chi_{[0,1)}(t)+3\chi_{[1,2)}(t), & \text{if } y\in[2,3). \end{cases}$$

We express G_t as

$$\begin{aligned} G_t(y) = {} & \left[5\chi_{[0,1)}(t)+\chi_{[1,2)}(t)\right]\chi_{[0,1)}(y)+\left[4\chi_{[0,1)}(t)+2\chi_{[1,2)}(t)\right]\chi_{[1,2)}(y) \\ & +\left[6\chi_{[0,1)}(t)+3\chi_{[1,2)}(t)\right]\chi_{[2,3)}(y). \end{aligned}$$

Thus

$$G_t(y) = \begin{cases} 6\chi_{[2,3)}(y)+5\chi_{[0,1)}(y)+4\chi_{[1,2)}(y), & \text{if } t\in[0,1) \\ 3\chi_{[2,3)}(y)+2\chi_{[1,2)}(y)+\chi_{[0,1)}(y), & \text{if } t\in[1,2). \end{cases}$$

Since $\hat{f}(s,t) = (G_t)^{*y}(s)$, we have

$$\begin{aligned} (G_t)^{*y}(s) & = \begin{cases} 6\chi_{[0,1)}(s)+5\chi_{[1,2)}(s)+4\chi_{[2,3)}(s), & \text{if } t\in[0,1) \\ 3\chi_{[0,1)}(s)+2\chi_{[1,2)}(s)+\chi_{[2,3)}(s), & \text{if } t\in[1,2) \end{cases} \\ & = \hat{f}(s,t). \end{aligned}$$

Looking at f_2^* and $\hat{f}$, we see that

$$f_2^*(s,t) \neq \hat{f}(s,t).$$

Remember the definition of the classical Lorentz space: If v is a weight on $\mathbb{R}^+$ and $0<p<\infty$,

$$\Lambda^p(v) = \left\{ f:\mathbb{R}^n\to\mathbb{C} : \|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p\, v(t)\,dt\right)^{1/p} < \infty \right\}.$$

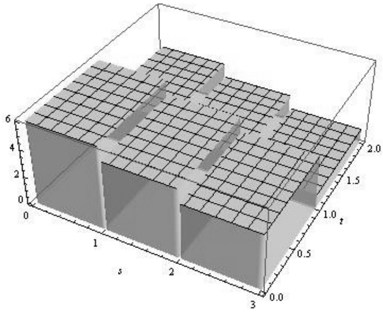

Figure 5: *The graph of* $\hat{f}$ *in Example* 3.

A good exposition on Lorentz spaces may be found in [20].

DEFINITION 4. We say that a measurable function f on $\mathbb{R}^2$ belongs to the two-dimensional Lorentz space $\Lambda_2^p(w)$ if the norm

$$\|f\|_{\Lambda_2^p(w)} := \left(\int_{\mathbb{R}_+^2} (f_2^*(x))^p \, w(x) \, dx \right)^{1/p}, \tag{9}$$

is finite. Here w is a non-negative function, locally integrable over $\mathbb{R}_+^2$, not identically 0.

For more information on the spaces $\Lambda_2^p(w)$ see [8] and the references therein.

3. Weighted composition operator on $\Lambda_2^p(w)$

In this section the boundedness, compactness and closed range of the weighted composition operator on the space $\Lambda_2^p(w)$ are characterized.

DEFINITION 5. Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space, $T : X \to X$ be a measurable transformation (i.e. $T^{-1}(A) \in \mathscr{A}$ for each $A \in \mathscr{A}$) and non-singular (i.e. $\mu\left(T^{-1}(A)\right) = 0$ for all $A \in \mathscr{A}$ with $\mu(A) = 0$, which means that μT^{-1} is absolutely continuous with respect to μ $(\mu T^{-1} \ll \mu))$ and $\mu : X \to \mathbb{C}$ be a measurable function. The linear transformation $W_{u,T}$ is defined as follows:

$$\begin{aligned} W_{u,T} : \mathscr{F}(X,\mathscr{A}) &\to \mathscr{F}(X,\mathscr{A}) \\ f &\mapsto W_{u,T}(f) = u \circ T \cdot f \circ T, \end{aligned}$$

where

$$\begin{aligned} W_{u,T} : X &\to \mathbb{C} \\ x &\mapsto \left(W_{u,T}(f)\right)(x) = u(T(x)) \cdot f(T(x)). \end{aligned}$$

If the operator $W_{u,T}$ is bounded and has range in $\Lambda_2^p(w)$, then it is called the weighted composition operator on $\Lambda_2^p(w)$.

REMARK 2.

1. If $u=1$, then $W_{u,T}=W_{1,T}=C_T : f\mapsto f\circ T$ is called the composition operator induced by T.
2. If $T=I_X$, identity on X, then $W_{u,T}=W_{u,I_X}=M_u : f\mapsto u\cdot f$ is called the multiplication operator induced by u.
3. Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space, $T:X\to X$ be a measurable and non-singular transformation and $u:X\to\mathbb{C}$ be a measurable function, then T and u induce a weighted composition operator that is well defined on $\mathscr{F}(X,\mathscr{A})$.

   Indeed, remember that $\mathscr{F}(X,\mathscr{A})$ is a set of functions classes where two functions belong to a same class if they are equal almost everywhere with respect to μ. That is to say,
   $$f\cong g \Leftrightarrow \mu(\{x\in X : f(x)\neq g(x)\})=0.$$
   Let $f,g\in\mathscr{F}(X,\mathscr{A})$ such that $f\cong g$. Then
   $$\begin{aligned} &\mu(\{x\in X : f(x)\neq g(x)\})=0\\ \Rightarrow\quad &\mu(\{x\in X : (uf)(x)\neq (ug)(x)\})=0. \end{aligned}$$
   (We are assuming that $u\neq 0$, otherwise $M_u=M_0=0$, which is not of interest.) Now,
   $$\begin{aligned} &x_0\in\{x\in X : u(T(x))f(T(x))\neq u(T(x))g(T(x))\}\\ \Leftrightarrow\quad & u(T(x_0))f(T(x_0))\neq u(T(x_0))g(T(x_0))\\ \Leftrightarrow\quad & u(y_0)f(y_0)\neq u(y_0)g(y_0), y_0=T(x_0)\\ \Leftrightarrow\quad & y_0=T(x_0)\in\{x\in X : u(x)f(x)\neq u(x)g(x)\}\\ \Leftrightarrow\quad & x_0\in T^{-1}\left(\{x\in X : (uf)(x)\neq (ug)(x)\}\right). \end{aligned}$$
   Therefore
   $$\begin{aligned} &\mu\left(\{x\in X : u(T(x))f(T(x))\neq u(T(x))g(T(x))\}\right)\\ &\qquad\qquad =\mu\left(T^{-1}\left(\{x\in X : (uf)(x)\neq (ug)(x)\}\right)\right)=0, \end{aligned}$$
   because of the non-singularity of T. So,
   $$\begin{aligned} \left(W_{u,T}f\right)(x)&=u(T(x))f(T(x))\\ &=u(T(x))g(T(x))\\ &=\left(W_{u,T}g\right)(x)\quad \mu-\text{a.e.} \end{aligned}$$
   Consequently, $W_{u,T}$ is well defined on the classes of $\mathscr{F}(X,\mathscr{A})$.

From now on, $(X,\mathscr{A},\mu)=\left(\mathbb{R}^2,\mathscr{B},m_2\right)$, which is a σ-finite measure space.

3.1. Boundedness

THEOREM 2. *Let $u:\mathbb{R}^2\to\mathbb{C}$ be a measurable function. Suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is a non-singular measurable transformation. Also, suppose that there exists a constant $b\geqslant 1$ such that $m_1\left(T_x^{-1}(E)\right)\leqslant bm_1(E_x)$ for all $E\subset\mathbb{R}^2$. Then*

$$W_{u,T}:f\mapsto W_{u,T}f=W_{u,T}(f)=u\circ T\cdot f\circ T,$$

is bounded on $\Lambda_2^p(w)$ if $u\in L_\infty(\mathbb{R}^2)$. Moreover,

$$\|W_{u,T}\|\leqslant b^{1/p}\|u\|_\infty.$$

Besides, if $T_x^{-1}(F)\supset F_x$ for all $F\subset\mathbb{R}^2$, then

$$\|W_{u,T}\|=b^{1/p}\|u\|_\infty.$$

Proof. We are going to use iterated rearrangement, since Theorem 1 ensures that $h_2^*=\tilde{h}$, where $\tilde{h}=\left((h_x)^{*y}\right)^{*x}$ and $h_x(y)=h(x,y)$ (sometimes h_x is called the x-section of h.)

For $(x,y)\in\mathbb{R}^2$, we know that

$$\begin{aligned}(W_{u,T}f)_x&=(u\circ T\cdot f\circ T)_x(y)\\&=(u\circ T\cdot f\circ T)(x,y)\\&=(u\circ T)(x,y)\cdot(f\circ T)(x,y)\\&=u(T(x,y))\cdot f(T(x,y))\\&=u(T_x(y))\cdot f(T_x(y))\\&=(u\circ T_x)(y)\cdot(f\circ T_x)(y).\end{aligned}$$

So,

$$\begin{aligned}D_{(W_{u,T}f)_x}(\lambda)&=m_1\left(\{y\in\mathbb{R}:|(u\circ T\cdot f\circ T)_x(y)|>\lambda\}\right)\\&=m_1\left(\{y\in\mathbb{R}:|u(T_x(y))\cdot f(T_x(y))|>\lambda\}\right)\\&=m_1\left(\left\{(x,y)\in\mathbb{R}^2:|u(T_x(y))\cdot f(T_x(y))|>\lambda\right\}_x\right).\end{aligned}$$

In the other hand, since

$$\begin{aligned}&y_0\in\left\{(x,y)\in\mathbb{R}^2:|u(T_x(y))\cdot f(T_x(y))|>\lambda\right\}_x\\\Leftrightarrow\;&\{y\in\mathbb{R}:|u(T_x(y))\cdot f(T_x(y))|>\lambda\}\\\Leftrightarrow\;&|u(T_x(y_0))f(T_x(y_0))|>\lambda\\\Leftrightarrow\;&|u(z_0)f(z_0)|>\lambda,z_0=T_x(y_0)\\\Leftrightarrow\;&z_0=T_x(y_0)\in\left\{(x,y)\in\mathbb{R}^2:|u(x,y)f(x,y)|>\lambda\right\}\\\Leftrightarrow\;&y_0\in T_x^{-1}\left(\left\{(x,y)\in\mathbb{R}^2:|u(x,y)f(x,y)|>\lambda\right\}\right)\\\Leftrightarrow\;&y_0\in\left(T^{-1}\left(\left\{(x,y)\in\mathbb{R}^2:|u(x,y)f(x,y)|>\lambda\right\}\right)\right)_x.\end{aligned}$$

Then

$$\begin{aligned}\{y \in \mathbb{R} : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\} &= T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)f(x,y)| > \lambda\right\}\right)\\ &= \left(T^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)f(x,y)| > \lambda\right\}\right)\right)_x.\end{aligned}$$

Hence

$$\begin{aligned}D_{\left(W_{u,T}f\right)_x} &= m_1\left(\{y \in \mathbb{R} : |u(T_x(y)) \cdot f(T_x(y))| > \lambda\}\right)\\ &= m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)f(x,y)| > \lambda\right\}\right)\right)\\ &= m_1\left(\left(T^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)f(x,y)| > \lambda\right\}\right)\right)_x\right). \qquad (10)\end{aligned}$$

Next, since $|u(x,y)| \leqslant \|u\|_\infty \ \forall\, (x,y) \in \mathbb{R}^2$, it holds that, in particular,

$$\begin{aligned}&(x_0,y_0) \in \left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}^{\complement}\\ &\Rightarrow \|u\|_\infty |f(x_0,y_0)| \leqslant \lambda\\ &\Rightarrow |u(x_0,y_0)||f(x_0,y_0)| \leqslant \lambda\\ &\Rightarrow (x_0,y_0) \in \left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\}^{\complement}.\end{aligned}$$

Therefore

$$\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\} \subset \left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\},$$

then

$$T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\}\right) \subset T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right),$$

and so

$$\begin{aligned}&m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : |u(x,y)||f(x,y)| > \lambda\right\}\right)\right)\\ &\qquad \leqslant m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right)\right).\end{aligned}$$

Thus,

$$D_{\left(W_{u,T}f\right)_x}(\lambda) \leqslant m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right)\right). \qquad (11)$$

Let $E = \left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\} \subset \mathbb{R}^2$. By hypothesis we know that

$$m_1\left(T_x^{-1}(E)\right) \leqslant b m_1\left(E_x\right). \qquad (12)$$

Going back to (11),

$$\begin{aligned}D_{\left(W_{u,T}f\right)_x}(\lambda) &\leqslant m_1\left(T_x^{-1}\left(\left\{(x,y) \in \mathbb{R}^2 : \|u\|_\infty |f(x,y)| > \lambda\right\}\right)\right)\\ &\leqslant b m_1\left(\{y \in \mathbb{R} : \|u\|_\infty |f(x,y)| > \lambda\}\right)\\ &= b D_{\|u\|_\infty f_x}(\lambda).\end{aligned}$$

Now, for any $t \geqslant 0$,

$$D_{(W_{u,T}f)_x}(\lambda) \leqslant bD_{\|u\|_\infty f_x}(\lambda)$$
$$\Leftrightarrow \quad \left\{\lambda > 0 : D_{\|u\|_\infty f_x}(\lambda) \leqslant \frac{t}{b}\right\} \subset \left\{\lambda > 0 : D_{(W_{u,T}f)_x}(\lambda) \leqslant t\right\}.$$

Which implies that

$$\begin{aligned}
(W_{u,T}f)_x^*(t) &= \inf\left\{\lambda > 0 : D_{(W_{u,T}f)_x}(\lambda) \leqslant t\right\} \\
&\leqslant \inf\left\{\lambda > 0 : D_{\|u\|_\infty f_x}(\lambda) \leqslant \frac{t}{b}\right\} \\
&= \inf\left\{\lambda > 0 : m_1\left(\{y \in \mathbb{R} : \|u\|_\infty |f_x(y)| > \lambda\}\right) \leqslant \frac{t}{b}\right\} \\
&= \inf\left\{\lambda > 0 : m_1\left(\left\{y \in \mathbb{R} : |f_x(y)| > \frac{\lambda}{\|u\|_\infty}\right\}\right) \leqslant \frac{t}{b}\right\}, \quad r = \frac{\lambda}{\|u\|_\infty} \\
&= \inf\left\{r\|u\|_\infty > 0 : m_1\left(\{y \in \mathbb{R} : |f_x(y)| > r\}\right) \leqslant \frac{t}{b}\right\} \\
&= \|u\|_\infty \inf\left\{r > 0 : m_1\left(\{y \in \mathbb{R} : |f_x(y)| > r\}\right) \leqslant \frac{t}{b}\right\} \\
&= \|u\|_\infty \inf\left\{r > 0 : D_{f_x}(r) \leqslant \frac{t}{b}\right\} \\
&= \|u\|_\infty (f_x)^*\left(\frac{t}{b}\right).
\end{aligned}$$

Now, using Theorem 1, we rearrange with respect to x to obtain

$$(W_{u,T}f)_2^*(s,t) \leqslant \|u\|_\infty f_2^*\left(s, \frac{t}{b}\right),$$

then

$$\|W_{u,T}f\|_{\Lambda_2^p(w)} \leqslant b^{1/p}\|u\|_\infty \|f\|_{\Lambda_2^p(w)},$$

from where

$$\|W_{u,T}f\| \leqslant b^{1/p}\|u\|_\infty. \tag{13}$$

Now, let us see under what conditions $\|W_{u,T}f\| = b^{1/p}\|u\|_\infty$.

Let $B_\varepsilon = \left\{x \in \mathbb{R}^2 : |u(T(x))| \geqslant b^{1/p}\|u\|_\infty - \varepsilon\right\}$ (note that $m_2(B_\varepsilon) > 0$). Then,

$$|u(T(x,y))\chi_{B_\varepsilon}(T(x,y))| \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\chi_{B_\varepsilon}(T(x,y)). \tag{14}$$

On the other hand, for a fixed x,

$$\chi_{B_\varepsilon}(T(x,y)) = \chi_{B_\varepsilon}(T_x(y)) = \begin{cases} 1, & \text{if } T_x(y) \in B_\varepsilon \\ 0, & \text{if } T_x(y) \notin B_\varepsilon \end{cases} = \begin{cases} 1, & \text{if } y \in T_x^{-1}(B_\varepsilon) \\ 0, & \text{if } y \notin T_x^{-1}(B_\varepsilon) \end{cases} = \chi_{T_x^{-1}(B_\varepsilon)}(y).$$

And

$$\chi_{B_\varepsilon}(x,y) = (\chi_{B_\varepsilon})_x(y) = \begin{cases} 1, & \text{if } (x,y) \in B_\varepsilon \\ 0, & \text{if } (x,y) \notin B_\varepsilon \end{cases} = \begin{cases} 1, & \text{if } y \in (B_\varepsilon)_x \\ 0, & \text{if } y \notin (B_\varepsilon)_x \end{cases} = \chi_{(B_\varepsilon)_x}(y).$$

By hypothesis, $T_x^{-1}(B_\varepsilon) \supset (B_\varepsilon)_x$, then

$$\chi_{B_\varepsilon}(T(x,y)) = \chi_{T_x^{-1}(B_\varepsilon)}(y) \geqslant \chi_{(B_\varepsilon)_x}(y) = \chi_{B_\varepsilon}(x,y).$$

Going back to (14), we have

$$\begin{aligned} & |u(T(x,y))\chi_{B_\varepsilon}(T(x,y))| \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\chi_{B_\varepsilon}(x,y) \\ \Rightarrow \quad & |u(T_x(y))\chi_{B_\varepsilon}(T_x(y))| \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)(\chi_{B_\varepsilon})_x(y) \\ \Rightarrow \quad & \left|(W_{u,T}\chi_{B_\varepsilon})_x(y)\right| \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)(\chi_{B_\varepsilon})_x(y). \end{aligned}$$

Next we use Theorem 1 in order to calculate the two-dimensional rearrangement in an iterated way, so

$$\begin{aligned} & (W_{u,T}\chi_{B_\varepsilon})_x^{*y}(t) \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)(\chi_{B_\varepsilon})_x^{*y}(t) \\ \Rightarrow \quad & (W_{u,T}\chi_{B_\varepsilon})_2^{*}(s,t) \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)(\chi_{B_\varepsilon})_2^{*}(s,t) \\ \Rightarrow \quad & \|W_{u,T}\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)} \geqslant \left(b^{1/p}\|u\|_\infty - \varepsilon\right)\|\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)} \\ \Rightarrow \quad & \frac{\|W_{u,T}\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)}}{\|\chi_{B_\varepsilon}\|_{\Lambda_2^p(w)}} \geqslant b^{1/p}\|u\|_\infty - \varepsilon \\ \Rightarrow \quad & \|W_{u,T}\| \geqslant b^{1/p}\|u\|_\infty - \varepsilon. \end{aligned}$$

Since the above inequality is valid for all $\varepsilon > 0$, then

$$\|W_{u,T}\| \geqslant b^{1/p}\|u\|_\infty. \tag{15}$$

Combining (13) and (15), it follows that

$$\|W_{u,T}\| = b^{1/p}\|u\|_\infty. \quad \square$$

THEOREM 3. *Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation such that $T(A_\varepsilon) \subset A_\varepsilon$ for all $\varepsilon > 0$, where $A_\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : |u(x,y)| > \varepsilon\right\}$. If $W_{u,T}$ is bounded on $\Lambda_2^p(w)$, then $u \in L_\infty(\mathbb{R}^2)$.*

Proof. Suppose that $W_{u,T}$ is bounded on $\Lambda_2^p(w)$ and $u \notin L_\infty(\mathbb{R}^2)$. Let

$$A_n = \left\{(x,y) \in \mathbb{R}^2 : |u(x,y)| > n\right\}.$$

Since $u \notin L_\infty(\mathbb{R}^2)$, we have that $m_2(A_n) > 0 \ \forall\, n \in \mathbb{N}$.

Besides, $T(A_n) \subset A_n$ implies $A_n \subset T^{-1}(A_n)$, then $(A_n)_x \subset \left(T^{-1}(A_n)\right)_x$ and therefore $\chi_{(A_n)_x}(y) \leqslant \chi_{\left(T^{-1}(A_n)\right)_x}(y)$ for all $y \in \mathbb{R}$. Let $\lambda > 0$ and $y \in (A_n)_x$. We have

$$\left|\chi_{(A_n)_x}(y)\right| > \lambda \Rightarrow \left|\chi_{\left(T^{-1}(A_n)\right)_x}(y)\right| > \lambda.$$

Also,

$$y \in (A_n)_x \Rightarrow (x,y) \in A_n \Rightarrow T(x,y) \in A_n \Rightarrow |u(T(x,y))| > n \Rightarrow |u(T_x(y)) > n|,$$

then we have

$$\left|u(T_x(y))\,\chi_{\left(T^{-1}(A_n)\right)_x}(y)\right| > n\lambda.$$

Thereupon

$$\begin{aligned}
&\left\{y \in \mathbb{R} : \left|\chi_{(A_n)_x}(y)\right| > \lambda\right\} \subset \left\{y \in \mathbb{R} : \left|u(T_x(y))\,\chi_{\left(T^{-1}(A_n)\right)_x}(y)\right| > n\lambda\right\} \\
\Rightarrow\ & m_1\left(\left\{y \in \mathbb{R} : \left|\chi_{(A_n)_x}(y)\right| > \lambda\right\}\right) \leqslant m_1\left(\left\{y \in \mathbb{R} : \left|u(T_x(y))\,\chi_{\left(T^{-1}(A_n)\right)_x}(y)\right| > n\lambda\right\}\right).
\end{aligned} \tag{16}$$

Since

$$\begin{aligned}
\chi_{\left(T^{-1}(A_n)\right)_x}(y) &= \begin{cases} 1, & \text{if } y \in \left(T^{-1}(A_n)\right)_x \\ 0, & \text{if } y \notin \left(T^{-1}(A_n)\right)_x \end{cases} = \begin{cases} 1, & \text{if } y \in T_x^{-1}(A_n) \\ 0, & \text{if } y \notin T_x^{-1}(A_n) \end{cases} \\
&= \begin{cases} 1, & \text{if } T_x(y) \in A_n \\ 0, & \text{if } T_x(y) \in A_n \end{cases} = \chi_{A_n}(T_x(y)) = (\chi_{A_n} \circ T_x)(y).
\end{aligned}$$

Returning to (16),

$$m_1\left(\left\{y \in \mathbb{R} : \left|\chi_{(A_n)_x}(y)\right| > \lambda\right\}\right) \leqslant m_1\left(\left\{y \in \mathbb{R} : |u(T_x(y))\,\chi_{A_n}(T_x(y))| > n\lambda\right\}\right).$$

In consequence

$$m_1\left(\left\{y \in \mathbb{R} : \left|\chi_{(A_n)_x}(y)\right| > \lambda\right\}\right) \leqslant m_1\left(\left\{y \in \mathbb{R} : |(W_{u,T_x}\chi_{A_n})(y)| > n\lambda\right\}\right).$$

Therefore,

$$\begin{aligned}
& & D_{\chi_{(A_n)_x}}(\lambda) &\leqslant D_{W_{u,T_x}\chi_{A_n}}(n\lambda) \\
&\Rightarrow & D_{\chi_{(A_n)_x}}(\lambda) &\leqslant D_{\frac{1}{n}W_{u,T_x}\chi_{A_n}}(\lambda) \\
&\Rightarrow & \left\{\lambda > 0 : D_{\frac{1}{n}W_{u,T_x}\chi_{A_n}}(\lambda) \leqslant t\right\} &\subset \left\{\lambda > 0 : D_{\chi_{(A_n)_x}}(\lambda) \leqslant t\right\} \\
&\Rightarrow & \inf\left\{\lambda > 0 : D_{\chi_{(A_n)_x}}(\lambda) \leqslant t\right\} &\leqslant \inf\left\{\lambda > 0 : D_{\frac{1}{n}W_{u,T_x}\chi_{A_n}}(\lambda) \leqslant t\right\} \\
&\Rightarrow & \left[\chi_{(A_n)_x}\right]^*(t) &\leqslant \left[\frac{1}{n}W_{u,T_x}\chi_{A_n}\right]^*(t) \\
&\Rightarrow & n\left[\chi_{(A_n)_x}\right]^*(t) &\leqslant [W_{u,T_x}\chi_{A_n}]^*(t).
\end{aligned} \tag{17}$$

Note that

$$\chi_{(A_n)_x}(y) = \begin{cases} 1, & \text{if } y \in (A_n)_x \\ 0, & \text{if } y \notin (A_n)_x \end{cases} = \begin{cases} 1, & \text{if } (x,y) \in A_n \\ 0, & \text{if } (x,y) \notin A_n \end{cases} = \chi_{A_n}(x,y) = (\chi_{A_n})_x(y).$$

Also

$$\begin{aligned}\left(W_{u,T_x}\chi_{A_n}\right)(y) &= u\left(T_x(y)\right)\chi_{A_n}\left(T_x(y)\right) = u\left(T(x,y)\right)\chi_{A_n}\left(T(x,y)\right)\\ &= \left(W_{u,T}\chi_{A_n}\right)(x,y) = \left(W_{u,T}\chi_{A_n}\right)_x(y).\end{aligned}$$

Returning to (17), we obtain

$$n\left[\left(\chi_{A_n}\right)_x\right]^*(t) \leqslant \left[\left(W_{u,T}\chi_{A_n}\right)_x\right]^*(t).$$

By Theorem 1, we rearrange with respect to x to obtain

$$n\left(\chi_{A_n}\right)_2^*(s,t) \leqslant \left(W_{u,T}\chi_{A_n}\right)_2^*(s,t),$$

which implies

$$n\left\|\chi_{A_n}\right\|_{\Lambda_2^p(w)} \leqslant \left\|W_{u,T}\chi_{A_n}\right\|_{\Lambda_2^p(w)}.$$

We conclude that given $n\in\mathbb{N}$, there exists $\chi_{A_n}\in\Lambda_2^p(w)$ such that

$$\left\|W_{u,T}\chi_{A_n}\right\|_{\Lambda_2^p(w)} > n\left\|\chi_{A_n}\right\|_{\Lambda_2^p(w)}.$$

Hence $W_{u,T}$ is not bounded, which contradicts the hypothesis of the theorem, so $u\in L_\infty(\mathbb{R}^2)$. $\square$

The next result follows as a consequence from the last two theorems.

THEOREM 4. *Let $u:\mathbb{R}^2\to\mathbb{C}$ be a measurable function. Suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is a non-singular measurable transformation that satisfies the following conditions:*

1. *There exists a constant $b\geqslant 1$ such that*

$$m_1\left(T_x^{-1}(E)\right) \leqslant b m_1\left(E_x\right),\ \textit{for all } E\subset\mathbb{R}^2.$$

2. *$T(A_\varepsilon)\subset A_\varepsilon$ for all $\varepsilon>0$, with $A_\varepsilon=\left\{(x,y)\in\mathbb{R}^2:|u(x,y)|>\varepsilon\right\}$.*

Then $W_{u,T}$ is bounded on $\Lambda_2^p(w)$ if and only if $u\in L_\infty(\mathbb{R}^2)$.

3.2. Compactness

In this section compactness of the weighted composition operator on the space $\Lambda_2^p(w)$ is characterized.

THEOREM 5. *Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a non-singular measurable transformation for which there exist constants $b\geqslant 1$ and $\delta>0$ such that $\delta m_1\left(E_x\right)\leqslant m_1\left(T_x^{-1}(E)\right)\leqslant b m_1\left(E_x\right)$ for all $E\subset\mathbb{R}^2$. If $f\in\Lambda_2^p(w)$, then*

$$\alpha\left\|M_u f\right\|_{\Lambda_2^p(w)} \leqslant \left\|W_{u,T} f\right\|_{\Lambda_2^p(w)} \leqslant \beta\left\|M_u f\right\|_{\Lambda_2^p(w)},$$

with $\alpha=\delta^{1/p}$, $\beta=b^{1/p}$ $\forall f\in\Lambda_2^p(w)$.

Proof. Let $E=\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}$ and let $t>0$. By equation (10) and the inequality (13), we have

$$\begin{aligned}
\left[(W_{u,T}f)_x\right]^*(t) &= \inf\{\lambda>0: m_1(\{y\in\mathbb{R}:|u(T_x(y))f(T_x(y))|>\lambda\})\leqslant t\}\\
&= \inf\left\{\lambda>0: m_1\left(\left(T^{-1}\left(\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}\right)\right)_x\right)\leqslant t\right\},\\
&\qquad\text{by (10)}\\
&= \inf\left\{\lambda>0: m_1\left(\left(T^{-1}(E)\right)_x\right)\leqslant t\right\}\\
&\leqslant \inf\{\lambda>0: bm_1(E_x)\leqslant t\},\ \text{by (13)}\\
&= \inf\left\{\lambda>0: bm_1\left(\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}_x\right)\leqslant t\right\}\\
&= \inf\left\{\lambda>0: m_1\left(\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}_x\right)\leqslant \frac{t}{b}\right\}\\
&= \inf\left\{\lambda>0: D_{(uf)_x}(\lambda)\leqslant \frac{t}{b}\right\}\\
&= [(uf)_x]^*\left(\frac{t}{b}\right)\\
&= [(M_uf)_x]^*\left(\frac{t}{b}\right).
\end{aligned}$$

Rearranging with respect to x, we obtain

$$(W_{u,T}f)_2^*(s,t)\leqslant (M_uf)_2^*\left(s,\frac{t}{b}\right).$$

Hence, if $1<p<\infty$, it holds that

$$\|W_{u,T}f\|_{\Lambda_2^p(w)}\leqslant b^{1/p}\|M_uf\|_{\Lambda_2^p(w)}=\beta\|M_uf\|_{\Lambda_2^p(w)}. \tag{18}$$

On the other hand, we know that

$$D_{(W_{u,T}f)_x}(\lambda)=m_1\left(\left(T^{-1}\left(\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}\right)\right)_x\right),$$

and

$$\left[(W_{u,T}f)_x\right]^*(t)=\inf\left\{\lambda>0: m_1\left(\left(T^{-1}\left(\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}\right)\right)_x\right)\leqslant t\right\}.$$

Let $S=\left\{(x,y)\in\mathbb{R}^2:u(x,y)\neq 0\right\}$. From the hypothesis we know that for all $F\in\mathscr{B}, F\subset S$

$$m_1\left(T_x^{-1}(F)\right)\geqslant \delta m_1(F_x).$$

Now, let $G=\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}$ (note that $G\subset S$), then

$$\left\{\lambda>0: m_1\left(T_x^{-1}(G)\right)\leqslant t\right\}\subset\{\lambda>0:\delta m_1(G_x)\leqslant t\},$$

and it holds that

$$\begin{aligned}
\left[(W_{u,T}f)_x\right]^*(t) &= \inf\left\{\lambda>0: m_1\left(T_x^{-1}\left(\left\{(x,y)\in\mathbb{R}^2:|(uf)(x,y)|>\lambda\right\}\right)\right)\leqslant t\right\}\\
&= \inf\left\{\lambda>0: m_1\left(T_x^{-1}(G)\right)\leqslant t\right\}\\
&\geqslant \inf\{\lambda>0:\delta m_1(G_x)\leqslant t\}
\end{aligned}$$

$$\begin{aligned}
&= \inf\left\{\lambda > 0 : \delta m_1\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}_x\right) \leqslant t\right\}\\
&= \inf\left\{\lambda > 0 : m_1\left(\left\{(x,y)\in\mathbb{R}^2 : |(uf)(x,y)| > \lambda\right\}_x\right) \leqslant \frac{t}{\delta}\right\}\\
&= \inf\left\{\lambda > 0 : D_{(uf)_x} \leqslant \frac{t}{\delta}\right\}\\
&= [(uf)_x]^*\left(\frac{t}{\delta}\right)\\
&= [(M_u f)_x]^*\left(\frac{t}{\delta}\right).
\end{aligned}$$

Consequently,

$$\left[(W_{u,T}f)_x\right]^*(t) \geqslant [(M_u f)_x]^*\left(\frac{t}{\delta}\right).$$

Now, we invoke Theorem 1, and rearrange with respect to x, then

$$(W_{u,T}f)_2^*(s,t) \geqslant M_u f_2^*\left(s, \frac{t}{\delta}\right),$$

from where

$$\|W_{u,T}f\|_{\Lambda_2^p(w)} \geqslant \delta^{1/p}\|M_u f\|_{\Lambda_2^p(w)}. \tag{19}$$

Looking at inequalities (18) and (19), it follows that for any $f\in\Lambda_2^p(w)$,

$$\alpha\|M_u f\|_{\Lambda_2^p(w)} \leqslant \|W_{u,T}f\|_{\Lambda_2^p(w)} \leqslant \beta\|M_u f\|_{\Lambda_2^p(w)}. \quad \square \tag{20}$$

LEMMA 1. *Let M_u be a compact operator. For $\varepsilon > 0$, we define*

$$A_\varepsilon(u) = \left\{x\in\mathbb{R}^2 : |u(x,y)| > \varepsilon\right\} \quad \textit{and} \quad L^w_{A_\varepsilon(u)} = \left\{f\chi_{A_\varepsilon(u)} : f\in\Lambda_2^p(w)\right\}.$$

Then $L^w_{A_\varepsilon(u)}$ is an invariant closed subspace of $\Lambda_2^p(w)$ under M_u. Moreover, $M_u|_{L^w_{A_\varepsilon(u)}}$ is a compact operator.

Proof. Suppose that $h,k\in L^w_{A_\varepsilon(u)}$, then $h = f\chi_{A_\varepsilon(u)}$ and $k = g\chi_{A_\varepsilon(u)}$ with $f,g\in\Lambda_2^p(w)$.

For scalars α,β, it holds that

$$\alpha h + \beta k = \alpha\left(f\chi_{A_\varepsilon(u)}\right) + \beta\left(g\chi_{A_\varepsilon(u)}\right) = (\alpha f)\chi_{A_\varepsilon(u)} + (\beta g)\chi_{A_\varepsilon(u)} = (\alpha f + \beta g)\chi_{A_\varepsilon(u)},$$

where $(\alpha f + \beta g)\in\Lambda_2^p(w)$. Hence $\alpha h + \beta k\in L^w_{A_\varepsilon(u)}$ and then $L^w_{A_\varepsilon(u)}$ is a vector subspace of $\Lambda_2^p(w)$.

Besides, if $h\in L^w_{A_\varepsilon(u)}$ with $h = f\chi_{A_\varepsilon(u)}$, $f\in\Lambda_2^p(w)$, then

$$M_u h = uh = u\left(f\chi_{A_\varepsilon(u)}\right) = (uf)\chi_{A_\varepsilon(u)},$$

where $uf\in\Lambda_2^p(w)$. Therefore $M_u h\in L^w_{A_\varepsilon(u)}$, which implies that $L^w_{A_\varepsilon(u)}$ is an invariant subspace of $\Lambda_2^p(w)$ under the operator M_u.

Then, since the restriction of a compact operator to an invariant closed subspace is again a compact operator, we conclude that $M_u|_{L^w_{A_\varepsilon(u)}}$ is a compact operator. $\square$

THEOREM 6. *Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a non-singular measurable transformation for which there exist constants $b\geqslant 1$ and $\delta>0$ such that $\delta m_1(E_x)\leqslant m_1\left(T_x^{-1}(E)\right)\leqslant bm_1(E_x)$ for all $E\subset\mathbb{R}^2$. Suppose that $u:\mathbb{R}^2\to\mathbb{C}$ is a measurable function such that $W_{u,T}$ is bounded on $\Lambda_2^p(w)$. The following assertions are equivalent:*

1. *$W_{u,T}$ is compact.*

2. *M_u is compact.*

3. *$L^w_{A_\varepsilon(u)}$ has finite dimension for $\varepsilon>0$.*

Proof. 1. $\Leftrightarrow$ 2. This follows from inequalities (20) of Theorem 5 and a theorem from functional analysis, which states the following: Let X and Y be Banach spaces, S and T bounded linear operators form X to Y. If there exists $\alpha>0$ such that $\|S\|\leqslant\alpha\|T\|$ and T is compact for all $x\in X$, then S is compact.

2. $\Rightarrow$ 3. Let $\varepsilon>0$, then

$$\begin{aligned}|u(x,y)|\geqslant\varepsilon \quad &\Rightarrow\quad \left|uf\chi_{A_\varepsilon(u)}(x,y)\right|\geqslant\left|\varepsilon f\chi_{A_\varepsilon(u)}(x,y)\right|,(x,y)\in\mathbb{R}^2\\ &\Rightarrow\quad \varepsilon\left(f\chi_{A_\varepsilon(u)}\right)_2^*(s,t)\leqslant\left(uf\chi_{A_\varepsilon(u)}\right)_2^*(s,t)=\left(M_uf\chi_{A_\varepsilon(u)}\right)_2^*(s,t).\end{aligned}$$

Hence, for $1<p<\infty$,

$$\left\|M_uf\chi_{A_\varepsilon(u)}\right\|_{\Lambda_2^p(w)}\geqslant\varepsilon\left\|f\chi_{A_\varepsilon(u)}\right\|_{\Lambda_2^p(w)}.$$

Which implies that $M_u|_{L^w_{A_\varepsilon(u)}}$ is bounded below, therefore is invertible. Hence, since $M_u|_{L^w_{A_\varepsilon(u)}}$ is compact, the dimension of $L^w_{A_\varepsilon(u)}$ is finite.

3. $\Rightarrow$ 2. Suppose that $L^w_{A_\varepsilon(u)}$ has finite dimension for each $\varepsilon>0$, in particular, for all $n\in\mathbb{N}$, $L^w_{A_{\frac{1}{n}}(u)}$ has finite dimension.

Define, for each $n\in\mathbb{N}$, $u_n:\mathbb{R}^2\to\mathbb{C}$ as follows

$$u_n(x,y)=\begin{cases}u(x,y) & \text{if } (x,y)\in A_{\frac{1}{n}}(u)\\ 0 & \text{if } (x,y)\notin A_{\frac{1}{n}}(u).\end{cases}$$

Then, for each $f\in\Lambda_2^p(w)$, it holds that

$$\begin{aligned}|(u_n-u)f|&=|u_n-u||f|\leqslant\frac{1}{n}f\\ &\Rightarrow((u_n-u)f)_2^*(s,t)\leqslant\frac{1}{n}f_2^*(s,t)\\ &\Rightarrow\|(u_n-u)f\|_{\Lambda_2^p(w)}\leqslant\frac{1}{n}\|f\|_{\Lambda_2^p(w)}.\end{aligned}$$

Therefore

$$\begin{aligned}\|M_{u_n} - M_u\| &= \sup_{\substack{f \in \Lambda_2^p(w) \\ \|f\|_{\Lambda_2^p(w)}=1}} \|M_{u_n} f - M_u f\|_{\Lambda_2^p(w)} \\ &= \sup_{\substack{f \in \Lambda_2^p(w) \\ \|f\|_{\Lambda_2^p(w)}=1}} \|M_{u_n - u} f\|_{\Lambda_2^p(w)} \\ &= \sup_{\substack{f \in \Lambda_2^p(w) \\ \|f\|_{\Lambda_2^p(w)}=1}} \|(u_n - u) f\|_{\Lambda_2^p(w)} \\ &\leqslant \frac{1}{n} \|f\|_{\Lambda_2^p(w)} \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

Hence M_{u_n} converges to M_u uniformly. Next, since $L^w_{A_\varepsilon(u)}$ has finite dimension, we have that M_{u_n} is a finite range operator. Then, the linear operator M_{u_n} is bounded and it has finite range, from which compactness of M_{u_n} follows.

Finally, the uniform convergence of the compact operators M_{u_n} to M_u implies that M_u is a compact operator. $\square$

3.3. Closed range

In this section the closed range of the weighted composition operator on the space $\Lambda_2^p(w)$ is characterized.

DEFINITION 6. Let $T : X \to Y$ be a linear operator between normed spaces. T it is said to be bounded below if

$$\exists\, m > 0 \ \text{ such that } \ m\|x\| \leqslant \|Tx\|, \ \forall\, x \in X.$$

Let us see when the operator $W_{u,T}$ is 1-1.

THEOREM 7. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation and $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Then $W_{u,T} : \Lambda_2^p(w) \to \Lambda_2^p(w)$ is 1-1 if and only if $u \circ T \neq 0$ and T is surjective.*

Proof. $(\Rightarrow)$

1. Suppose that T is not surjective. Then there exist $F \subset \mathbb{R}^2 \setminus T(\mathbb{R}^2)$ such that $m_2(F) < \infty$ and therefore $0 \neq \chi_F \in \Lambda_2^p(w)$. Now,

$$W_{u,T}(\chi_F)(x) = u(T(x)) \cdot \chi_F(T(x));$$

since $\chi_F(T(x)) = 0$ (because $T(x) \notin F$), we obtain

$$\begin{aligned} & W_{u,T}(\chi_F)(x) = 0, \quad \forall\, x \in \mathbb{R}^2 \\ \Rightarrow \quad & W_{u,T}(\chi_F) = 0, \ \text{ with } \ \chi_F \neq 0. \end{aligned}$$

So $\ker(W_{u,T}) \neq \{0\}$ and then $W_{u,T}$ is not 1-1. In conclusion T is surjective.

2. Suppose that $u \circ T = 0$. Let

$$E = \left\{x \in \mathbb{R}^2 : (u \circ T)(x) = 0\right\}, \quad \text{with} \quad m_2(E) > 0.$$

Then there exists $A \subset \mathbb{R}^2$ such that $T^{-1}(A) \subset E$ and $0 < m_2(A) < \infty$ (since $0 < m_2(T^{-1}(A)) < m_2(E)$, by being m_2 non-atomic, and $m_2(A) > 0$, by being T non-singular), so $\chi_A \in \Lambda_2^p(w)$.

Consider

$$\begin{aligned} W_{u,T}\left(\chi_A\right)(x) &= \left(u \circ T\right)(x) \cdot \left(\chi_A \circ T\right)(x) \\ &= u\left(T(x)\right) \cdot \chi_A\left(T(x)\right) \\ &= u\left(T(x)\right) \cdot \chi_{T^{-1}(A)}(x). \end{aligned}$$

- If $T(x) \notin A$, then $\chi_A\left(T(x)\right) = 0$, therefore

$$W_{u,T}\left(\chi_A\right)(x) = 0.$$

- If $T(x) \in A$, then $x \in T^{-1}(A) \subset E$, so $(u \circ T)(x) = 0$, therefore

$$W_{u,T}\left(\chi_A\right)(x) = 0.$$

Thus,

$$W_{u,T}\left(\chi_A\right)(x) = 0, \quad \forall\, x \in \mathbb{R}^2.$$

Hence $0 \neq \chi_A \in \ker\left(W_{u,T}\right) \neq \{0\}$ and so $W_{u,T}$ is not 1-1. In conclusion, $u \circ T = 0$ m_2-a.e.

$(\Leftarrow)$ Let $y \in \mathbb{R}^2$, since T is onto, there exists $x \in \mathbb{R}^2$ such that $Tx = y$. So,

$$\begin{aligned} W_{u,T} f = W_{u,T} g, &\quad \text{with} \quad f, g \in \Lambda_2^p(w) \\ &\Rightarrow \left(W_{u,T} f\right)(x) = \left(W_{u,T} g\right)(x), x \in \mathbb{R}^2 \\ &\Rightarrow (u \circ T)(x) f(Tx) = (u \circ T)(x) g(Tx) \\ &\Rightarrow f(Tx) = g(Tx), \;\text{ since }\; u \circ T \neq 0 \Rightarrow (u \circ T)(x) \neq 0 \\ &\Rightarrow f(y) = g(y), \forall\, y \in \mathbb{R}^2 \\ &\Rightarrow f = g. \end{aligned}$$

Thus, $W_{u,T}$ is 1-1. $\square$

In the following results, we will denote $S = \operatorname{supp}(u) = \left\{x \in \mathbb{R}^2 : u(x) \neq 0\right\}$, the support of u.

COROLLARY 1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation and $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Then $W_{u,T} : L_S^w \to L_S^w$ is 1-1, where $L_S^w = \left\{f \chi_S : f \in \Lambda_2^p(w)\right\}$.*

Proof. Consider $W_{u,T}\left(\bar{f}\right) = 0$, where $\bar{f} = f \chi_S \in L_S^w$. Then

$$
\begin{aligned}
0 = W_{u,T}(\bar{f}) &= u(T(x))\,\bar{f}(T(x)) \\
&= u(T(x))\,f(T(x))\,\chi_S(T(x)) \\
&\Rightarrow u(T(x))\,f(T(x)) = 0 \\
&\Rightarrow f(T(x)) = 0, \forall\, T(x) \in S, \quad \text{since} \quad T(x) \in S \Leftrightarrow u(T(x)) \neq 0 \\
&\Rightarrow f(T(x))\,\chi_S(T(x)) = 0 \\
&\Rightarrow (f\chi_S)(T(x)) = 0, \forall\, T(x) \in S \\
&\Rightarrow \bar{f} = 0. \quad \square
\end{aligned}
$$

COROLLARY 2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation and $u : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function. Then $W_{u,T} : L_S^w \to L_S^w$ is bounded below if and only if $W_{u,T}$ has closed range.*

Proof. It is well-known that a linear operator between Banach spaces is bounded below if and only if it is injective and has closed range. The result follows since $W_{u,T}$ is 1-1 on L_S^w. $\square$

THEOREM 8. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-singular measurable transformation for which there exist constants $b \geqslant 1$ and $\delta > 0$ such that $\delta m_1(E_x) \leqslant m_1\left(T_x^{-1}(E)\right) \leqslant bm_1(E_x)$ for all $E \subset \mathbb{R}^2$. Suppose that $u : \mathbb{R}^2 \to \mathbb{C}$ is a measurable function. If $W_{u,T}$ is bounded on L_S^w, then $W_{u,T}$ is bounded below on $L_S(w)$ if and only if there exists $r > 0$ such that $|u(x)| \geqslant r$ a.e. in S.*

Proof. $(\Rightarrow)$ Suppose that $W_{u,T}$ is bounded below. There exists $m > 0$ such that

$$
\|W_{u,T} f\|_{\Lambda_2^p(w)} \geqslant m \|f\|_{\Lambda_2^p(w)}, f \in L_S^w. \tag{21}
$$

Let $r > 0$ such that $r < m$. Let $R = \{x \in S : |u(x)| < r\}$ and suppose that $0 < m_2(E) < \infty$. Then, $\chi_E \in L_S^w$. Now,

$$
\begin{aligned}
& |u\chi_E(x)| \leqslant |r\chi_E(x)|, \ \forall\, x \in \mathbb{R}^2 \\
\Rightarrow \quad & (u\chi_E)_2^*(s,t) \leqslant (r\chi_E)_2^*(s,t) \\
\Rightarrow \quad & \|u\chi_E\|_{\Lambda_2^p(w)} \leqslant \|r\chi_E\|_{\Lambda_2^p(w)} = r\|\chi_E\|_{\Lambda_2^p(w)}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|W_{u,T}\chi_E\|_{\Lambda_2^p(w)} &\leqslant \|M_u\chi_E\|_{\Lambda_2^p(w)}, \quad \text{by using (20)} \\
&= \|u\chi_E\|_{\Lambda_2^p(w)} \\
&\leqslant r\|\chi_E\|_{\Lambda_2^p(w)} \\
&< m\|\chi_E\|_{\Lambda_2^p(w)},
\end{aligned}
$$

which contradicts (21). Hence $m_2(E) = 0$, that is to say, there exists $r > 0$ such that $|u(x)| \geqslant r$ a.e. in S.

($\Leftarrow$) Let $r>0$ such that $|u(x)| \geqslant r$ a.e. in S, then

$$|(uf\chi_S)(x)| \geqslant |(rf\chi_S)(x)|, \ \forall f \in \Lambda_2^p(w).$$

From where

$$\|uf\chi_S\|_{\Lambda_2^p(w)} \geqslant \|rf\chi_S\|_{\Lambda_2^p(w)} = r\|f\chi_S\|_{\Lambda_2^p(w)}.$$

Thus, by inequality (20) in Theorem 5, we obtain

$$\begin{aligned}\|W_{u,T}f\chi_S\|_{\Lambda_2^p(w)} &\geqslant \delta^{1/p}\|M_uf\chi_S\|_{\Lambda_2^p(w)} \\ &= \delta^{1/p}\|uf\chi_S\|_{\Lambda_2^p(w)} \\ &\geqslant \delta^{1/p}r\|f\chi_S\|_{\Lambda_2^p(w)},\end{aligned}$$

which means that $W_{u,T}$ is bounded below. $\square$

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René Erlín Castillo
Department of Mathematics
Universidad Nacional de Colombia
Bogotá, Colombia
e-mail: recastillo@unal.edu.co

Héctor Camilo Chaparro
Department of Mathematics
Universidad Nacional de Colombia
Bogotá, Colombia
e-mail: hcchaparrog@unal.edu.co