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ESTIMATES FOR TSALLIS RELATIVE OPERATOR ENTROPY

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Abstract. We give the tight bounds of Tsallis relative operator entropy by using Hermite-Hadamard's inequality. Some reverse inequalities related to Young's inequality are also given. In addition, operator inequalities for normalized positive linear map with Tsallis relative operator entropy are given.

1. Introduction and preliminaries

The operator theory related to inequalities in Hilbert space is studied in many papers. Let A,B be two operators in a Hilbert space $\mathscr{H}$. An operator A is said to be strictly positive (denoted by $A>0$) if A is positive and invertible. For two strictly positive operators A,B and $p\in[0,1]$, p-power mean $A\#_pB$ is defined by

$$A\#_pB := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^pA^{\frac{1}{2}},$$

and we remark that $A\#_pB = A^{1-p}B^p$ if A commutes with B. We also use the symbol $A\natural_rB := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^rA^{\frac{1}{2}}$ for $r\in\mathbb{R}$. The weighted operator arithmetic mean is defined by

$$A\nabla_pB := (1-p)A+pB,$$

for any $p\in[0,1]$.

The relative operator entropy $S(A|B)$ in [5] is defined by

$$S(A|B) := A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

where A and B are two invertible positive operators on a Hilbert space. As a parametric extension of the relative operator entropy, Yanagi, Kuriyama and Furuichi [19] defined Tsallis relative operator entropy which is an operator version of Tsallis relative entropy in quantum system due to Abe [1], also see [9]:

$$T_p(A|B) := \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^pA^{\frac{1}{2}}-A}{p}, \quad p\in(0,1].$$

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Notice that $T_p(A|B)$ can be written by using the notation of $A\#_pB$ as follows:

$$T_p(A|B) := \frac{A\#_pB - A}{p}, \quad p \in (0,1].$$

The relation between relative operator entropy $S(A|B)$ and Tsallis relative operator entropy $T_p(A|B)$ was considered in [9, 20], as follows:

$$A - AB^{-1}A \leqslant T_{-p}(A|B) \leqslant S(A|B) \leqslant T_p(A|B) \leqslant B - A. \tag{1}$$

Some deeper properties of Tsallis relative operator entropy were proved in [6, 11, 13, 20].

The main result of the present paper is a set of bounds that are complementary to (1). Some of our inequalities improve well-known ones. Among other inequalities, it is shown that if A, B are invertible positive operators and $p \in (0,1]$, then

$$\begin{aligned} A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-I\right)A^{\frac{1}{2}} &\leqslant T_p(A|B) \\ &\leqslant \frac{1}{2}\left(A\#_pB - A\natural_{p-1}B + B - A\right), \end{aligned}$$

which is a considerable refinement of (1), where I is the identity operator. We also prove a reverse inequality involving Tsallis relative operator entropy $T_p(A|B)$.

2. Refinements of the inequalities (1) via Hermite-Hadamard inequality

An important ingredient in our approach is the following:

Let f be a convex function on $[a,b] \subseteq \mathbb{R}$; the well-known Hermite-Hadamard's inequality can be expressed as

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2}. \tag{2}$$

THEOREM 1. *For any invertible positive operators A and B such that $A \leqslant B$, and $p \in (0,1]$ we have*

$$\begin{aligned} A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-I\right)A^{\frac{1}{2}} &\leqslant T_p(A|B) \\ &\leqslant \frac{1}{2}\left(A\#_pB - A\natural_{p-1}B + B - A\right). \end{aligned} \tag{3}$$

Proof. Consider the function $f(t) = t^{p-1}$, $p \in (0,1]$. It is easy to check that $f(t)$ is convex on $[1,\infty)$. Bearing in mind the fact

$$\int_1^x t^{p-1}dt = \frac{x^p - 1}{p},$$

and utilizing the left-hand side of Hermite-Hadamard inequality, one can see that

$$\left(\frac{x+1}{2}\right)^{p-1}(x-1) \leqslant \frac{x^p-1}{p}, \tag{4}$$

where $x \geqslant 1$ and $p \in (0,1]$. On the other hand, it follows from the right-hand side of Hermite-Hadamard inequality that

$$\frac{x^p-1}{p} \leqslant \left(\frac{x^{p-1}+1}{2}\right)(x-1), \tag{5}$$

for each $x \geqslant 1$ and $p \in (0,1]$.

Replacing x by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (4) and (5), and multiplying $A^{\frac{1}{2}}$ on both sides, we get the desired result (3). □

REMARK 1. Simple calculation gives for all $x \geqslant 1$ and $p \in (0,1]$,

$$0 \leqslant 1-\frac{1}{x} \leqslant \left(\frac{x+1}{2}\right)^{p-1}(x-1) \leqslant \frac{x^p-1}{p} \leqslant \left(\frac{x^{p-1}+1}{2}\right)(x-1) \leqslant x-1, \tag{6}$$

which means

$$\begin{aligned} 0 \leqslant A-AB^{-1}A &\leqslant A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-I\right)A^{\frac{1}{2}} \\ &\leqslant T_p(A|B) \\ &\leqslant \frac{1}{2}\left(A\#_p B - A\natural_{p-1}B + B - A\right) \\ &\leqslant B-A, \end{aligned}$$

for any invertible positive operators A and B such that $A \leqslant B$, and $p \in (0,1]$. Therefore, our inequalities (3) improve the inequalities (1) for the case $A \leqslant B$.

PROPOSITION 1. For $x \geqslant 1$ and $\frac{1}{2} \leqslant p \leqslant 1$,

$$\frac{x-1}{\sqrt{x}} \leqslant \left(\frac{x+1}{2}\right)^{p-1}(x-1). \tag{7}$$

Proof. In order to prove (7), we set the function $f_p(x) \equiv \left(\frac{x+1}{2}\right)^{p-1} - \frac{1}{\sqrt{x}}$ where $x \geqslant 1$ and $\frac{1}{2} \leqslant p \leqslant 1$. Since $\frac{df_p(x)}{dp} = \left(\frac{x+1}{2}\right)^{p-1}\ln\left(\frac{x+1}{2}\right)$, $\frac{df_p(x)}{dp} \geqslant 0$ for $x \geqslant 1$. Thus, we have $f_p(x) \geqslant f_{1/2}(x) = \frac{\sqrt{2x}-\sqrt{x+1}}{\sqrt{x(x+1)}} \geqslant 0$ for $x \geqslant 1$. Therefore, we have the inequality (7). □

REMARK 2. The first inequality (3) gives tight lower bound for the Tsallis relative operator entropy $T_p(A|B)$ more than the eighth inequality in [8, Theorem 2.8 (i)], due to Proposition 1.

COROLLARY 1. *For any invertible positive operators A and B such that $A \geqslant B$, and $p \in (0,1]$, we have*

$$
\begin{aligned}
A\#_p B - A\natural_{p-1}B &\leqslant \frac{1}{2}\left(A\#_p B - A\natural_{p-1}B + B - A\right) \leqslant T_p\left(A|B\right) \\
&\leqslant A^{\frac{1}{2}}\left(\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}\right)^{p-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-I\right)A^{\frac{1}{2}} \\
&\leqslant A\natural_{p+1}B - A\#_p B \leqslant 0.
\end{aligned}
$$

Proof. Put $t = \frac{1}{x} \leqslant 1$ in the inequalities (6). $\square$

REMARK 3. By numerical computations, we have $\left(\frac{x^{p-1}+1}{2}\right)(x-1) - \frac{2(x-1)}{x+1} \simeq -0.83219$ for $p = \frac{2}{3}, x = 0.01$, and we also have $\left(\frac{x^{p-1}+1}{2}\right)(x-1) - \frac{2(x-1)}{x+1} \simeq 0.216868$ for $p = \frac{2}{3}, x = 0.1$. Thus we do not have ordering between $\left(\frac{x^{p-1}+1}{2}\right)(x-1)$ and $\frac{2(x-1)}{x+1}$ for $0 < x \leqslant 1$ and $\frac{1}{2} \leqslant p \leqslant 1$. So there is no ordering between the second inequality in Corollary 1 and the sixth inequality in [8, Theorem 2.8 (ii)].

THEOREM 2. *For any invertible positive operators A and B such that $A \leqslant B$ and $p \in (0,1]$, we have the following inequalities,*

$$L_p(A,B) + K_p(A,B) \leqslant T_p(A|B) \leqslant R_p(A,B) + K_p(A,B), \tag{8}$$

and

$$J_p(A,B) - 2R_p(A,B) \leqslant T_p(A|B) \leqslant J_p(A,B) - 2L_p(A,B), \tag{9}$$

where

$$
\begin{aligned}
&K_p(A,B) \equiv A^{1/2}\left(\tfrac{A^{-1/2}BA^{-1/2}+I}{2}\right)^{p-1}\left(A^{-1/2}BA^{-1/2}-I\right)A^{1/2}, \\
&J_p(A,B) \equiv \tfrac{1}{2}\left(A\#_p B - A\natural_{p-1}B + B - A\right), \\
&L_p(A,B) \equiv \tfrac{1}{24}(p-1)(p-2)\left(A\#_p B - 3A\natural_{p-1}B + 3A\natural_{p-2}B - A\natural_{p-3}B\right), \\
&R_p(A,B) \equiv \tfrac{1}{24}(p-1)(p-2)\left(A\natural_3 B - 3A\natural_2 B + 3B - A\right).
\end{aligned}
$$

Proof. According to [18, Theorem 1], if $f : [a,b] \to \mathbb{R}$ is a twice differentiable function that there exists real constants m and M so that $m \leqslant f'' \leqslant M$, then

$$m\frac{(b-a)^2}{24} \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leqslant M\frac{(b-a)^2}{24}, \tag{10}$$

$$m\frac{(b-a)^2}{12} \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant M\frac{(b-a)^2}{12}. \tag{11}$$

Putting $f(t) = t^{p-1}$ with $p \in (0,1]$ and $a = 1$, $b = x$ in the above inequalities, then we have the desired results by a similar way to the proof of Theorem 1. $\square$

REMARK 4. The first inequality of (8) and the second inequality of (9) give tighter bounds of Tsallis relative entropy $T_p(A|B)$ than those in the inequalities (3), because of the following reasons.

(i) The first inequality of (10) gives tight lower bound more than the first inequality of Hermite-Hadamard's inequality (2).

(ii) The first inequality of (11) gives tight upper bound more than the second inequality of Hermite-Hadamard's inequality (2).

3. Some reverse inequalities via Young type inequalities

The scalar Young's inequality says that if $a,b>0$, then

$$a^{1-p}b^p \leqslant (1-p)a+pb, \quad p\in[0,1].$$

The following inequalities provide a refinement and a multiplicative reverse for Young's inequality with Kantorovich constant:

$$K^r(h,2)\,a^{1-p}b^p \leqslant (1-p)a+pb \leqslant K^R(h,2)\,a^{1-p}b^p, \tag{12}$$

where $a,b>0,\ p\in[0,1],\ r=\min\{p,1-p\},\ R=\max\{p,1-p\},\ K(h,2)=\frac{(h+1)^2}{4h}$ and $h=\frac{b}{a}$. Notice that the first inequality in (12) was obtained by Zou *et al.* in [21, Corollary 3] while the second one was obtained by Liao *et al.* [17, Corollary 2.2].

In [14, 15], Kittaneh and Manasrah obtained another refinement and reverse of Young's inequality:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2+a^{1-p}b^p \leqslant (1-p)a+pb \leqslant R\left(\sqrt{a}-\sqrt{b}\right)^2+a^{1-p}b^p, \tag{13}$$

where $a,b>0,\ p\in[0,1],\ r=\min\{p,1-p\},\ R=\max\{p,1-p\}$. Further refinements and generalizations of Young's inequality have been obtained in [2, 7].

More interesting things happen when we apply these considerations to the operators. For instance, from the inequality (12) it follows that:

PROPOSITION 2. Let A,B be two invertible positive operators such that $I<h'I\leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant hI$ or $0<hI\leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leqslant h'I<I$, then

$$K^r\left(h',2\right)A\#_pB \leqslant A\nabla_pB \leqslant K^R(h,2)\,A\#_pB, \tag{14}$$

where $p\in[0,1],\ r=\min\{p,1-p\},\ R=\max\{p,1-p\}$.

Proof. The choice $a=1$ and $b=x$ reduces (12) to the inequality

$$K^r(x,2)\,x^p \leqslant (1-p)+px \leqslant K^R(x,2)\,x^p.$$

Whence

$$\min_{1<h'\leqslant x\leqslant h} K^r(x,2)\,T^p \leqslant (1-p)I+pT \leqslant \max_{1<h'\leqslant x\leqslant h} K^R(x,2)\,T^p, \tag{15}$$

for any positive operator T such that $I < h'I \leqslant T \leqslant hI$. On choosing $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (15) we get

$$\min_{1<h'\leqslant x\leqslant h} K^r(x,2)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p \leqslant (1-p)I + pA^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant \max_{1<h'\leqslant x\leqslant h} K^R(x,2)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p.$$

Since $K(x,2)$ is increasing for $x > 1$ we can write

$$K^r\left(h',2\right)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p \leqslant (1-p)I + pA^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant K^R(h,2)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p. \quad (16)$$

Multiplying both sides by $A^{\frac{1}{2}}$ to inequality (16), we obtain the required inequality (14).

Another case follows from the fact that $K(x,2)$ is decreasing for $x < 1$. □

Ando's inequality [3, Theorem 3] says that if A,B are positive operators and Φ is a positive linear mapping, then

$$\Phi(A\#_pB) \leqslant \Phi(A)\#_p\Phi(B), \quad p \in [0,1]. \quad (17)$$

Concerning inequality (17), we have the following corollary:

COROLLARY 2. *Let A,B be two invertible positive operators such that $I < h'I \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant hI$ or $0 < hI \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant h'I < I$. Let Φ be positive linear map on $\mathscr{B}(\mathscr{H})$, then*

$$\begin{aligned}\frac{K^r(h',2)}{K^R(h,2)}\Phi(A\#_pB) &\leqslant \frac{1}{K^R(h,2)}\Phi(A\nabla_pB) \leqslant \Phi(A)\#_p\Phi(B)\\ &\leqslant \frac{1}{K^r(h',2)}\Phi(A\nabla_pB) \leqslant \frac{K^R(h,2)}{K^r(h',2)}\Phi(A\#_pB),\end{aligned} \quad (18)$$

where $p \in [0,1]$, $r = \min\{p, 1-p\}$ and $R = \max\{p, 1-p\}$.

Proof. If we apply positive linear map Φ in (14) we infer

$$K^r\left(h',2\right)\Phi(A\#_pB) \leqslant \Phi(A\nabla_pB) \leqslant K^R(h,2)\Phi(A\#_pB). \quad (19)$$

On the other hand, if we take $A = \Phi(A)$ and $B = \Phi(B)$ in (14) we can write

$$K^r\left(h',2\right)\Phi(A)\#_p\Phi(B) \leqslant \Phi(A\nabla_pB) \leqslant K^R(h,2)\Phi(A)\#_p\Phi(B). \quad (20)$$

Now, combining inequality (19) and inequality (20), we deduce the desired inequalities (18). □

REMARK 5. It is well-known that the generalized Kantorovich constant $K(h,p)$ [12, Definition 1] is defined by

$$K(h,p) := \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p, \quad (21)$$

for all $p \in \mathbb{R}$. By virtue of a generalized Kantorovich constant, in the matrix setting, Bourin *et al.* in [4, Theorem 6] gave the following reverse of Ando's inequality for a positive linear map:

Let A and B be positive operators such that $mA \leqslant B \leqslant MA$, and let Φ be a positive linear map. Then

$$\Phi(A)\#_p\Phi(B) \leqslant \frac{1}{K(h,p)}\Phi(A\#_pB), \quad p \in [0,1] \tag{22}$$

where $h = \frac{M}{m}$. The above result naturally extends one proved in Lee [16, Theorem 4] for $p = \frac{1}{2}$.

Of course the constant $\frac{K^R(h,2)}{K^r(h',2)}$ is not better than $\frac{1}{K\left(\frac{h}{h'},p\right)}$. Concerning the sharpness of the estimate (22), see [4, Lemma 7]. However our bounds on $\Phi(A)\#_p\Phi(B)$ are calculated by the original Kantorovich constant $K(h,2)$ without the generalized one $K(h,p)$. It is also interesting our bounds on $\Phi(A)\#_p\Phi(B)$ are expressed by $\Phi(A\nabla_pB)$ with only one constant either h or h'.

After discussion on inequalities related to the operator mean with positive linear map, we give a result on Tsallis relative operator entropy (which is the main theme in this paper) with a positive linear map. It is well-known that Tsallis relative operator entropy has the following information monotonicity:

$$\Phi(T_p(A|B)) \leqslant T_p(\Phi(A)|\Phi(B)). \tag{23}$$

Utilizing (13), we have the following counterpart of (23):

THEOREM 3. *Let A,B be two invertible positive operators. Let Φ be normalized positive linear map on $\mathscr{B}(\mathscr{H})$, then*

$$\begin{aligned} &\frac{2r}{p}(\Phi(A\nabla B) - \Phi(A)\#\Phi(B)) + T_p(\Phi(A)|\Phi(B)) \\ &\leqslant \Phi(B-A) \leqslant \frac{2R}{p}(\Phi(A\nabla B) - \Phi(A\#B)) + \Phi(T_p(A|B)), \end{aligned} \tag{24}$$

where $p \in (0,1]$, $r = \min\{p,1-p\}$ and $R = \max\{p,1-p\}$.

Proof. Using the method of the proof of Proposition 2 and Corollary 2 for the inequality (13), we can obtain that

$$\begin{aligned} &2r(\Phi(A\nabla B) - \Phi(A)\#\Phi(B)) + \Phi(A)\#_p\Phi(B) \\ &\leqslant \Phi(A\nabla_pB) \leqslant 2R(\Phi(A\nabla B) - \Phi(A\#B)) + \Phi(A\#_pB). \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} &\frac{2r}{p}(\Phi(A\nabla B) - \Phi(A)\#\Phi(B)) + \frac{\Phi(A)\#_p\Phi(B) - \Phi(A)}{p} \\ &\leqslant \Phi(B-A) \leqslant \frac{2R}{p}(\Phi(A\nabla B) - \Phi(A\#B)) + \frac{\Phi(A\#_pB) - \Phi(A)}{p}. \end{aligned}$$

Apparently, the above inequality is equivalent to inequality (24). The proof is completed. □

The following example may render the above statement clearer.

EXAMPLE 1. Let $\Phi(X) = U^*XU\ (X \in \mathscr{M}_2)$ where $U = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$. If $A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$ and $p = \frac{1}{4}$ we get

$$\frac{2r}{p}\left(\Phi\left(A\nabla B\right) - \Phi\left(A\right)\#\Phi\left(A\right)\right) + \left(T_p\left(\Phi\left(A\right)|\Phi\left(A\right)\right)\right) = \begin{pmatrix} 0.486 & 0.443 \\ 0.443 & 5.638 \end{pmatrix}$$

$$\Phi\left(B - A\right) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 6.5 \end{pmatrix}$$

and

$$\frac{2R}{p}\left(\Phi\left(A\nabla B\right) - \Phi\left(A\#B\right)\right) + \Phi\left(T_p\left(A|B\right)\right) = \begin{pmatrix} 0.562 & 0.951 \\ 0.951 & 13.521 \end{pmatrix}$$

which shows that inequality (24) is true.

Tsallis relative entropy $D_p\left(A||B\right)$ for two positive operators A and B is defined by:

$$D_p\left(A||B\right) := \frac{Tr\left[A\right] - Tr\left[A^{1-p}B^p\right]}{p}, \quad p \in (0,1].$$

In information theory, relative entropy (divergence) is usually defined for density operators which are positive operators with unit trace. However we consider Tsallis relative entropy defined for positive operators to derive the relation with Tsallis relative operator entropy. If A and B are positive operators, then

$$Tr\left[A - B\right] \leqslant D_p\left(A||B\right) \leqslant -Tr\left[T_p\left(A|B\right)\right], \quad p \in (0,1]. \tag{25}$$

Note that the first inequality of (25) is due to Furuta [13, Proposition F] and the second inequality is due to Furuichi *et al.* [10, Theorem 2.2].

As a direct consequence of Theorem 3, we have the following interesting relation.

COROLLARY 3. *Let A, B be two positive operators on a finite dimensional Hilbert space $\mathscr{H}$, then*

$$\begin{aligned} &\frac{2R}{p}\left(Tr\left[A\#B\right] - \frac{Tr\left[A+B\right]}{2}\right) - Tr\left[T_p\left(A|B\right)\right] \\ &\leqslant Tr\left[A-B\right] \leqslant \frac{2r}{p}\left(\sqrt{Tr\left[A\right]Tr\left[B\right]} - \frac{Tr\left[A+B\right]}{2}\right) + D_p\left(A||B\right), \end{aligned} \tag{26}$$

where $p \in (0,1]$, $r = \min\{p, 1-p\}$ *and* $R = \max\{p, 1-p\}$.

Proof. Taking $\Phi(X) = \frac{1}{\dim \mathscr{H}} Tr[X]$ in (24), we have

$$\begin{aligned} &\frac{2r}{p}\left(\frac{Tr[A+B]}{2} - \sqrt{Tr[A]\,Tr[B]}\right) + \frac{(Tr[A])^{1-p}(Tr[B])^{p} - Tr[A]}{p} \\ &\leqslant Tr[B-A] \leqslant \frac{2R}{p}\left(\frac{Tr[A+B]}{2} - Tr[A\#B]\right) + Tr[T_p(A|B)]. \end{aligned} \tag{27}$$

It is not too difficult to see that (27) can be also reformulated in the following way

$$\begin{aligned} &\frac{2R}{p}\left(Tr[A\#B] - \frac{Tr[A+B]}{2}\right) - Tr[T_p(A|B)] \\ &\leqslant Tr[A-B] \leqslant \frac{2r}{p}\left(\sqrt{Tr[A]\,Tr[B]} - \frac{Tr[A+B]}{2}\right) + \frac{Tr[A] - (Tr[A])^{1-p}(Tr[B])^{p}}{p}. \end{aligned}$$

Now, having in mind that $Tr\left[A^{1-p}B^{p}\right] \leqslant (Tr[A])^{1-p}(Tr[B])^{p}$, we infer

$$\begin{aligned} &\frac{2R}{p}\left(Tr[A\#B] - \frac{Tr[A+B]}{2}\right) - Tr[T_p(A|B)] \\ &\leqslant Tr[A-B] \leqslant \frac{2r}{p}\left(\sqrt{Tr[A]\,Tr[B]} - \frac{Tr[A+B]}{2}\right) + \frac{Tr[A] - Tr\left[A^{1-p}B^{p}\right]}{p}, \end{aligned}$$

which, in turn, leads to (26). □

REMARK 6. If A and B are density operators, then the second inequality of (26) implies the non-negativity of Tsallis relative entropy, $D_p(A||B) \geqslant 0$. If $p \in [\frac{1}{2}, 1]$, then the first inequality of (26) implies

$$Tr[T_{1/2}(A|B)] \leqslant Tr[T_p(A|B)].$$

If $p \in (0, \frac{1}{2}]$, then the first inequality of (26) also implies

$$(1-p)Tr[T_{1/2}(A|B)] + (2p-1)Tr[B-A] \leqslant pTr[T_p(A|B)].$$

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