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L_p–MIXED AFFINE SURFACE AREA

TIAN LI AND WEIDONG WANG

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Abstract. Lutwak introduced the notion of L_p-affine surface area by L_p-mixed volume and obtained some related inequalities. In this article, based on the L_p-mixed quermassintegrals, we define the concept of the L_p-mixed affine surface area and extend some of Lutwak's result.

1. Introduction

We say that K is a convex body if K is a compact, convex subset in n-dimensional Euclidean space $\mathbb{R}^n$ with non-empty interior. The set of all convex bodies in $\mathbb{R}^n$ is written as $\mathscr{K}^n$, and its subset $\mathscr{K}_o^n$ denotes the set of convex bodies containing the origin in their interiors. Similarly, $\mathscr{K}_c^n$ denotes the set of convex bodies with centroid at the origin. Let $\mathscr{F}^n$ ($\mathscr{F}_o^n$) denote the subset of $\mathscr{K}^n$ ($\mathscr{K}_o^n$) that have a positive continuous curvature function. Besides $\mathscr{S}_o^n$ denotes the set of star bodies (with respect to the origin) and $\mathscr{S}_c^n$ denotes the set of star bodies whose centroid lie at the origin in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of the body K, for the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

The notion of classical affine surface area was defined first by Blaschke ([1]). For a smooth convex body K in $\mathbb{R}^3$, the affine surface area, $\Omega(K)$, of K is given by

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{3/4} dS(u),$$

where $f(K,\cdot)$ denotes the curvature function of $K \in \mathscr{K}^n$, and $dS(\cdot)$ denotes the infinitesimal of Lebesgue measure $S(\cdot)$ on the unit sphere S^{n-1}. Later, $\Omega(K)$ was naturally considered for sufficiently smooth K in $\mathscr{K}^n$ by Leichtweiss ([6]) as

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} dS(u).$$

In 1989, Leichtweiβ ([7]) extended the domain of Ω: $\mathscr{F}^n \to (0,+\infty)$ from $\mathscr{F}^n$ to $\mathscr{K}^n$ as follows: For $K \in \mathscr{K}^n$, the affine surface area, $\Omega(K)$, of K is defined by

$$n^{-\frac{1}{n}} \Omega(K)^{\frac{n+1}{n}} = \inf\{nV_1(K,Q^*)V(Q)^{\frac{1}{n}} : Q \in \mathscr{S}_o^n\}.$$

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Here Q^* denotes the polar body of Q and $V_1(M,N)$ denotes the mixed volume of convex bodies M and N.

Based on the classical affine surface area, Lutwak (see [12]) introduced the classical notion of mixed affine surface area and obtained some isoperimetric inequalities for this notion. During the past three decades, the investigations of the classical affine surface area have received great attention from many articles (see [2, 3, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 20, 34, 35, 36]).

In 1996, according to the L_p-mixed volume, Lutwak ([15]) introduced the notion of L_p-affine surface area. For $K \in \mathcal{K}_o^n$, $p \geqslant 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}.$$

Here $V_p(M,N)$ denotes the L_p-mixed volume of $M,N \in \mathcal{K}_o^n$. Obviously, for $p=1$, $\Omega_p(K)$ is the classical affine surface area $\Omega(K)$.

In 2007, Wang and Leng ([24]) introduced the notion of ith L_p-mixed affine surface area and obtained results related to it. Regarding the studies of the L_p-affine surface areas also see ([21, 22, 25, 26, 27, 28, 30, 31, 32, 33]).

Based on the definition of L_p-affine surface area, Lutwak ([15]) proved the following results:

THEOREM 1.A$_a$. *If $p \geqslant 1$ and $K \in \mathcal{K}_o^n$, then*

$$\left[\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right]^{\frac{1}{p}} \leqslant V(K)V(K^*). \tag{1.1}$$

Here $\left[\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right]^{\frac{1}{p}}$ *is called the L_p-affine surface area ratio of K.*

THEOREM 1.A$_b$. *If $p \geqslant 1$ and $K \in \mathcal{F}_o^n$, then*

$$\left[\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right]^{\frac{1}{p}} \leqslant V(K)V(K^*), \tag{1.2}$$

with equality if and only if K^* and $\Lambda_p K$ are dilates.

THEOREM 1.B$_a$. *If $K \in \mathcal{K}_o^n$, $1 \leqslant p < q$, then*

$$\left[\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\Omega_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right]^{\frac{1}{q}}. \tag{1.3}$$

THEOREM 1.B$_b$. *If $K \in \mathcal{F}_o^n$, $1 \leqslant p < q$, then*

$$\left[\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\Omega_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right]^{\frac{1}{q}}, \tag{1.4}$$

with equality if and only if $\Lambda_p K$ and $\Lambda_q K$ are dilates.

THEOREM 1.C$_a$. *If $K \in \mathscr{K}_o^n$, $1 \leqslant p < q < r$, then*

$$\Omega_q(K)^{(n+q)(r-p)} \leqslant \Omega_p(K)^{(n+p)(r-q)}\Omega_r(K)^{(n+r)(q-p)}. \tag{1.5}$$

THEOREM 1.C$_b$. *If $K \in \mathscr{F}_o^n$, $1 \leqslant p < q < r$, then*

$$\Omega_q(K)^{(n+q)(r-p)} \leqslant \Omega_p(K)^{(n+p)(r-q)}\Omega_r(K)^{(n+r)(q-p)}. \tag{1.6}$$

with equality if and only if $\Lambda_p K$ and $\Lambda_r K$ are dilates.

The main purpose of this article is to define the notion of L_p-mixed affine surface area by Lutwak's L_p-mixed quermassintegrals (see [14]). Then by this notion we establish some inequalities which are extensions of parts of Lutwak's results in [12] and [15].

Associated with the L_p-mixed quermassintegrals (see [14]), we first give the notion of L_p-mixed affine surface area as follows:

DEFINITION 1.1. For $K \in \mathscr{K}_o^n$, $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, the L_p-mixed affine surface area, $\Omega_{p,i}(K)$, of K is defined by

$$n^{-\frac{p}{n-i}}\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}} = \inf\{nW_{p,i}(K, Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathscr{S}_o^n\}. \tag{1.7}$$

Here $W_{p,i}(M,N)$ denotes the L_p-mixed quermassintegrals of $M, N \in \mathscr{K}_o^n$.

Note that above definition is different from the notion of ith L_p-mixed affine surface area in ([24])

We easily see that if $i = 0$ in (1.7), then $\Omega_{p,i}(K)$ is just the L_p-affine surface area.

Further, we establish some inequalities for the L_p-mixed affine surface area which extend the results of Theorems 1.A$_a$-1.C$_b$. Our main results can be stated as follows:

THEOREM 1.1$_a$. *If $K \in \mathscr{K}_o^n$, $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, then*

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant W_i(K)\widetilde{W}_i(K^*). \tag{1.8}$$

Here $\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}}$ may be called the i-th L_p-mixed affine surface area ratio of K.

THEOREM 1.1$_b$. *If $K \in \mathscr{F}_o^n$, $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, then*

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant W_i(K)\widetilde{W}_i(K^*), \tag{1.9}$$

with equality if and only if K^* and $\Lambda_{p,i}K$ are dilates.

THEOREM 1.2$_a$. *If $K \in \mathscr{K}_o^n$, $1 \leqslant p < q$ and $i = 0, 1, \cdots, n-1$, then*

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\Omega_{q,i}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right]^{\frac{1}{q}}. \tag{1.10}$$

THEOREM 1.2$_b$. *If* $K \in \mathscr{F}_o^n$, $1 \leqslant p < q$ *and* $i = 0, 1, \cdots, n-1$, *then*

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\Omega_{q,i}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right]^{\frac{1}{q}}, \tag{1.11}$$

with equality if and only if $\Lambda_{p,i}K$ *and* $\Lambda_{q,i}K$ *are dilates.*

THEOREM 1.3$_a$. *If* $K \in \mathscr{K}_o^n$, $1 \leqslant p < q < r$ *and* $i = 0, 1, \cdots, n-1$, *then*

$$\Omega_{q,i}(K)^{(n+q-i)(r-p)} \leqslant \Omega_{p,i}(K)^{(n+p-i)(r-q)}\Omega_{r,i}(K)^{(n+r-i)(q-p)}. \tag{1.12}$$

THEOREM 1.3$_b$. *If* $K \in \mathscr{F}_o^n$, $1 \leqslant p < q < r$ *and* $i = 0, 1, \cdots, n-1$, *then*

$$\Omega_{q,i}(K)^{(n+q-i)(r-p)} \leqslant \Omega_{p,i}(K)^{(n+p-i)(r-q)}\Omega_{r,i}(K)^{(n+r-i)(q-p)}, \tag{1.13}$$

with equality if and only if $\Lambda_{p,i}K$ *and* $\Lambda_{r,i}K$ *are dilates.*

The proofs of Theorems 1.1$_a$-1.3$_b$ will be completed in section 3 of this paper.

2. Notations and background materials

2.1. Support function, radial functions and polar set

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [4, 19])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y. Obviously, $h(\lambda K, \cdot) = \lambda h(K, \cdot)$, where λ is a positive constant.

If K is a compact star-shaped (with respect to the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [4, 19])

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is a nonempty subset and contains the origin in $\mathbb{R}^n$, then the polar set, E^*, of E is defined by (see [4, 19])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in E\}.$$

It is easily verified that $(K^*)^* = K$ for all $K \in \mathscr{K}_o^n$. Moreover, for $K \in \mathscr{S}_o^n$ and all $u \in S^{n-1}$,

$$\rho_K(u) = \frac{1}{h_{K^*}(u)}. \tag{2.1}$$

2.2. L_p-mixed surface area measure and L_p-mixed curvature image

The L_p-mixed surface area measure of convex bodies is introduced by Lutwak (see[14]). For $K \in \mathscr{K}_o^n$, real $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, the L_p-mixed surface area measure, $S_{p,i}(K,\cdot)$, of K is defined by

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h^{1-p}(K,\cdot). \tag{2.2}$$

Equation (2.2) is the Radon-Nikodym derivative of the L_p-surface area measure $S_{p,i}(K,\cdot)$ with the respect to the surface area measure $S_i(K,\cdot)$.

For $i = 0, 1, \cdots, n-1$, we say $K \in \mathscr{K}^n$ has a curvature function $f_i(K,\cdot) : S^{n-1} \to \mathbb{R}$, if measure $S_i(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_i(K,\cdot)}{dS} = f_i(K,\cdot). \tag{2.3}$$

Let $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$. A convex body $K \in \mathscr{K}_o^n$ is said to have a generalized L_p-curvature function (see[18]), $f_{p,i}(K,\cdot) : S^{n-1} \to \mathbb{R}$, if measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_{p,i}(K,\cdot)}{dS} = f_{p,i}(K,\cdot). \tag{2.4}$$

Obviously, $f_{p,0}(K,\cdot) = f_p(K,\cdot)$. Here $f_p(K,\cdot)$ is the L_p-curvature function of $K \in \mathscr{K}_o^n$ (see [15]).

Also, from (2.2), (2.3) and (2.4), we know that for $K \in \mathscr{F}_o^n$,

$$f_{p,i}(K,\cdot) = h^{1-p}(K,\cdot) f_i(K,\cdot). \tag{2.5}$$

Meanwhile, according to the definition of L_p-curvature image, Lu and Wang ([18]) gave the definition of L_p-mixed curvature image as follows: For $K \in \mathscr{F}_o^n$, $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, the L_p-mixed curvature image, $\Lambda_{p,i} K \in \mathscr{S}_o^n$, of K is defined by

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i} = \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n} f_{p,i}(K,\cdot). \tag{2.6}$$

If $i = 0$ in (2.6), then

$$\Lambda_{p,0} K = \Lambda_p K.$$

Here $\Lambda_p K$ is called L_p-curvature image that was established by Lutwak (see[15]).

2.3. Quermassintegrals and L_p-mixed quermassintegrals

For $K \in \mathscr{K}^n$, $i = 0, 1, \cdots, n-1$, the quermassintegrals, $W_i(K)$, of K are defined by (see [4, 19])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_i(K,u).$$

Here $S_i(K,\cdot)$ $(i=0,1,\cdots,n-1)$ are the area measure of $K\in\mathscr{K}^n$. Obviously,

$$W_0(K)=\frac{1}{n}\int_{S^{n-1}}h_K(u)dS(K,u)=V(K).$$

For $K,L\in\mathscr{K}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-Minkowski linear combination (also called Firey combination), $\lambda\cdot K+_p\mu\cdot L\in\mathscr{K}_o^n$, of K and L is defined by (see[14])

$$h(\lambda\cdot K+_p\mu\cdot L,\cdot)^p=\lambda h(K,\cdot)^p+\mu h(L,\cdot)^p.$$

Here '$+_p$' denotes the L_p-Minkowski addition and '$\cdot$' denotes the Firey scalar multiplication.

Associated with L_p-mixed surface area measure, Lutwak ([14]) defined the L_p-mixed quermassintegrals (also called mixed p-quermassintegrals). For $K,L\in\mathscr{K}_o^n$, $p\geqslant 1$, $i=0,1,\cdots,n-1$, the L_p-mixed quermassintegrals, $W_{p,i}(K,L)$, of K and L are given by (see [14])

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^p(u)dS_{p,i}(K,u). \tag{2.7}$$

From (2.7), it follows immediately that for each $K\in\mathscr{K}_o^n$ and all $p\geqslant 1$,

$$W_{p,i}(K,K)=W_i(K). \tag{2.8}$$

Let $i=0$ in (2.7), the L_p-mixed volume, $V_p(K,L)$, of $K,L\in\mathscr{K}_o^n$ is given by

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^p(u)dS_p(K,u).$$

If $i=0$ and $p=1$ in (2.7), then the mixed volume, $V_1(K,L)$, of convex bodies K and L is defined by

$$V_1(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L(u)dS(K,u).$$

2.4. Dual quermassintegrals and L_p-dual mixed quermassintegrals

For $K\in\mathscr{S}_o^n$ and real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K is defined by (see[9])

$$\widetilde{W}_i(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)dS(u). \tag{2.9}$$

Obviously, for $i=0$,

$$\widetilde{W}_0(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(u)dS(u)=V(K).$$

For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-harmonic radial combination, $\lambda\times K\widetilde{+}_{-p}\mu\times L\in\mathscr{S}_o^n$, of K and L is defined by (see[15])

$$\rho(\lambda\times K\widetilde{+}_{-p}\mu\times L,\cdot)^{-p}=\lambda\rho(K,\cdot)^{-p}+\mu\rho(L,\cdot)^{-p}.$$

Associated with the L_p-harmonic radial combination of star bodies, Wang and Leng (see[23]) introduced the notion of L_p-dual mixed quermassintegrals as follows: For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K,L)$, of K and L is defined by

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)dS(u). \tag{2.10}$$

From (2.9) and (2.10), it follows immediately that for each $K \in \mathscr{S}_o^n$ and all $p \geqslant 1$,

$$\widetilde{W}_{-p,i}(K,K) = \widetilde{W}_i(K). \tag{2.11}$$

The Minkowski's inequality for the L_p-dual mixed quermassintegrals is (see[23]):

Let $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and real $i \neq n$, then for $i < n$ or $n < i < n+p$,

$$\widetilde{W}_{-p,i}(K,L) \geqslant \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(L)^{-\frac{p}{n-i}}, \tag{2.12}$$

for $i > n+p$, the inequality (2.12) is reversed. Equality holds in each inequality if and only if K and L are dilates.

3. Proofs of Theorems

In this section, we complete the proofs of Theorems 1.1_a-1.3_b. Here, we first give a property of the L_p-mixed affine surface area $\Omega_{p,i}(K)$ as follows:

THEOREM 3.1. *If $K \in \mathscr{F}_o^n$, $p \geqslant 1$ and $i = 0,1,\cdots,n-1$, then*

$$\Omega_{p,i}(K)^{n+p-i} = n^{n+p-i}\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p. \tag{3.1}$$

LEMMA 3.1. [18] *If $K \in \mathscr{F}_o^n$, $p \geqslant 1$ and $i = 0,1,\cdots,n-1$, then for any $Q \in \mathscr{S}_o^n$,*

$$\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q) = \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n} W_{p,i}(K,Q^*). \tag{3.2}$$

Proof of Theorem 3.1. According to (2.12), for $i < n$, we have for any $Q \in \mathscr{S}_o^n$

$$\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q) \geqslant \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(Q)^{-\frac{p}{n-i}}. \tag{3.3}$$

Using (3.2), (3.3) and definition (1.7), yield

$$\begin{aligned}
n^{-\frac{p}{n-i}}\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}} &= \inf\{nW_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathscr{S}_o^n\} \\
&= \inf\left\{\frac{n\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathscr{S}_o^n\right\} \\
&\geqslant \inf\left\{\frac{n\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(Q)^{-\frac{p}{n-i}}\widetilde{W}_i(Q)^{\frac{p}{n-i}} : Q \in \mathscr{S}_o^n\right\} \\
&= \inf\left\{n\omega_n\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}} : Q \in \mathscr{S}_o^n\right\} \\
&= n\omega_n\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}}.
\end{aligned} \tag{3.4}$$

On the other hand, combining with definition (1.7) and (3.2), we obtain for any $Q\in\mathscr{S}_o^n$

$$n^{-\frac{p}{n-i}}\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}}\leqslant nW_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}}$$
$$=\frac{n\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)\widetilde{W}_i(Q)^{\frac{p}{n-i}}. \tag{3.5}$$

Taking Q for $\Lambda_{p,i}K$ in (3.5) and using (2.11), then

$$n^{-\frac{p}{n-i}}\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}}\leqslant n\omega_n\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}}. \tag{3.6}$$

From (3.4) and (3.6), we see that

$$n^{-\frac{p}{n-i}}\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}}=n\omega_n\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}}.$$

i.e.

$$\Omega_{p,i}(K)^{n+p-i}=n^{n+p-i}\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p.$$

This yields (3.1). □

Let $i=0$ in Theorem 3.1 to get the following result which was obtained by Lutwak (see [15]).

COROLLARY 3.1. *Suppose $K\in\mathscr{F}_o^n$, $p\geqslant 1$, then*

$$\Omega_p(K)=n\omega_n^{\frac{n}{n+p}}V(\Lambda_pK)^{\frac{p}{n+p}}.$$

Proof of Theorem 1.1$_a$. From definition (1.7), we obtain for any $Q\in\mathscr{S}_o^n$,

$$\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}}\leqslant n^{\frac{n+p-i}{n-i}}W_{p,i}(K,Q^*)\widetilde{W}_i(Q)^{\frac{p}{n-i}}, \tag{3.7}$$

Since $Q^*\in\mathscr{K}_o^n$, thus taking $Q^*=K$ in (3.7), then

$$\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}}\leqslant n^{\frac{n+p-i}{n-i}}W_i(K)\widetilde{W}_i(K^*)^{\frac{p}{n-i}},$$

Hence

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}}\leqslant W_i(K)\widetilde{W}_i(K^*).$$

This gives (1.8). □

Proof of Theorem 1.1$_b$. Let $Q^*=K$ in (3.2), and together (2.8) and inequality (2.12), we have that for all $i<n$,

$$W_i(K)=\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_{-p,i}(\Lambda_{p,i}K,K^*)$$
$$\geqslant\frac{\omega_n}{\widetilde{W}_i(\Lambda_{p,i}K)}\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(K^*)^{-\frac{p}{n-i}}.$$

Then

$$\left[\frac{\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p}{W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant W_i(K)\widetilde{W}_i(K^*).$$

Using (3.1), we get

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant W_i(K)\widetilde{W}_i(K^*). \tag{3.8}$$

According to the equality condition of (2.12), we see that equality holds in (3.8) if and only if K^* and $\Lambda_{p,i}K$ are dilates. This yields (1.9). □

In order to prove Theorem 1.2_a, the following lemma obtained by Wei and Wang (see[29]) is needed.

LEMMA 3.2. *If* $K,L \in \mathscr{K}_o^n$, $1 \leqslant p < q$ *and* $i = 0,1,\cdots,n-1$, *then*

$$\left[\frac{W_{p,i}(K,L)}{W_i(K)}\right]^{\frac{1}{p}} \leqslant \left[\frac{W_{q,i}(K,L)}{W_i(K)}\right]^{\frac{1}{q}},$$

with equality if and only if K *and* L *are dilates.*

Proof of Theorem 1.2_a. For any $Q \in \mathscr{S}_o^n$, we have $Q^* \in \mathscr{K}_o^n$. Hence, by Lemma 3.2, we get for $1 \leqslant p < q$,

$$\left[\frac{W_{p,i}(K,Q^*)}{W_i(K)}\right]^{\frac{1}{p}} \leqslant \left[\frac{W_{q,i}(K,Q^*)}{W_i(K)}\right]^{\frac{1}{q}}, \tag{3.9}$$

with equality if and only if K is a dilate of Q^*.

Combining with definition (1.7) and (3.9), if $i = 0,1,\cdots,n-1$, then we yield

$$\begin{aligned}
\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} &= \left[\frac{\Omega_{p,i}(K)^{\frac{n+p-i}{n-i}}}{n^{\frac{n+p-i}{n-i}}W_i(K)^{\frac{n-p-i}{n-i}}}\right]^{\frac{n-i}{p}} \\
&= \inf\left\{\left[\frac{W_{p,i}(K,Q^*)}{W_i(K)}\right]^{\frac{n-i}{p}} W_i(K)\widetilde{W}_i(Q) : Q \in \mathscr{S}_o^n\right\} \\
&\leqslant \inf\left\{\left[\frac{W_{q,i}(K,Q^*)}{W_i(K)}\right]^{\frac{n-i}{q}} W_i(K)\widetilde{W}_i(Q) : Q \in \mathscr{S}_o^n\right\} \\
&= \left[\frac{\Omega_{q,i}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right]^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\Omega_{q,i}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right]^{\frac{1}{q}}.$$

This yields (1.10). □

Proof of Theorem 1.2$_b$. From (3.2), we have

$$\left[\frac{W_{p,i}(K,Q^*)}{W_i(K)}\right]^{\frac{1}{p}} = \left[\frac{\omega_n \widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)}{W_i(K)\widetilde{W}_i(\Lambda_{p,i}K)}\right]^{\frac{1}{p}},$$

$$\left[\frac{W_{q,i}(K,Q^*)}{W_i(K)}\right]^{\frac{1}{q}} = \left[\frac{\omega_n \widetilde{W}_{-q,i}(\Lambda_{q,i}K,Q)}{W_i(K)\widetilde{W}_i(\Lambda_{q,i}K)}\right]^{\frac{1}{q}}.$$

Using (3.9), we get that for $1 \leqslant p < q$,

$$\left[\frac{\omega_n \widetilde{W}_{-p,i}(\Lambda_{p,i}K,Q)}{W_i(K)\widetilde{W}_i(\Lambda_{p,i}K)}\right]^{\frac{1}{p}} \leqslant \left[\frac{\omega_n \widetilde{W}_{-q,i}(\Lambda_{q,i}K,Q)}{W_i(K)\widetilde{W}_i(\Lambda_{q,i}K)}\right]^{\frac{1}{q}}. \tag{3.10}$$

Taking $Q = \Lambda_{q,i}K$ in (3.10), and using (2.11) and inequality (2.12), we obtain

$$\begin{aligned}\left[\frac{\omega_n}{W_i(K)}\right]^{\frac{1}{q}} &\geqslant \left[\frac{\omega_n \widetilde{W}_{-p,i}(\Lambda_{p,i}K,\Lambda_{q,i}K)}{W_i(K)\widetilde{W}_i(\Lambda_{p,i}K)}\right]^{\frac{1}{p}} \\ &\geqslant \left[\frac{\omega_n \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}}\widetilde{W}_i(\Lambda_{q,i}K)^{-\frac{p}{n-i}}}{W_i(K)}\right]^{\frac{1}{p}}.\end{aligned}$$

So, we get

$$\left[\frac{\omega_n \widetilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n-i}}}{W_i(K)}\right]^{\frac{1}{p}} \leqslant \left[\frac{\omega_n \widetilde{W}_i(\Lambda_{q,i}K)^{\frac{q}{n-i}}}{W_i(K)}\right]^{\frac{1}{q}}.$$

From (3.1), we have

$$\left[\frac{\Omega_{p,i}(K)^{n+p-i}}{n^{n+p-i}W_i(K)^{n-p-i}}\right]^{\frac{1}{p}} \leqslant \left[\frac{\Omega_{q,i}(K)^{n+q-i}}{n^{n+q-i}W_i(K)^{n-q-i}}\right]^{\frac{1}{q}}. \tag{3.11}$$

According to the equality condition of (2.12), we see that equality holds in (3.11) if and only if $\Lambda_{p,i}K$ and $\Lambda_{q,i}K$ are dilates. This yields (1.11). □

Proof of Theorem 1.3$_a$. Since for any $Q_1, Q_2 \in \mathscr{S}_o^n$, there exists $Q_3 \in \mathscr{S}_o^n$ such that

$$\rho(Q_3,\cdot)^{q(r-p)} = \rho(Q_1,\cdot)^{p(r-q)}\rho(Q_2,\cdot)^{r(q-p)}. \tag{3.12}$$

Then for any $u \in S^{n-1}$, this yields

$$\rho_{Q_3}(u)^{n-i} = \rho_{Q_1}(u)^{\frac{(n-i)p(r-q)}{q(r-p)}}\rho_{Q_2}(u)^{\frac{(n-i)r(q-p)}{q(r-p)}}.$$

Since $1 \leqslant p < q < r$, then $\frac{q(r-p)}{p(r-q)} > 1$. According to the Hölder's integral inequality (see[5]) and definition (2.9), we get

$$\begin{aligned}
&\widetilde{W}_i(Q_1)^{\frac{p(r-q)}{q(r-p)}}\widetilde{W}_i(Q_2)^{\frac{r(q-p)}{q(r-p)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}}\rho_{Q_1}(u)^{n-i}dS(u)\right]^{\frac{p(r-q)}{q(r-p)}}\left[\frac{1}{n}\int_{S^{n-1}}\rho_{Q_2}(u)^{n-i}dS(u)\right]^{\frac{r(q-p)}{q(r-p)}} \\
&= \left\{\frac{1}{n}\int_{S^{n-1}}\left[\rho_{Q_1}(u)^{\frac{(n-i)p(r-q)}{q(r-p)}}\right]^{\frac{q(r-p)}{p(r-q)}}dS(u)\right\}^{\frac{p(r-q)}{q(r-p)}} \\
&\quad\times\left\{\frac{1}{n}\int_{S^{n-1}}\left[\rho_{Q_2}(u)^{\frac{(n-i)r(q-p)}{q(r-p)}}\right]^{\frac{q(r-p)}{r(q-p)}}dS(u)\right\}^{\frac{r(q-p)}{q(r-p)}} \\
&\geqslant \frac{1}{n}\int_{S^{n-1}}\rho_{Q_1}(u)^{\frac{(n-i)p(r-q)}{q(r-p)}}\rho_{Q_2}(u)^{\frac{(n-i)r(q-p)}{q(r-p)}}dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}}\rho_{Q_3}(u)^{n-i}dS(u) = \widetilde{W}_i(Q_3).
\end{aligned}$$

Since $q(r-p) > 0$, then

$$\widetilde{W}_i(Q_3)^{q(r-p)} \leqslant \widetilde{W}_i(Q_1)^{p(r-q)}\widetilde{W}_i(Q_2)^{r(q-p)}. \tag{3.13}$$

From (3.12), we see that for any $u \in S^{n-1}$,

$$\rho_{Q_3}(u)^{-q}h_K(u)^{1-q} = [\rho_{Q_1}(u)^{-p}h_K(u)^{1-p}]^{\frac{r-q}{r-p}}[\rho_{Q_2}(u)^{-r}h_K(u)^{1-r}]^{\frac{q-p}{r-p}}. \tag{3.14}$$

Then for $1 \leqslant p < q < r$, i.e. $\frac{r-p}{r-q} > 1$, according to the Hölder's integral inequality, (2.1), (2.7) and (3.14), we get

$$\begin{aligned}
&W_{p,i}(K,Q_1^*)^{\frac{r-q}{r-p}}W_{r,i}(K,Q_2^*)^{\frac{q-p}{r-p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}}h_{Q_1^*}(u)^p h_K(u)^{1-p}dS_i(K,u)\right]^{\frac{r-q}{r-p}} \\
&\quad\times\left[\frac{1}{n}\int_{S^{n-1}}h_{Q_2^*}(u)^r h_K(u)^{1-r}dS_i(K,u)\right]^{\frac{q-p}{r-p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}}[\rho_{Q_1}(u)^{-p}h_K(u)^{1-p}]^{\frac{r-q}{r-p}\frac{r-p}{r-q}}dS_i(K,u)\right]^{\frac{r-q}{r-p}} \\
&\quad\times\left[\frac{1}{n}\int_{S^{n-1}}[\rho_{Q_2}(u)^{-r}h_K(u)^{1-r}]^{\frac{q-p}{r-p}\frac{r-p}{q-p}}dS_i(K,u)\right]^{\frac{q-p}{r-p}} \\
&\geqslant \frac{1}{n}\int_{S^{n-1}}\rho_{Q_3}(u)^{-q}h_K(u)^{1-q}dS_i(K,u) \\
&= W_{q,i}(K,Q_3^*),
\end{aligned}$$

i.e.

$$W_{q,i}(K,Q_3^*)^{r-p} \leqslant W_{p,i}(K,Q_1^*)^{r-q}W_{r,i}(K,Q_2^*)^{q-p}. \tag{3.15}$$

Hence, combining with (3.13) and (3.15), we get for $i=0,1,\cdots,n-1$,

$$[W_{q,i}(K,Q_3^*)\widetilde{W}_i(Q_3)^{\frac{q}{n-i}}]^{r-p}$$
$$\leqslant [W_{p,i}(K,Q_1^*)\widetilde{W}_i(Q_1)^{\frac{p}{n-i}}]^{r-q}[W_{r,i}(K,Q_2^*)\widetilde{W}_i(Q_2)^{\frac{r}{n-i}}]^{q-p}.$$

This together with (1.7) yields

$$\Omega_{q,i}(K)^{(n+q-i)(r-p)} \leqslant \Omega_{p,i}(K)^{(n+p-i)(r-q)}\Omega_{r,i}(K)^{(n+r-i)(q-p)}.$$

This gives (1.12). □

Finally, we give the proof of Theorem 1.3_b. The following lemma is required.

LEMMA 3.3. *If $K\in\mathscr{F}_o^n$, $1\leqslant p<q<r$ and $i=0,1,\cdots,n-1$, then*

$$\widetilde{W}_i(\Lambda_{q,i}K)^{q(r-p)} \leqslant \widetilde{W}_i(\Lambda_{p,i}K)^{p(r-q)}\widetilde{W}_i(\Lambda_{r,i}K)^{r(q-p)}, \tag{3.16}$$

with equality if and only if $\Lambda_{p,i}K$ and $\Lambda_{r,i}K$ are dilates.

Proof. From the formula (2.5), it follows that for $i=0,1,\cdots,n-1$,

$$f_{q,i}(K,\cdot)^{r-p} = f_{p,i}(K,\cdot)^{r-q}f_{r,i}(K,\cdot)^{q-p}.$$

Thus, by (2.6), we get for any $u\in S^{n-1}$,

$$\widetilde{W}_i(\Lambda_{q,i}K)^{p-r}\rho(\Lambda_{q,i}K,u)^{(n+q-i)(r-p)}$$
$$=[\widetilde{W}_i(\Lambda_{p,i}K)^{q-r}\rho(\Lambda_{p,i}K,u)^{(n+p-i)(r-q)}][\widetilde{W}_i(\Lambda_{r,i}K)^{p-q}\rho(\Lambda_{r,i}K,u)^{(n+r-i)(q-p)}],$$

that is

$$\begin{aligned}&\widetilde{W}_i(\Lambda_{q,i}K)^{\frac{(p-r)(n-i)}{(n+q-i)(r-p)}}\rho(\Lambda_{q,i}K,u)^{n-i}\\ &=[\widetilde{W}_i(\Lambda_{p,i}K)^{\frac{(q-r)(n-i)}{(n+q-i)(r-p)}}(\rho(\Lambda_{p,i}K,u)^{n-i})^{\frac{(n+p-i)(r-q)}{(n+q-i)(r-p)}}]\\ &\quad\times[\widetilde{W}_i(\Lambda_{r,i}K)^{\frac{(p-q)(n-i)}{(n+q-i)(r-p)}}(\rho(\Lambda_{r,i}K,u)^{n-i})^{\frac{(n+r-i)(q-p)}{(n+q-i)(r-p)}}].\end{aligned} \tag{3.17}$$

Using the Hölder inequality and (2.9) in (3.17), we obtain

$$\widetilde{W}_i(\Lambda_{q,i}K)^{q(r-p)} \leqslant \widetilde{W}_i(\Lambda_{p,i}K)^{p(r-q)}\widetilde{W}_i(\Lambda_{r,i}K)^{r(q-p)}. \tag{3.18}$$

From the equality condition of the Hölder inequality, we see that equality holds in (3.18) if and only if $\Lambda_{p,i}K$ and $\Lambda_{r,i}K$ are dilates. This yields (3.16). □

Proof of Theorem 1.3$_b$. From (3.1) and (3.16), we have

$$\begin{aligned}\Omega_{q,i}(K)^{(n+q-i)(r-p)} &= [n^{n+q-i}\omega_n^{n-i}]^{r-p}\widetilde{W}_i(\Lambda_{q,i}K)^{q(r-p)}\\ &\leqslant [n^{n+q-i}\omega_n^{n-i}]^{r-p}\widetilde{W}_i(\Lambda_{p,i}K)^{p(r-q)}\widetilde{W}_i(\Lambda_{r,i}K)^{r(q-p)}\\ &=\Omega_{p,i}(K)^{(n+p-i)(r-q)}\Omega_{r,i}(K)^{(n+r-i)(q-p)}.\end{aligned} \tag{3.19}$$

According to the equality condition of (3.16), we know that equality holds in (3.19) if and only if $\Lambda_{p,i}K$ and $\Lambda_{r,i}K$ are dilates. This gives (1.13). □

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Tian Li
Department of Mathematics
China Three Gorges University
Yichang, 443002, P. R. China
e-mail: 13997717035@163.com

Weidong Wang
Department of Mathematics
China Three Gorges University
Yichang, 443002, P. R. China
e-mail: wdwxh722@163.com