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NEW LOWER BOUNDS FOR ARITHMETIC, GEOMETRIC, HARMONIC MEAN INEQUALITIES AND ENTROPY UPPER BOUND

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Abstract. In this paper the arithmetic, geometric and harmonic mean inequalities are refined. New lower bounds for the corresponding inequalities which are better than the previous ones are obtained. As an application for the results, a strong upper bound for Shannon's entropy is presented. The new entropy upper bound improves the basic results of Simic (2009) and Ţăpuş and Popescu (2012).

1. Introduction

For $n \geqslant 2$, let x_i, $i = 1, 2, \cdots, n$ be positive real numbers, and let w_i, $i = 1, 2, \cdots, n$ be positive weights such that: $\sum_{i=1}^n w_i = 1$. We denote by A_n, G_n and H_n be the (weighted) arithmetic, geometric and harmonic means of the $x_i{}'s$, that is,

$$A_n = \sum_{i=1}^n w_i x_i, \quad G_n = \prod_{i=1}^n x_i^{w_i}, \quad H_n = \left(\sum_{i=1}^n \frac{w_i}{x_i}\right)^{-1}.$$

It is well known that

$$H_n \leqslant G_n \leqslant A_n.$$

with the inequalities being strict unless all $x_i{}'s$ are equal.

The arithmetic, geometric and harmonic mean inequalities have received a great deal of attention of mathematicians. In 1978, Cartwright and Field [4] prove

$$\frac{1}{2\max_{1\leqslant i\leqslant n}\{x_i\}} \sum_{i=1}^n w_i (x_i - A_n)^2 \leqslant A_n - G_n \leqslant \frac{1}{2\min_{1\leqslant i\leqslant n}\{x_i\}} \sum_{i=1}^n w_i (x_i - A_n)^2.$$

In 1997, Alzer [3] improves the bounds as follows

$$A_n - G_n \geqslant \frac{1}{2\max_{1\leqslant i\leqslant n}\{x_i\}} \sum_{i=1}^n w_i (x_i - G_n)^2, \tag{1}$$

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In 2003, Mercer [11] obtains better bounds as follows

$$\sum_{i=1}^{n} \frac{w_i(x_i - G_n)^2}{x_i + \max(x_i, G_n)} \leqslant A_n - G_n \leqslant \sum_{i=1}^{n} \frac{w_i(x_i - G_n)^2}{x_i + \min(x_i, G_n)}, \tag{2}$$

and has other bounds for arithmetic-harmonic, geometric-harmonic mean inequalities

$$\frac{1}{A_n} \sum_{i=1}^{n} \frac{w_i(x_i - A_n)^2}{x_i + \max(x_i, A_n)} \leqslant \log A_n - \log G_n \leqslant \frac{1}{A_n} \sum_{i=1}^{n} \frac{w_i(x_i - A_n)^2}{x_i + \min(x_i, A_n)}, \tag{3}$$

$$\sum_{i=1}^{n} \frac{w_i}{x_i} \frac{(x_i - H_n)^2}{H_n + \max(x_i, H_n)} \leqslant \log G_n - \log H_n \leqslant \sum_{i=1}^{n} \frac{w_i}{x_i} \frac{(x_i - H_n)^2}{H_n + \min(x_i, H_n)}. \tag{4}$$

All the above equalities occur if and only if all $x_i's$ are equal. Later there are a considerable number of other extensions and refinements (cf. Aldaz [1, 2], Fujiwara and Ozawa [6], Gao [7, 8], Mercer [9, 10], Parkash and Kakkar [12]).

In this paper, we establish new lower bounds for the arithmetic, geometric and harmonic mean inequalities. As an application of our new lower bounds, we refine the work of Simic [13] and Ţăpuş and Popescu [14], obtain a more precise upper bound for Shannon's entropy.

2. Main results

Now we introduce a new inequality, where the term "log" refers to the natural logarithm.

LEMMA 1. *For $x > 0$,*

$$x - 1 - \log x \geqslant \frac{2(x-1)^2(x+2)}{3(x+1)^2}. \tag{5}$$

The equality holds if and only if $x = 1$.

Proof. Let $f(x) = x - 1 - \log x - \frac{2(x-1)^2(x+2)}{3(x+1)^2}$. Direct computing yields $f'(x) = \frac{(x-1)^3(x+3)}{3x(x+1)^3}$. This shows $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Next because $f(1) = 0$ holds, then $f(x) > 0$ for $0 < x < 1$ as well as $x > 1$. So the assertion of the lemma follows. □

REMARK 1. When $x > 0$, there exists the standard inequality $\log x \leqslant x - 1$ formerly. From the above lemma, we can refine the inequality into the form $\log x \leqslant x - 1 - \frac{2(x-1)^2(x+2)}{3(x+1)^2}$.

THEOREM 1. *The following lower bound holds, with equality occurring if and only if all $x_i's$ are equal.*

$$A_n - G_n \geqslant \sum_{i=1}^{n} \frac{2w_i(x_i - G_n)^2(x_i + 2G_n)}{3(x_i + G_n)^2}. \tag{6}$$

Proof. Substituting $x = \frac{x_i}{G_n}$ into (5), multiplying by w_i and summing, we obtain

$$\sum_{i=1}^{n} w_i \left(\frac{x_i}{G_n} - 1 - \log\left(\frac{x_i}{G_n}\right)\right) \geqslant \sum_{i=1}^{n} \frac{2w_i(\frac{x_i}{G_n}-1)^2(\frac{x_i}{G_n}+2)}{3(\frac{x_i}{G_n}+1)^2}, \tag{7}$$

and the equality holds if and only if all $x_i's$ are equal. Observing $G_n = \prod_{i=i}^{n} x_i^{w_i}$,

$$\sum_{i=1}^{n} w_i \log\left(\frac{x_i}{G_n}\right) = \sum_{i=1}^{n} \log \frac{x_i^{w_i}}{G_n^{w_i}} = \log \frac{\prod_{i=i}^{n} x_i^{w_i}}{G_n^{\sum_{i=1}^{n} w_i}} = \log \frac{G_n}{G_n} = 0.$$

Then we rewrite (7) into the following inequality

$$\frac{\sum_{i=1}^{n} w_i x_i}{G_n} - 1 \geqslant \frac{1}{G_n} \sum_{i=1}^{n} \frac{2w_i(x_i - G_n)^2(x_i+2G_n)}{3(x_i+G_n)^2}.$$

Since $A_n = \sum_{i=1}^{n} w_i x_i$, we have

$$\frac{A_n}{G_n} - 1 \geqslant \frac{1}{G_n} \sum_{i=1}^{n} \frac{2w_i(x_i - G_n)^2(x_i+2G_n)}{3(x_i+G_n)^2},$$

or equivalently,

$$A_n - G_n \geqslant \sum_{i=1}^{n} \frac{2w_i(x_i - G_n)^2(x_i+2G_n)}{3(x_i+G_n)^2}.$$

So the inequality (6) follows and the equality holds if and only if all $x_i's$ are equal. □

THEOREM 2. *The following lower bound holds, with equality occurring if and only if all $x_i's$ are equal.*

$$\log A_n - \log G_n \geqslant \frac{1}{A_n} \sum_{i=1}^{n} \frac{2w_i(x_i - A_n)^2(x_i+2A_n)}{3(x_i+A_n)^2}. \tag{8}$$

Proof. Substituting $x = \frac{x_i}{A_n}$ into (5), multiplying by w_i and summing, we obtain

$$\sum_{i=1}^{n} w_i \left(\frac{x_i}{A_n} - 1 - \log\left(\frac{x_i}{A_n}\right)\right) \geqslant \sum_{i=1}^{n} \frac{2w_i(\frac{x_i}{A_n}-1)^2(\frac{x_i}{A_n}+2)}{3(\frac{x_i}{A_n}+1)^2}, \tag{9}$$

and the equality holds if and only if all $x_i's$ are equal. We rewrite (9) into the following inequality

$$\frac{\sum_{i=1}^{n} w_i x_i}{A_n} - 1 - \log \frac{\prod_{i=1}^{n} x_i^{w_i}}{A_n^{\sum_{i=1}^{n} w_i}} \geqslant \frac{1}{A_n} \sum_{i=1}^{n} \frac{2w_i(x_i - A_n)^2(x_i+2A_n)}{3(x_i+A_n)^2}.$$

Since $A_n = \sum_{i=1}^{n} w_i x_i$, $G_n = \prod_{i=i}^{n} x_i^{w_i}$, we have

$$\log A_n - \log G_n \geqslant \frac{1}{A_n} \sum_{i=1}^{n} \frac{2w_i(x_i - A_n)^2(x_i+2A_n)}{3(x_i+A_n)^2}.$$

This is exactly the inequality (8) and the equality holds if and only if all $x_i's$ are equal. □

THEOREM 3. *The following lower bound holds, with equality occurring if and only if all x_i's are equal.*

$$\log G_n - \log H_n \geqslant \sum_{i=1}^{n} \frac{2w_i(x_i-H_n)^2(2x_i+H_n)}{3x_i(x_i+H_n)^2}. \tag{10}$$

Proof. Substituting $x = \frac{H_n}{x_i}$ into (5), multiplying by w_i and summing, we obtain

$$\sum_{i=1}^{n} w_i \left(\frac{H_n}{x_i} - 1 - \log\left(\frac{H_n}{x_i}\right)\right) \geqslant \sum_{i=1}^{n} \frac{2w_i(\frac{H_n}{x_i}-1)^2(\frac{H_n}{x_i}+2)}{3(\frac{H_n}{x_i}+1)^2}, \tag{11}$$

and the equality holds if and only if all $x_i's$ are equal. We rewrite (11) into the following inequality

$$H_n\left(\sum_{i=1}^{n}\frac{w_i}{x_i}\right) - 1 - \log\frac{H_n^{\sum_{i=1}^{n} w_i}}{\prod_{i=1}^{n} x_i^{w_i}} \geqslant \sum_{i=1}^{n} \frac{2w_i(x_i-H_n)^2(2x_i+H_n)}{3x_i(x_i+H_n)^2}.$$

Since $G_n = \prod_{i=i}^{n} x_i^{w_i}$, $H_n = \left(\sum_{i=1}^{n}\frac{w_i}{x_i}\right)^{-1}$, we have

$$\log G_n - \log H_n \geqslant \sum_{i=1}^{n} \frac{2w_i(x_i-H_n)^2(2x_i+H_n)}{3x_i(x_i+H_n)^2}.$$

This is exactly the inequality (10) and the equality holds if and only if all $x_i's$ are equal. □

The next two theorems will show that the new bound (6) is better than the previous lower bounds (1) and (2).

THEOREM 4. *The following inequality holds, with equality occurring if and only if all x_i's are equal.*

$$\sum_{i=1}^{n} \frac{2w_i(x_i-G_n)^2(x_i+2G_n)}{3(x_i+G_n)^2} \geqslant \frac{1}{2\max_{1\leqslant i\leqslant n}\{x_i\}} \sum_{i=1}^{n} w_i(x_i-G_n)^2. \tag{12}$$

Proof. Let $m := \max_{1\leqslant i\leqslant n}\{x_i\}$ and $g(x) = \frac{x+2G_n}{(x+G_n)^2}$, $x>0$. Straightforward differentiation shows that $g'(x) = -\frac{x+3G_n}{(x+G_n)^3} < 0$. So the function $g(x)$ is decreasing. Obviously for all $x_i's$, $x_i \leqslant m$ hold, then we have $g(x_i) \geqslant g(m)$ or $\frac{x_i+2G_n}{(x_i+G_n)^2} \geqslant \frac{m+2G_n}{(m+G_n)^2}$ and

$$\sum_{i=1}^{n} \frac{2w_i(x_i-G_n)^2(x_i+2G_n)}{3(x_i+G_n)^2} \geqslant \frac{2(m+2G_n)}{3(m+G_n)^2} \sum_{i=1}^{n} w_i(x_i-G_n)^2. \tag{13}$$

Let $h(x) = \frac{2(m+2x)}{3(m+x)^2}$, $x>0$. Straightforward differentiation shows that $h'(x) = -\frac{4x}{3(m+x)^3} < 0$. So the function $h(x)$ is decreasing. Obviously $G_n \leqslant m$ holds, then we have $h(G_n) \geqslant h(m)$ or $\frac{2(m+2G_n)}{3(m+G_n)^2} \geqslant \frac{2(m+2m)}{3(m+m)^2} = \frac{1}{2m}$. Using (13) we can obtain the assertion of this theorem. □

THEOREM 5. *The following inequality holds, with equality occurring if and only if all $x_i{}'s$ are equal.*

$$\sum_{i=1}^{n}\frac{2w_i(x_i-G_n)^2(x_i+2G_n)}{3(x_i+G_n)^2}\geqslant\sum_{i=1}^{n}\frac{w_i(x_i-G_n)^2}{x_i+\max(x_i,G_n)}. \tag{14}$$

Proof. Let $h_i(x)=\frac{2(x_i+2x)}{3(x_i+x)^2}$, $x>0$, $i=1,2,\cdots,n$. Straightforward differentiation shows that $h_i'(x)=-\frac{4x}{3(x_i+x)^3}<0$. So the functions $h_i(x)$ are all decreasing. Obviously for all $x_i{}'s$, $G_n\leqslant\max(x_i,G_n)$ hold, then we have $h_i(G_n)\geqslant h_i(\max(x_i,G_n))$ or

$$\frac{2(x_i+2G_n)}{3(x_i+G_n)^2}\geqslant\frac{2(x_i+2\max(x_i,G_n))}{3(x_i+\max(x_i,G_n))^2} \tag{15}$$

As $\max(x_i,G_n)\geqslant x_i$, we have $2(x_i+2\max(x_i,G_n))\geqslant 3(x_i+\max(x_i,G_n))$. From this the inequality is obtained as follows

$$\frac{2(x_i+2\max(x_i,G_n))}{3(x_i+\max(x_i,G_n))^2}\geqslant\frac{3(x_i+\max(x_i,G_n))}{3(x_i+\max(x_i,G_n))^2}=\frac{1}{x_i+\max(x_i,G_n)} \tag{16}$$

Using the inequalities (15) and (16) we obtain $\frac{2(x_i+2G_n)}{3(x_i+G_n)^2}\geqslant\frac{1}{x_i+\max(x_i,G_n)}$. And then the assertion of the theorem follows. □

The next theorem will show that the new bound (8) is better than the previous lower bound (3).

THEOREM 6. *The following inequality holds, with equality occurring if and only if all $x_i{}'s$ are equal.*

$$\frac{1}{A_n}\sum_{i=1}^{n}\frac{2w_i(x_i-A_n)^2(x_i+2A_n)}{3(x_i+A_n)^2}\geqslant\frac{1}{A_n}\sum_{i=1}^{n}\frac{w_i(x_i-A_n)^2}{x_i+\max(x_i,A_n)}. \tag{17}$$

Proof. Substituting A_n for G_n in (15) and (16), the assertion of the theorem follows by using the similar method. □

The next theorem will show that the new bound (10) is better than the previous lower bound (4).

THEOREM 7. *The following inequality holds, with equality occurring if and only if all $x_i{}'s$ are equal.*

$$\sum_{i=1}^{n}\frac{2w_i(x_i-H_n)^2(2x_i+H_n)}{3x_i(x_i+H_n)^2}\geqslant\sum_{i=1}^{n}\frac{w_i}{x_i}\frac{(x_i-H_n)^2}{H_n+\max(x_i,H_n)}. \tag{18}$$

Proof. Substituting H_n for G_n in (15) and (16), the assertion of the theorem follows by using the similar method. □

REMARK 2. We can conclude that the key point of the bounds for the arithmetic, geometric and harmonic mean inequalities is the bounds of the function $x-1-\log x$. In this paper, we obtain the new lower bound as the Lemma 1. So if we are able to find better upper bound such as $x-1-\log x \leqslant U(x)(x-1)^2$ than the previous upper bound in [11], then we may have new refinement of the upper bounds for the arithmetic, geometric and harmonic mean inequalities. In view of this, we conjecture that to find sharper $U(x)$ is the focus of the following analysis and discussion.

3. An application to refine entropy upper bound

In information theory [5], if the discrete probability distribution P is given by $P(X=i)=p_i,\ p_i>0,\ i=1,2,\cdots,n$, s.t. $\sum_{i=1}^n p_i=1$, then the Shannon's entropy is defined as $H(P):=\sum_{i=1}^n p_i\log\frac{1}{p_i}$. In 2009, Simic presents the corresponding entropy upper bound in [13] as follows:

$$\begin{aligned} 0 \leqslant m(\mu,\nu) &:= \mu\log\left(\frac{2\mu}{\mu+\nu}\right)+\nu\log\left(\frac{2\nu}{\mu+\nu}\right) \\ &\leqslant \log n - H(P) \leqslant \log\left(\frac{(\mu+\nu)^2}{4\mu\nu}\right) := M(\mu,\nu), \end{aligned} \tag{19}$$

where $\mu=\min_{1\leqslant i\leqslant n}\{p_i\}$ and $\nu=\max_{1\leqslant i\leqslant n}\{p_i\}$. In 2012, Ţăpuş and Popescu [14] obtain the sharper entropy upper bound based on Simic's work:

$$H(P)\leqslant \log n - \max_{1\leqslant\mu_1<\mu_2<\cdots<\mu_{n-1}\leqslant n}\log\left[\left(\frac{n-1}{\sum_{i=1}^{n-1}p_{\mu_i}}\right)^{\sum_{i=1}^{n-1}p_{\mu_i}}\left(\prod_{i=1}^{n-1}p_{\mu_i}^{p_{\mu_i}}\right)\right]. \tag{20}$$

In 2013, Parkash and Kakkar [12] obtain new inequalities using the arithmetic-geometric-harmonic mean inequality and improve the condition for the above bound.

In this section we will obtain a more precise entropy upper bound as a result of Theorem 2.

LEMMA 2. *Let* $f_\alpha(x):=\frac{2(x-\alpha)^2(x+2\alpha)}{3\alpha(x+\alpha)^2}+\log x,\ \alpha>0$. *Then* f_α *is a concave function on* $(0,+\infty)$.

Proof. Straightforward differentiation shows

$$f''_\alpha(x)=-\frac{(x-\alpha)^2(x^2+6\alpha x+\alpha^2)}{x^2(x+\alpha)^4}\leqslant 0.$$

So the function $f_\alpha(x)$ is a concave function on $(0,+\infty)$. □

LEMMA 3. *If* f_α *is defined as above,* $j\in\{2,\cdots,n-1\}$, *and the notation* T_j *is defined as follows:*

$$T_j := \max_{1\leqslant\mu_1<\mu_2<\cdots<\mu_j\leqslant n}\left[\left(\sum_{i=1}^{j}w_{\mu_i}\right)f_{A_n}\left(\frac{\sum_{i=1}^{j}w_{\mu_i}x_{\mu_i}}{\sum_{i=1}^{j}w_{\mu_i}}\right)-\sum_{i=1}^{j}w_{\mu_i}f_{A_n}\left(x_{\mu_i}\right)\right],$$

then we have

$$0 \leqslant T_2 \leqslant T_3 \leqslant \cdots \leqslant T_{n-1}.$$

Proof. Because f_{A_n} is concave on $(0,+\infty)$ by Lemma 2, using Jensen's inequality we can easily have $T_2 \geqslant 0$. Next we will show that for any $j \in \{2,\cdots,n-2\}$, $T_j \leqslant T_{j+1}$. Let us consider that the maximum of the expression

$$\left(\sum_{i=1}^{j} w_{\mu_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{j} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{j} w_{\mu_i}}\right) - \sum_{i=1}^{j} w_{\mu_i} f_{A_n}\left(x_{\mu_i}\right)$$

is obtained for $\mu_i = \eta_i,\ \eta_i \in \{1,2,\cdots,n\},\ i = 1,2,\cdots,j$. Then it is enough to prove that

$$\left(\sum_{i=1}^{j} w_{\eta_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{j} w_{\eta_i} x_{\eta_i}}{\sum_{i=1}^{j} w_{\eta_i}}\right) - \sum_{i=1}^{j} w_{\eta_i} f_{A_n}\left(x_{\eta_i}\right)$$
$$\leqslant \left(\sum_{i=1}^{j+1} w_{\eta_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{j+1} w_{\eta_i} x_{\eta_i}}{\sum_{i=1}^{j+1} w_{\eta_i}}\right) - \sum_{i=1}^{j+1} w_{\eta_i} f_{A_n}\left(x_{\eta_i}\right)$$

for any $\eta_{j+1} \in \{1,2,\cdots,n\} \setminus \{\eta_1,\cdots,\eta_j\}$. The above inequality is equivalent to

$$w_{\eta_{j+1}} f_{A_n}\left(x_{\eta_{j+1}}\right) + \left(\sum_{i=1}^{j} w_{\eta_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{j} w_{\eta_i} x_{\eta_i}}{\sum_{i=1}^{j} w_{\eta_i}}\right) \leqslant \left(\sum_{i=1}^{j+1} w_{\eta_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{j+1} w_{\eta_i} x_{\eta_i}}{\sum_{i=1}^{j+1} w_{\eta_i}}\right).$$

Multiplying by $\left(\sum_{i=1}^{j+1} w_{\eta_i}\right)^{-1}$, we have

$$\frac{w_{\eta_{j+1}}}{\sum_{i=1}^{j+1} w_{\eta_i}} f_{A_n}\left(x_{\eta_{j+1}}\right) + \frac{\sum_{i=1}^{j} w_{\eta_i}}{\sum_{i=1}^{j+1} w_{\eta_i}} f_{A_n}\left(\frac{\sum_{i=1}^{j} w_{\eta_i} x_{\eta_i}}{\sum_{i=1}^{j} w_{\eta_i}}\right) \leqslant f_{A_n}\left(\frac{\sum_{i=1}^{j+1} w_{\eta_i} x_{\eta_i}}{\sum_{i=1}^{j+1} w_{\eta_i}}\right).$$

This inequality follows from Jensen's inequality for the concave function $f_{A_n}(x)$. So we obtain the assertion of the lemma. □

THEOREM 8. *Let* $C := \frac{1}{A_n} \sum_{i=1}^{n} \frac{2w_i(x_i - A_n)^2(x_i + 2A_n)}{3(x_i + A_n)^2}$ *and* A_n, G_n *be as defined above, then the following estimates hold, with equality occurring if and only if all* x_i'*s are equal.*

$$C \leqslant C + T_2 \leqslant C + T_3 \leqslant \cdots \leqslant C + T_{n-1} \leqslant \log A_n - \log G_n. \tag{21}$$

Proof. Using Lemma 3, we have

$$C \leqslant C + T_2 \leqslant C + T_3 \leqslant \cdots \leqslant C + T_{n-1}.$$

Next we prove the last inequality of (21). Choose arbitrary $x_{\mu_i} \in \{x_1, x_2, \cdots, x_n\}$ such that $1 \leqslant \mu_1 < \mu_2 < \cdots < \mu_{n-1} \leqslant n$ with corresponding weights $\{w_{\mu_1}, w_{\mu_2} \cdots w_{\mu_{n-1}}\}$, and let $x_{\mu_n} = \{x_1, x_2, \cdots, x_n\} \setminus \{x_{\mu_1}, x_{\mu_2}, \cdots, x_{\mu_{n-1}}\}$. Using the inequality (8) for $W_1 = w_{\mu_n}$, $W_2 = \sum_{i=1}^{n-1} w_{\mu_i}$, $X_1 = x_{\mu_n}$, $X_2 = \frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}}$, $W_1X_1 + W_2X_2 = A_n$, we have

$$\begin{aligned}
\log A_n &= \log\left(\sum_{i=1}^{n} w_i x_i\right) = \log\left(\left(\sum_{i=1}^{n-1} w_{\mu_i}\right)\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} + w_{\mu_n} x_{\mu_n}\right) \\
&\geqslant \frac{1}{A_n}\left[\frac{2\left(\sum_{i=1}^{n-1} w_{\mu_i}\right)\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} - A_n\right)^2\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} + 2A_n\right)}{3\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} + A_n\right)^2}\right. \\
&\quad \left. + \frac{2w_{\mu_n}(x_{\mu_n} - A_n)^2(x_{\mu_n} + 2A_n)}{3(x_{\mu_n} + A_n)^2}\right] + \log\left(x_{\mu_n}^{w_{\mu_n}}\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}}\right)^{\sum_{i=1}^{n-1} w_{\mu_i}}\right) \\
&= \frac{1}{A_n}\frac{2\left(\sum_{i=1}^{n-1} w_{\mu_i}\right)\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} - A_n\right)^2\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} + 2A_n\right)}{3\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} + A_n\right)^2} \\
&\quad + \frac{1}{A_n}\sum_{i=1}^{n}\frac{2w_i(x_i - A_n)^2(x_i + 2A_n)}{3(x_i + A_n)^2} - \frac{1}{A_n}\sum_{i=1}^{n-1}\frac{2w_{\mu_i}(x_{\mu_i} - A_n)^2(x_{\mu_i} + 2A_n)}{3(x_{\mu_i} + A_n)^2} \\
&\quad + \log G_n - \sum_{i=1}^{n-1} w_{\mu_i}\log x_{\mu_i} + \left(\sum_{i=1}^{n-1} w_{\mu_i}\right)\log\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}} \\
&= \log G_n + C + \left(\sum_{i=1}^{n-1} w_{\mu_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}}\right) - \sum_{i=1}^{n-1} w_{\mu_i} f_{A_n}\left(x_{\mu_i}\right)
\end{aligned}$$

Because $\mu_i \in \{1, 2, \cdots, n\}$ are arbitrary, we have

$$\begin{aligned}
\log A_n &\geqslant \log G_n + C \\
&\quad + \max_{1 \leqslant \mu_1 < \mu_2 < \cdots < \mu_{n-1} \leqslant n}\left[\left(\sum_{i=1}^{n-1} w_{\mu_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{n-1} w_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} w_{\mu_i}}\right) - \sum_{i=1}^{n-1} w_{\mu_i} f_{A_n}\left(x_{\mu_i}\right)\right] \\
&= \log G_n + C + T_{n-1}.
\end{aligned}$$

Then the last inequality of (21) follows. □

Obviously, the inequalities (21) refine the inequality (8). By using Theorem 6, we can obtain the new upper entropy bound.

THEOREM 9. *We have*

$$H(P) \leqslant \log n - \frac{1}{n}\sum_{i=1}^{n}\frac{2(1 - np_i)^2(1 + 2np_i)}{3(1 + np_i)^2} - \max_{1 \leqslant \mu_1 < \mu_2 < \cdots < \mu_{n-1} \leqslant n}\{R(\mu) + S(\mu)\}, \quad (22)$$

where

$$R(\mu) := \log\left[\left(\frac{n-1}{\sum_{i=1}^{n-1} p_{\mu_i}}\right)^{\sum_{i=1}^{n-1} p_{\mu_i}} \left(\prod_{i=1}^{n-1} p_{\mu_i}^{p_{\mu_i}}\right)\right],$$

$$S(\mu) := \frac{2(n-1-n\sum_{i=1}^{n-1} p_{\mu_i})^2(n-1+2n\sum_{i=1}^{n-1} p_{\mu_i})}{3n(n-1+n\sum_{i=1}^{n-1} p_{\mu_i})^2} - \frac{1}{n}\sum_{i=1}^{n-1}\frac{2(1-np_{\mu_i})^2(1+2np_{\mu_i})}{3(1+np_{\mu_i})^2}.$$

Proof. Applying the last inequality (21) with $w_i = p_i$, $x_i = 1/p_i$, after some calculations by using $A_n = n$ we can obtain the inequality (22). □

REMARK 3. Let $\phi(x) := \frac{2(n-1)(1-nx)^2(1+2nx)}{3n(1+nx)^2}$. We can easily obtain $\phi(x)$ is convex for $x > 0$ by the second derivative $\phi''(x) = \frac{16(n-1)n^2x}{(1+nx)^4} > 0$. Hence, by Jensen's inequality we have $S(\mu) \leqslant 0$.

THEOREM 10. *The estimation* (22) *is better than* (20)*, i. e.,*

$$\begin{aligned} &\max_{1\leqslant\mu_1<\mu_2<\cdots<\mu_{n-1}\leqslant n}\{R(\mu)\} \\ \leqslant&\frac{1}{n}\sum_{i=1}^{n}\frac{2(1-np_i)^2(1+2np_i)}{3(1+np_i)^2} + \max_{1\leqslant\mu_1<\mu_2<\cdots<\mu_{n-1}\leqslant n}\{R(\mu)+S(\mu)\}. \end{aligned} \tag{23}$$

Proof. Let us consider that the maximum of $R(\mu)$ is obtained for $\mu_i = \eta_i$, $\eta_i \in \{1,2,\cdots,n\}$, $i = 1,2,\cdots,n-1$, and let $\eta_n = \{1,2,\cdots,n\}\setminus\{\eta_1,\cdots,\eta_{n-1}\}$. Then we obtain

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\frac{2(1-np_i)^2(1+2np_i)}{3(1+np_i)^2} + \max_{1\leqslant\mu_1<\mu_2<\cdots<\mu_{n-1}\leqslant n}\{R(\mu)+S(\mu)\} \\ &- \max_{1\leqslant\mu_1<\mu_2<\cdots<\mu_{n-1}\leqslant n}\{R(\mu)\} \\ \geqslant&\frac{1}{n}\sum_{i=1}^{n}\frac{2(1-np_i)^2(1+2np_i)}{3(1+np_i)^2} + S(\eta) \\ =&\frac{1}{n}\sum_{i=1}^{n}\frac{2(1-np_i)^2(1+2np_i)}{3(1+np_i)^2} + \frac{2(n-1-n\sum_{i=1}^{n-1} p_{\eta_i})^2(n-1+2n\sum_{i=1}^{n-1} p_{\eta_i})}{3n(n-1+n\sum_{i=1}^{n-1} p_{\eta_i})^2} \\ &- \frac{1}{n}\sum_{i=1}^{n-1}\frac{2(1-np_{\eta_i})^2(1+2np_{\eta_i})}{3(1+np_{\eta_i})^2} \\ =&\frac{2(1-np_{\eta_n})^2(1+2np_{\eta_n})}{3n(1+np_{\eta_n})^2} + \frac{2(n-1-n\sum_{i=1}^{n-1} p_{\eta_i})^2(n-1+2n\sum_{i=1}^{n-1} p_{\eta_i})}{3n(n-1+n\sum_{i=1}^{n-1} p_{\eta_i})^2} \geqslant 0. \end{aligned}$$

So the assertion of the theorem follows. □

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