Mathematical
Inequalities
& Applications
Volume 21, Number 1 (2018), 1–16
doi:10.7153/mia-2018-21-01

NECESSARY CONDITIONS FOR THE BOUNDEDNESS OF LINEAR AND BILINEAR COMMUTATORS ON BANACH FUNCTION SPACES

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(*Communicated by I. Perić*)

Abstract. In this article we extend recent results by the first author [3] on the necessity of BMO for the boundedness of commutators on the classical Lebesgue spaces. We generalize these results to a large class of Banach function spaces. We show that with modest assumptions on the underlying spaces and on the operator T, if the commutator $[b,T]$ is bounded, then the function b is in BMO.

1. Introduction

The purpose of this paper is to extend a recent result of the first author [3] on necessary conditions for commutators to be bounded on the classical Lebesgue spaces. He showed that if T is a "nice" operator, and if (for example) the commutator $[b,T]$ is bounded on L^p, then $b \in BMO$. He also proved an analogous result for bilinear commutators. We generalize these results to a large collection of Banach function spaces. To do so requires the assumption of a geometric condition on the underlying spaces that is closely related to the boundedness of the Hardy-Littlewood maximal operator, and which holds in a large number of important special cases.

To state our results we recall some basic facts about Banach function spaces. For further information, see Bennett and Sharpley [1]. By a Banach function space X we mean a Banach space of measurable functions over $\mathbb{R}^n$ whose norm $\|\cdot\|_X$ satisfies the following for all $f, g \in X$:

1. $\|f\|_X = \||f|\|_X$;
2. if $|f| \leqslant |g|$ a.e., then $\|f\|_X \leqslant \|g\|_X$;
3. if $\{f_n\} \subset X$ is a sequence such that $|f_n|$ increases to $|f|$ a.e., then $\|f_n\|_X$ increases to $\|f\|_X$;
4. if $E \subset \mathbb{R}^n$ is bounded, then $\|\chi_E\|_X < \infty$;
5. if E is bounded, then $\int_E |f(x)|\,d\mu \leqslant C\|f\|_X$, where $C = C(E,X)$.

Mathematics subject classification (2010): 42B20, 42B35.
Keywords and phrases: BMO, commutators, singular integrals, fractional integrals, bilinear operators, weights, variable Lebesgue spaces.
The second author is supported by NSF Grant DMS-1362425 and research funds from the Dean of the College of Arts & Sciences, the University of Alabama.

Given a Banach function space X, there exists another Banach function space X', called the associate space of X, such that for all $f \in X$,

$$\|f\|_X \approx \sup_{\substack{g \in X' \\ \|g\|_{X'} \leqslant 1}} \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

In many (but not all) cases, the associate space is equal to the dual space X^*. The associate space, however, is always reflexive, in the sense that $(X')' = X$. Moreover, we have the following generalization of Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leqslant \|f\|_X \|g\|_{X'}.$$

Given a linear operator T, define the commutator $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$, where b is a locally integrable function. We can now state our first result.

THEOREM 1.1. *Given Banach function spaces X and Y, and $0 \leqslant \alpha < n$, suppose that for every cube Q,*

$$|Q|^{-\frac{\alpha}{n}} \|\chi_Q\|_{Y'} \|\chi_Q\|_X \leqslant C|Q|. \tag{1}$$

Let T be a linear operator defined on X which can be represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

for all $x \notin supp(f)$, where K is a homogeneous kernel of degree $n-\alpha$. Suppose further that there exists a ball $B \in \mathbb{R}^n$ on which $\frac{1}{K}$ has an absolutely convergent Fourier series. If the commutator satisfies $[b,T] : X \to Y$, then $b \in BMO(\mathbb{R}^n)$.

A wide variety of classical operators satisfy the hypotheses of Theorem 1.1. The kernel K is such that $\frac{1}{K}$ has an absolutely convergent Fourier series if it is non-zero on a ball B and has enough regularity: $K \in C^s(B)$ for $s > n/2$ is sufficient. (See Grafakos [17, Theorem 3.2.16]. For weaker sufficient conditions, see recent results by Móricz and Veres [31].) In the linear case this condition is satisfied by Calderón-Zygmund singular integrals of convolution type whose kernels are smooth, and in particular the Riesz transforms. It also includes the fractional integral operators (also referred to as Riesz potentials). For precise definitions, see Section 2 below.

To state our result in bilinear case we recall that there are two commutators to consider: if T is a bilinear operator and $b \in L^1_{loc}(\mathbb{R}^n)$, define

$$[b,T]_1(f,g)(x) = b(x)T(f,g)(x) - T(bf,g)(x),$$
$$[b,T]_2(f,g)(x) = b(x)T(f,g)(x) - T(f,bg)(x).$$

THEOREM 1.2. *Given Banach function spaces X_1, X_2, and Y, and $0 \leqslant \alpha < 2n$, suppose that for every cube Q,*

$$|Q|^{-\frac{\alpha}{n}} \|\chi_Q\|_{Y'} \|\chi_Q\|_{X_1} \|\chi_Q\|_{X_2} \leqslant C|Q|. \tag{2}$$

Let T be a bilinear operator defined on $X_1 \times X_2$ which can be represented by

$$T(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y,x-z)f(y)g(z)\,dydz$$

for all $x \notin supp(f) \cap supp(g)$, where K is a homogeneous kernel of degree $2n-\alpha$. Suppose further that there exists a ball $B \in \mathbb{R}^{2n}$ on which $\frac{1}{K}$ has an absolutely convergent Fourier series. If for $j=1$ or $j=2$, the bilinear commutator satisfies

$$[b,T]_j : X_1 \times X_2 \to Y,$$

then $b \in BMO(\mathbb{R}^n)$.

REMARK 1. Theorem 1.2 extends naturally to multilinear operators. We leave the statement and proof of this generalization to the interested reader.

REMARK 2. The restrictions on α are not actually necessary in the proofs of Theorems 1.1 and 1.2: we can take any $\alpha \in \mathbb{R}$. However, we are not aware of any operators for which Banach function space estimates hold that do not satisfy the given restrictions on α.

For the absolute convergence of multiple Fourier series, see the above references. In the bilinear case, Theorem 1.2 covers such operators as the bilinear Calderón-Zygmund operators with smooth kernels [20, 22, 21, 28] and the bilinear fractional integral operator [16, 19, 25, 30].

One drawback of Theorem 1.2 is that we must assume that the target space Y is a Banach function space. This is somewhat restrictive: even in the case of the Lebesgue spaces, bilinear operators satisfy inequalities of the form $T : L^{p_1} \times L^{p_2} \to L^p$ where $p < 1$. This assumption, however, is intrinsic to the statement and proof of our result, since we need to use the generalized Hölder's inequality. We are uncertain what the correct assumption should be when we assume that Y is only a quasi-Banach space.

REMARK 3. The necessity of BMO for the boundedness on Lebesgue spaces of commutators of certain multilinear singular integrals was recently proved in [29] using a completely different approach, but the authors were also required to assume that for the target space L^p, $p \geqslant 1$.

The remainder of this paper is split into two parts. We defer the actual proof of Theorems 1.1 and 1.2 to Section 3 and in fact we will only give the proof of the latter; the proof of the former is gotten by a trivial adaptation of the proof in the bilinear case. But first, in Section 2 we give a number of specific examples of Banach function spaces and consider the relationship between the conditions (1) and (2), and sufficient conditions for maximal operators and commutators to be bounded.

Throughout this paper, n will denote the dimension of the underlying space, $\mathbb{R}^n$. We will consider real-valued functions over $\mathbb{R}^n$. Cubes in $\mathbb{R}^n$ will always have their sides parallel to the coordinate axes. If we write $A \lesssim B$, we mean $A \leqslant cB$, where the constant c depends on the operator T, the Banach function spaces, and on the dimension n. These implicit constants may change from line to line. If we write $A \approx B$, then $A \lesssim B$ and $B \lesssim A$.

2. Examples of Banach function spaces

Averaging and maximal operators

The necessary condition (1) in Theorem 1.1 is closely related to a necessary condition for the boundedness of averaging operators and fractional maximal operators. For $0 \leqslant \alpha < n$, given a cube Q define the linear α-averaging operator

$$A_\alpha^Q f(x) = |Q|^{\frac{\alpha}{n}} \fint_Q f(y)\,dy \cdot \chi_Q(x).$$

We define the associated fractional maximal operator by

$$M_\alpha f(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \fint_Q |f(y)|\,dy \cdot \chi_Q(x).$$

We immediately have that for all Q, $|A_\alpha^Q f(x)| \leqslant M_\alpha f(x)$. We also make the analogous definitions in the bilinear case: for $0 \leqslant \alpha < 2n$,

$$A_\alpha^Q(f,g)(x) = |Q|^{\frac{\alpha}{n}} \fint_Q f(y)\,dy \fint_Q g(y)\,dy \cdot \chi_Q(x),$$

$$M_\alpha(f,g)(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \fint_Q |f(y)|\,dy \fint_Q |g(y)|\,dy \cdot \chi_Q(x).$$

Again, we have the pointwise bound $|A_\alpha^Q(f,g)(x)| \leqslant M_\alpha(f,g)(x)$. In both the linear and bilinear case, when $\alpha = 0$ we write M instead of M_0.

In the linear case the maximal operators are classical; the averaging operators were implicit but seem to have first been considered explicitly when $\alpha = 0$ in [24]. In the bilinear case, when $\alpha = 0$, the bilinear maximal operator was introduced in [28], and when $0 < \alpha < 2n$ in [30]. The bilinear averaging operators were first considered in [27]. The following result is due to Berezhnoi [2, Lemma 2.1] in the linear case and to Kokilashvili *et al.* [27, Theorem 2.1] in the bilinear case.

PROPOSITION 2.1. *Fix $0 \leqslant \alpha < n$. Given Banach function spaces X, Y, there exists a constant C such that for every cube Q,*

$$\|\chi_Q\|_Y \|\chi_Q\|_{X'} \leqslant C|Q|^{1-\frac{\alpha}{n}}; \tag{3}$$

if and only if

$$\|A_\alpha^Q f\|_Y \leqslant C\|f\|_X. \tag{4}$$

Similarly, in the bilinear case fix $0 \leqslant \alpha < 2n$. Given Banach function spaces X_1, X_2 and Y there exists a constant C such that for every cube Q,

$$\|\chi_Q\|_Y \|\chi_Q\|_{X_1'} \|\chi_Q\|_{X_2'} \leqslant C|Q|^{1-\frac{\alpha}{n}}; \tag{5}$$

if and only if

$$\|A_\alpha^Q(f,g)\|_Y \leqslant C\|f\|_{X_1}\|g\|_{X_2}. \tag{6}$$

By the pointwise estimates above, (4) holds whenever the fractional maximal operator satisfies $M_\alpha : X \to Y$, and the corresponding fact is true in the bilinear case. Moreover, when $\alpha = 0$ and $X = Y$, condition (3) is the same as (1). This yields the following important corollary to Theorem 1.1.

COROLLARY 2.2. *Given a Banach function space X, if $M : X \to X$, then (1) holds. Equivalently, if the maximal operator is bounded and T is an operator with a kernel that is homogenous of degree n, then a necessary condition for $[b,T] : X \to X$ is that $b \in BMO$.*

As we will discuss below, the assumption that the maximal operator is bounded on the Banach function space X is a natural one. Unfortunately, we cannot generalize Corollary 2.2 to the case $\alpha > 0$ for linear operators or to any bilinear operators acting on general Banach function spaces. However, we can prove that the conditions in Proposition 2.1 and the hypotheses in Theorems 1.1 and 1.2 are related in two important examples of Banach function spaces—the weighted and variable Lebesgue spaces—and for singular integrals of convolution type and fractional integral operators.

Before considering these spaces, we want to specify the operators we are interested in. In the linear setting, we will consider singular integrals of the form

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)\,dy,$$

where $x' = x/|x|^n$ and Ω is defined on S^{n-1}, has mean 0, and is sufficiently smooth. Examples include the Riesz transforms R_j, which have kernels $K_j(x) = \frac{x_j}{|x|^{n+1}}$. For $0 < \alpha < n$, we will consider the fractional integral operator: i.e., the convolution operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

For more information on both kinds of operators, see [17, 18].

In the bilinear setting, we consider singular integral operators of the form

$$T(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\Omega((y_1,y_2)')}{|(y_1,y_2)|^n} f(x-y_1)g(x-y_2)\,dy_1dy_2,$$

where Ω is defined on S^{2n-1}, has mean 0 and is sufficiently smooth. Examples include the multilinear Riesz transforms. For more on these operators, see [22]. We also consider the bilinear fractional integral operator, which is defined for $0 < \alpha < 2n$ by

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(y_1)g(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}\,dy_1dy_2.$$

For more on these operators see [16, 30].

For brevity, in the following sections we will refer to linear and bilinear singular integral operators whose kernels satisfy these hypotheses as regular operators.

Weighted Lebesgue spaces

In this section we apply Theorems 1.1 and 1.2 to the weighted Lebesgue spaces. Given a weight w (i.e., a non-negative, locally integrable function) and $1 < p < \infty$, we define the space $L^p(w)$ to be the set of all measurable functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \right)^{\frac{1}{p}} < \infty.$$

We say that a weight w is in the Muckenhoupt class A_p if for every cube Q,

$$⨍_Q w(x)\,dx \left(⨍_Q w(x)^{1-p'}\,dx \right)^{p-1} \leqslant C.$$

Then $L^p(w)$ is a Banach function space, and it is well known that the associate space is $L^{p'}(w^{1-p'})$. Further, the Hardy-Littlewood maximal operator is bounded on $L^p(w)$ if and only if $w \in A_p$.

For commutators, if $w \in A_p$ and if T is any Calderón-Zygmund singular integral operator (and not just the class of singular integrals described above), and if $b \in BMO$, then the commutator $[b,T] : L^p(w) \to L^p(w)$ [33]. Moreover, it is easy to see that the A_p condition is equivalent to

$$|Q|^{-1} \|\chi_Q\|_{L^p(w)} \|\chi_Q\|_{L^{p'}(w^{1-p'})} \leqslant C,$$

which is condition (1). Therefore, we have proved the following.

COROLLARY 2.3. *For $1 < p < \infty$ and $w \in A_p$, given a regular singular integral operator T and a function b, if $[b,T]$ is bounded on $L^p(w)$, then $b \in BMO$.*

For $0 < \alpha < n$ the corresponding weight condition is the $A_{p,q}$ condition. Given $1 < p < \frac{n}{\alpha}$, define q by $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. We say $w \in A_{p,q}$ if for every cube Q,

$$\left(⨍_Q w(x)^q\,dx \right)^{\frac{1}{q}} \left(⨍_Q w(x)^{-p'}\,dx \right)^{\frac{1}{p'}} \leqslant C.$$

We have that the fractional maximal operator satisfies $M_\alpha : L^p(w^p) \to L^q(w^q)$ if and only if $w \in A_{p,q}$ [32].

For commutators of the fractional integral operator I_α, if $w \in A_{p,q}$ and $b \in BMO$, $[b,I_\alpha] : L^p(w^p) \to L^q(w^q)$ [6]. The $A_{p,q}$ condition also implies (1), though unlike the case of A_p weights, this is less obvious. In this case we have $X = L^p(w^p)$ and $Y = L^q(w^q)$, so $Y' = L^{q'}(w^{-q'})$, and we can rewrite (1) as

$$\left(⨍_Q w^{-q'}\,dx \right)^{\frac{1}{q'}} \left(⨍_Q w^p\,dx \right)^{\frac{1}{p}} \leqslant C.$$

Here we use the fact that since $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$, $\frac{1}{p}+\frac{1}{q'}=1+\frac{\alpha}{n}$. Since $p<q$, $q'<p'$, so if we apply Hölder's inequality twice we get that

$$\left(\fint_Q w^{-q'}\,dx\right)^{\frac{1}{q'}}\left(\fint_Q w^{p}\,dx\right)^{\frac{1}{p}} \leqslant \left(\fint_Q w^{-p'}\,dx\right)^{\frac{1}{p'}}\left(\fint_Q w^{q}\,dx\right)^{\frac{1}{q}} \leqslant C.$$

COROLLARY 2.4. *Given* $0<\alpha<n$ *and* $1<p<\frac{n}{\alpha}$, *define* q *by* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$. *Given* $w\in A_{p,q}$ *and a function* b, *if the commutator* $[b,I_\alpha]:L^p(w^p)\to L^q(w^q)$, *then* $b\in BMO$.

We have similar results for bilinear operators, but they are much less complete. Given $1<p_1,p_2<\infty$, define the vector "exponent" $\vec{p}=(p_1,p_2,p)$, where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Given $\vec{p}$ and weights w_1, w_2, define the triple $\vec{w}=(w_1,w_2,w)$, where $w=w_1^{\frac{p}{p_1}}w_2^{\frac{p}{p_2}}$. Define the multilinear analog of the Muckenhoupt A_p weights as follows: given $\vec{p}$, we say that $\vec{w}\in A_{\vec{p}}$ if for every cube Q,

$$\left(\fint_Q w\,dx\right)^{\frac{1}{p}}\left(\fint_Q w_1^{1-p_1'}\,dx\right)^{\frac{1}{p_1'}}\left(\fint_Q w_2^{1-p_2'}\,dx\right)^{\frac{1}{p_2'}} \leqslant C.$$

These weights were introduced in [28], where they showed that the bilinear maximal operator satisfies $M:L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(w)$ if and only if $\vec{w}\in A_{\vec{p}}$. It is an immediate consequence of Hölder's inequality that if $w_1\in A_{p_1}$ and $w_2\in A_{p_2}$, then $\vec{w}\in A_{\vec{p}}$; however, this condition is not necessary.

Given a bilinear Calderón-Zygmund singular integral operator T and $b\in BMO$, we have that if $\vec{w}\in A_{\vec{p}}$, then for $i=1,2$, $[b,T]_i:L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(w)$ [28]. In light of the results in the linear case, it seems reasonable to conjecture that when $p>1$, the $A_{\vec{p}}$ condition implies (2), which in this case can be written as

$$\left(\fint_Q w^{1-p'}\,dx\right)^{\frac{1}{p'}}\left(\fint_Q w_1\,dx\right)^{\frac{1}{p_1}}\left(\fint_Q w_2\,dx\right)^{\frac{1}{p_2}} \leqslant C. \tag{7}$$

(Here we use the fact that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p'}=1$.) Written in this form, this condition can be viewed formally as the bilinear analog of the fact that in the linear case, if $w\in A_p$, then $w^{1-p'}\in A_{p'}$.

However, we cannot prove this in general, or even in the special case when $w_i\in A_{p_i}$, $i=1,2$. We can prove that (2) holds if we make the additional, stronger assumption that $w\in A_p$. This holds, for instance, if we assume that $w_1,w_2\in A_p\subset A_{p_i}$, $i=1,2$. (This inclusion holds since $A_q\subset A_r$ whenever $q<r$, and here $p<p_i$.) For then, since $w_i\in A_{p_i}$, by a multi-variable reverse Hölder inequality recently proved in [10] (and implicit in [12, Theorem 2.6]), we have that

$$\left(\fint_Q w_1\,dx\right)^{\frac{p}{p_1}}\left(\fint_Q w_2\,dx\right)^{\frac{p}{p_2}} \lesssim \fint_Q w\,dx.$$

But then, since $w \in A_p$,

$$\left(\fint_Q w^{1-p'}\,dx\right)^{\frac{1}{p'}} \left(\fint_Q w_1\,dx\right)^{\frac{1}{p_1}} \left(\fint_Q w_2\,dx\right)^{\frac{1}{p_2}} \approx \left(\fint_Q w\,dx\right)^{-\frac{1}{p}} \left(\fint_Q w_1\,dx\right)^{\frac{1}{p_1}} \left(\fint_Q w_2\,dx\right)^{\frac{1}{p_2}} \leqslant C.$$

COROLLARY 2.5. *Given $\vec{p}$ with $p>1$ and $\vec{w} \in A_{\vec{p}}$, suppose $w_i \in A_{p_i}$, $i=1,2$, and suppose $w \in A_p$. If T is a regular bilinear singular integral, and b is function such that for $i=1,2$, $[b,T]_i : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(w)$, then $b \in BMO$.*

For bilinear fractional integrals, the corresponding weight class was introduced in [30]. With the notation as before, given $0<\alpha<2n$ and $\vec{p}$, suppose that $\frac{1}{2}<p<\frac{n}{\alpha}$. Define q by $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$. If we define the vector weight $\vec{w}=(w_1,w_2,w)$, where now $w=w_1w_2$, then $\vec{w} \in A_{\vec{p},q}$ if

$$\left(\fint_Q w^q\,dx\right)^{\frac{1}{q}} \left(\fint_Q w_1^{-p_1'}\,dx\right)^{\frac{1}{p_1'}} \left(\fint_Q w_2^{-p_2'}\right)^{\frac{1}{p_2'}} \leqslant C.$$

The bilinear fractional maximal operator satisfies $M_\alpha : L^{p_1}(w^{p_1}) \times L^{p_2}(w_2) \to L^q(w^q)$ if and only if $\vec{w} \in A_{\vec{p},q}$. Similar to the $A_{\vec{p}}$ weights, if $w_i \in A_{p_i,q_i}$, where $q_i > p_i$, $i=1,2$, and $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$, then $\vec{w} \in A_{\vec{p},q}$.

For the commutators of the bilinear fractional integral operator, if $w \in A_{\vec{p},q}$, then for $i=1,2$, $[b,I_\alpha]_i : L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^q(w^q)$ [4, 5]. As we did for singular integrals, we conjecture that the $A_{\vec{p},q}$ condition implies (2), which in this case can be written as

$$\left(\fint_Q w^{-q'}\,dx\right)^{\frac{1}{q'}} \left(\fint_Q w_1^{p_1}\,dx\right)^{\frac{1}{p_1}} \left(\fint_Q w_2^{p_2}\,dx\right)^{\frac{1}{p_2}} \leqslant C. \tag{8}$$

(Here we use the fact that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{q'}=1+\frac{\alpha}{n}$.)

Arguing as in the case of bilinear fractional singular integrals, we can prove this if we assume that $w_i \in A_{p_i,q_i}$ and $w^q \in A_q$: i.e., that

$$\left(\fint_Q w^q\,dx\right)^{\frac{1}{q}} \left(\fint_Q w^{-q'}\,dx\right)^{\frac{1}{q'}} \leqslant C.$$

For then, again by the bilinear reverse Hölder inequality (since $w \in A_{p_i,q_i}$, $w \in A_\infty$, and this is sufficient for this inequality to hold) and Hölder's inequality (since $q_i > p_i$),

$$\left(\fint_Q w^q\,dx\right)^{\frac{1}{q}} \geqslant \left(\fint_Q w_1^{q_1}\,dx\right)^{\frac{1}{q_1}} \left(\fint_Q w_2^{q_2}\,dx\right)^{\frac{1}{q_2}} \geqslant \left(\fint_Q w_1^{p_1}\,dx\right)^{\frac{1}{p_1}} \left(\fint_Q w_2^{p_2}\,dx\right)^{\frac{1}{p_2}};$$

Inequality (8) follows at once.

We can eliminate the assumption that $w^q \in A_q$ if we restrict the range of α to $n \leqslant \alpha < 2n$. Suppose $w_i \in A_{p_i,q_i}$, $i=1,2$; then we have that $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n} \leqslant \frac{1-p}{p} < 1$

since $p > \frac{1}{2}$. Moreover, we have that $\frac{1}{q'} = \frac{1}{p_1'} + \frac{1}{p_2'} + \left(\frac{\alpha}{n} - 1\right) \geqslant \frac{1}{p_1'} + \frac{1}{p_2'}$. Therefore, we can apply Hölder's inequality three times to get that the left-hand side of (8) is bounded by

$$\left(\fint_Q w_1^{-p_1'}\,dx\right)^{\frac{1}{p_1'}} \left(\fint_Q w_2^{-p_2'}\,dx\right)^{\frac{1}{p_2'}} \left(\fint_Q w_1^{q_1}\,dx\right)^{\frac{1}{q_1}} \left(\fint_Q w_2^{q_2}\,dx\right)^{\frac{1}{q_2}} \leqslant C.$$

COROLLARY 2.6. *Given $0 < \alpha < 2n$, $\vec{p}$ such that $p < \frac{n}{\alpha}$, and $\vec{w}$ such that $w_i \in A_{p_i,q_i}$ with $q_i > p_i$, suppose either $w^q \in A_q$ or $n \leqslant \alpha < 2n$. If b is a function such that for $i = 1,2$, $[b, I_\alpha]_i : L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^q(w^q)$, then $b \in BMO$.*

REMARK 4. In Corollaries 2.5 and 2.6, we can interpret the hypotheses $w \in L^p(w)$ and $w^q \in L^q(w^q)$ as assuming the maximal operator is bounded on the target space $L^p(w)$ or $L^q(w^q)$. This should be compared to the assumptions in Corollaries 2.9 and 2.10 below.

Variable Lebesgue spaces

The variable Lebesgue spaces are a generalization of the classical L^p spaces. Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, we define $L^{p(\cdot)}$ to be the collection of all measurable functions such that

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

With this norm $L^{p(\cdot)}$ is a Banach function space; the associate space is (isomorphic to) $L^{p'(\cdot)}$, where we define $p'(\cdot)$ pointwise by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For brevity, we define

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

For complete information on these spaces, see [7].

The boundedness of the maximal operator depends (in a very subtle way) on the regularity of the exponent function $p(\cdot)$. A sufficient condition for $M : L^{p(\cdot)} \to L^{p(\cdot)}$ is that $p_- > 1$ and $p(\cdot)$ satisfies the log-Hölder continuity conditions locally and at infinity:

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leqslant \frac{C_0}{-\log(|x-y|)}, \quad |x-y| \leqslant \frac{1}{2},$$

and there exists a constant $1 \leqslant p_\infty \leqslant \infty$ such that

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leqslant \frac{C_\infty}{\log(e+|x|)}.$$

By Proposition 2.1, if $M : L^{p(\cdot)} \to L^{p(\cdot)}$, then for every cube Q,

$$\|\chi_Q\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{p'(\cdot)}} \leqslant C|Q|, \tag{9}$$

which is (1). However, we have a stronger result. Suppose $p(\cdot)$ is such that the maximal operator is bounded on $L^{p(\cdot)}$. Given a cube Q, if we define the exponents p_Q and p'_Q by

$$\frac{1}{p_Q} = \fint_Q \frac{1}{p(x)}\,dx, \qquad \frac{1}{p'_Q} = \fint_Q \frac{1}{p'(x)}\,dx,$$

then

$$\|\chi_Q\|_{p(\cdot)} \approx |Q|^{\frac{1}{p_Q}} \quad \text{and} \quad \|\chi_Q\|_{p'(\cdot)} \approx |Q|^{\frac{1}{p'_Q}}, \tag{10}$$

and the implicit constants are independent of Q [7, Proposition 4.66].

Let T be a Calderón-Zygmund singular integral operator. If $p(\cdot)$ is such that $1 < p_- \leqslant p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}$, then for all $b \in BMO$, $[b,T] : L^{p(\cdot)} \to L^{p(\cdot)}$ [8, Corollary 2.10].

COROLLARY 2.7. *Let $p(\cdot)$ be an exponent function such that $1 < p_- \leqslant p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}$. If T is a regular singular integral and b is a function such that $[b,T]$ is bounded on $L^{p(\cdot)}$, then $b \in BMO$.*

Given $0 < \alpha < n$ and $p(\cdot)$ such that $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$, define $q(\cdot)$ pointwise by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If there exists $q_0 > \frac{n}{n-\alpha}$ such that the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}$, then for all $b \in BMO$, $[b,I_\alpha] : L^{p(\cdot)} \to L^{q(\cdot)}$. (This result does not appear explicitly in the literature, but it is a straightforward application of known results. For instance, it follows by extrapolation, arguing as in [7, Theorem 5.46] but using the weighted norm inequalities for commutators from [6, Theorem 1.6].)

If the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}$, then by [7, Corollary 4.64], it is also bounded on $L^{q(\cdot)/q_0}$ and so on $L^{q(\cdot)}$ [7, Theorem 4.37]. If we let $\theta = \frac{1}{q_0}$, then we can write

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{\alpha}{n} = \frac{\theta}{q(x)/q_0} + \frac{1-\theta}{(1-\theta)\frac{n}{\alpha}}.$$

By our assumption on q_0, $r = (1-\theta)\frac{n}{\alpha} > 1$, and so the maximal operator is bounded on L^r. Hence, by interpolation (see [7, Theorem 3.38]) the maximal operator is bounded on $L^{p(\cdot)}$. Therefore, by (10),

$$|Q|^{-\frac{\alpha}{n}} \|\chi_Q\|_{L^{q'(\cdot)}} \|\chi_Q\|_{L^{p(\cdot)}} \lesssim |Q|^{-\frac{\alpha}{n}} |Q|^{\frac{1}{q'_Q}} |Q|^{\frac{1}{p_Q}} \lesssim |Q|.$$

So again, (1) holds.

COROLLARY 2.8. *Given $0 < \alpha < n$ and $p(\cdot)$ such that $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$, define $q(\cdot)$ pointwise by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. Suppose there exists $q_0 > \frac{n}{n-\alpha}$ such that the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}$. If b is such that $[b,I_\alpha] : L^{p(\cdot)} \to L^{q(\cdot)}$, then $b \in BMO$.*

We have similar results for bilinear operators. Suppose $p_1(\cdot)$, $p_2(\cdot)$ are such that $1 < (p_i)_- \leqslant (p_i)_+ < \infty$, $i = 1,2$, and define $p(\cdot)$ pointwise by $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$. Note that $p_- > \frac{1}{2}$. If we further assume that the (linear) maximal operator is bounded on $L^{p_i(\cdot)}$, $i=1,2$, then given any bilinear Calderón-Zygmund singular integral operator T, we have that $T : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)}$ [11, Corollary 4.1]. Moreover, given any $b \in BMO$, $[b,T]_i : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)}$. This is not proved explicitly in the literature, but the proof is the same as for bilinear singular integrals, using bilinear extrapolation and using the weighted inequalities for bilinear commutators in [28].

With the same assumptions we also have that $M : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)}$: by the generalized Hölder's inequality [7, Corollary 2.28],

$$\|M(f,g)\|_{p(\cdot)} \leqslant \|Mf \cdot Mg\|_{p(\cdot)} \lesssim \|Mf\|_{p_1(\cdot)} \|Mg\|_{p_2(\cdot)}.$$

Note that we pass from the bilinear to the linear maximal operator. Also, note that the generalized Hölder's inequality is only proved in [7] assuming $p_- \geqslant 1$, but for $p_- > 0$ it follows by a rescaling argument: cf. [13, Lemma 2.7].

If we further assume $p_- > 1$ and the maximal operator is bounded on $L^{p(\cdot)}$, then by (10) we have that

$$\|\chi_Q\|_{p'(\cdot)} \|\chi_Q\|_{p_1(\cdot)} \|\chi_Q\|_{p_2(\cdot)} \lesssim |Q|^{\frac{1}{p'_Q} + \frac{1}{(p_1)_Q} + \frac{1}{(p_2)_Q}} \leqslant C.$$

COROLLARY 2.9. *Suppose $p_1(\cdot)$, $p_2(\cdot)$ are such that $1 < (p_i)_- \leqslant (p_i)_+ < \infty$, $i=1,2$, and we define $p(\cdot)$ pointwise by $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$. Suppose further that $p_- > 1$ and the maximal operator is bounded on $L^{p(\cdot)}$ and $L^{p_i(\cdot)}$, $i=1,2$. If T is a regular bilinear singular integral and b is such that $[b,T]_i : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{p(\cdot)}$, then $b \in BMO$.*

To prove the analogous result for the bilinear fractional integral operator, fix $0 < \alpha < 2n$. Suppose $p_1(\cdot)$, $p_2(\cdot)$ are such that $1 < (p_i)_- \leqslant (p_i)_+ < \infty$, $i=1,2$, and again define $p(\cdot)$ pointwise by $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$. Define $q(\cdot)$ by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. Fix $0 < \alpha_1, \alpha_2 < n$ such that $\alpha_1 + \alpha_2 = \alpha$, and define $q_i(\cdot)$, $i=1,2$, by $\frac{1}{p_i(x)} - \frac{1}{q_i(x)} = \frac{\alpha_i}{n}$. If we assume that the maximal operator is bounded on $L^{q(\cdot)}$, and that there exist $q_i > \frac{n}{n-\alpha_i}$. $i=1,2$, such that the maximal operator is bounded on $L^{(q_i(\cdot)/q_i)'}$, then given any $b \in BMO$, $[b,I_\alpha]_i : L^{p_1(\cdot)} \times L^{p_2(\cdot)} \to L^{q(\cdot)}$. As in the linear case, this result has not been explicitly proved in the literature, but follows from known results. For all $w \in A_\infty$ and any $0 < p < \infty$, $\|I_\alpha(f,g)\|_{L^p(w)} \lesssim \|M_\alpha(f,g)\|_{L^p(w)}$ [30, Theorem 3.1]. Since the maximal operator is bounded on $L^{q(\cdot)}$, by extrapolation, $\|I_\alpha(f,g)\|_{q(\cdot)} \lesssim \|M_\alpha(f,g)\|_{q(\cdot)}$ [7, Theorem 5.24]. By the generalized Hölder's inequality,

$$\|M_\alpha(f,g)\|_{q(\cdot)} \leqslant \|M_{\alpha_1} f \cdot M_{\alpha_2} g\|_{q(\cdot)} \lesssim \|M_{\alpha_1} f\|_{q_1(\cdot)} \|M_{\alpha_2} g\|_{q_2(\cdot)} \lesssim \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)};$$

the last inequality follows from our assumptions on $q_i(\cdot)$ and [7, Remark 5.51].

In this case (2) becomes

$$|Q|^{-\frac{\alpha}{n}} \|\chi_Q\|_{q'(\cdot)} \|\chi_Q\|_{p_1(\cdot)} \|\chi_Q\|_{p_2(\cdot)} \leqslant C|Q|.$$

If we make the same assumptions on the exponents as used to prove the inequalities for the commutators, then arguing as we did above using (10), we get this inequality.

COROLLARY 2.10. *Fix $0<\alpha<2n$. Suppose $p_1(\cdot)$, $p_2(\cdot)$ are such that $1<(p_i)_-\leqslant(p_i)_+<\infty$, $i=1,2$, and again define $p(\cdot)$ pointwise by $\frac{1}{p(x)}=\frac{1}{p_1(x)}+\frac{1}{p_2(x)}$. Define $q(\cdot)$ by $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}$. Fix $0<\alpha_1,\alpha_2<n$ such that $\alpha_1+\alpha_2=\alpha$, and define $q_i(\cdot)$, $i=1,2$, by $\frac{1}{p_i(x)}-\frac{1}{q_i(x)}=\frac{\alpha_i}{n}$. Suppose that the maximal operator is bounded on $L^{q(\cdot)}$, and that there exist $q_i>\frac{n}{n-\alpha_i}$. $i=1,2$, such that the maximal operator is bounded on $L^{(q_i(\cdot)/q_i)'}$. Given any b, if $[b,I_\alpha]_i:L^{p_1(\cdot)}\times L^{p_2(\cdot)}\to L^{q(\cdot)}$, then $b\in BMO$.*

REMARK 5. Sufficient conditions for the boundedness of commutators of singular and fractional integrals on Orlicz spaces are known or can be readily proved using extrapolation: see [15, 26]. Similarly, such results can be proved in the weighted variable Lebesgue spaces and generalized Orlicz spaces (also known as Nakano spaces or Musielak-Orlicz spaces) using extrapolation: see [9, 14] for definitions and the corresponding extrapolation results. Results similar to those for the variable Lebesgue spaces above can be deduced in these settings – we leave the precise statements and proofs to the interested reader.

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. As we noted in the Introduction, the proof of Theorem 1.1 is gotten by a simple adaptation of this proof. Our proof actually closely follows the argument due to the first author in [3], which was in turn following the techniques of Janson in [23]. Here we will concentrate on the parts of the proof which change because we are working in the setting of Banach function spaces, and we refer the reader to [3] for further details.

Proof. We will assume without loss of generality that $[b,T]_1$ is bounded; the proof for the other commutator is identical.

Let $B=B((y_0,z_0),\delta\sqrt{2n})\subset\mathbb{R}^{2n}$ be a ball upon which $\frac{1}{K}$ has an absolutely convergent Fourier series. By the homogeneity of K, we may assume without loss of generality that $2\sqrt{n}<|(y_0,z_0)|<4\sqrt{n}$ and $\delta<1$. These conditions guarantee that $\overline{B}\cap\{0\}=\emptyset$, avoiding any potential singularity of K; this will be important below as it will let us use the integral representation of $[b,T]_1$.

Write the Fourier series of $\frac{1}{K}$ as

$$\frac{1}{K(y,z)}=\sum_j a_j e^{iv_j\cdot(y,z)}=\sum_j a_j e^{i(v_j^1,v_j^2)\cdot(y,z)};$$

note that the individual vectors $v_j=(v_j^1,v_j^2)\in\mathbb{R}^n\times\mathbb{R}^n$ do not play any significant role in the proof, and we introduce them simply to be precise.

Let $y_1=\delta^{-1}y_0$, $z_1=\delta^{-1}z_0$; then by homogeneity we have that for all $(y,z)\in B((y_1,z_1),\sqrt{2n})$,

$$\frac{1}{K(y,z)}=\frac{\delta^{-2n+\alpha}}{K(\delta y,\delta z)}=\delta^{-2n+\alpha}\sum_j a_j e^{i\delta v_j\cdot(y,z)}.$$

Let $Q = Q(x_0, r)$ be an arbitrary cube in $\mathbb{R}^n$, and set $\tilde{y} = x_0 + ry_1$, $\tilde{z} = x_0 + rz_1$. Define $Q' = Q(\tilde{y}, r)$ and $Q'' = Q(\tilde{z}, r)$.

It follows from the size conditions on y_0 and z_0 that Q and either Q' or Q'' are disjoint. To see that this is the case, note that the minimum size condition on (y_0, z_0) implies that $\max\{|y_0|, |z_0|\} \geqslant \sqrt{2n}$; without loss of generality, suppose that it is $|y_0|$. This in turn implies that the distance between x_0 and $\tilde{y}$ is greater than $r\sqrt{2n}$. Since Q and Q' each have side-length r, the distance of their centers from one another guarantees that they must be disjoint. If $|z_0|$ is larger, then we get the same conclusion for Q''.

As a consequence, $Q \cap Q' \cap Q'' = \emptyset$, which allows us to use the kernel representation of $[b, T]_1$ for $(x, y, z) \in Q \times Q' \times Q''$. Additionally, for $(x, y, z) \in Q \times Q' \times Q''$, $\left(\frac{x-y}{r}, \frac{x-z}{r}\right) \in B((y_1, z_1), \sqrt{2n})$, and so by our above calculations we may use the Fourier series representation of $1/K((x-y)/r, (x-z)/r)$ there as well. Further details of these calculations can be found in [3].

It also follows from the maximum size condition on (y_0, z_0) that

$$Q',\, Q'' \subset B\left(x_0, \left(\frac{r\sqrt{n}}{2}\left(1+\frac{8}{\delta}\right)\right)\right) \subset \sqrt{n}\left(1+\frac{8}{\delta}\right) Q. \tag{11}$$

To see this, note that the maximum size of y_0 or z_0 is $4\sqrt{n}$ which implies that the maximum distance from x_0 to $\tilde{y}$ or $\tilde{z}$ is $\frac{4r\sqrt{n}}{\delta}$. The containment in $B\left(x_0, \left(\frac{r\sqrt{n}}{2}\left(1+\frac{8}{\delta}\right)\right)\right)$ follows from this.

We can now estimate as follows. Fix Q and let $\sigma(x) = \operatorname{sgn}(b(x) - b_Q)$; then

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}|\, dx \\
&\quad = \int_Q (b(x) - b_{Q'})\sigma(x)\, dx \\
&\quad = \frac{1}{|Q''|}\frac{1}{|Q'|}\int_Q \int_{Q'} \int_{Q''} (b(x) - b(y))\sigma(x)\, dz dy dx \\
&\quad = r^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{r^{2n-\alpha} K(x-y, x-z)}{K\left(\frac{x-y}{r}, \frac{x-z}{r}\right)} \\
&\qquad \times \sigma(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z)\, dz dy dx \\
&\quad = \delta^{-2n+\alpha} r^{-\alpha} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y)) K(x-y, x-z) \sum_j a_j e^{i\frac{\delta}{r}\nu_j \cdot (x-y, x-z)} \\
&\qquad \times \sigma(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z)\, dz dy dx
\end{aligned}$$

Define the functions

$$\begin{aligned}
f_j(y) &= e^{-i\frac{\delta}{r}\nu_j^1 \cdot y} \chi_{Q'}(y) \\
g_j(z) &= e^{-i\frac{\delta}{r}\nu_j^2 \cdot z} \chi_{Q''}(z) \\
h_j(x) &= e^{i\frac{\delta}{r}\nu_j \cdot (x,x)} \sigma(x)\chi_Q(x).
\end{aligned}$$

Note that the norm of each of these functions in any Banach function space will be the same as the norm of its support. We can now continue the above estimate:

$$\begin{aligned}
&\int_Q |b(x)-b_{Q'}|\,dx \\
&\qquad = \delta^{-2n+\alpha} r^{-\alpha} \sum_j a_j \int_{\mathbb{R}^n} h_j(x) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (b(x)-b(y)) \\
&\qquad\qquad \times K(x-y,x-z) f_j(y) g_j(z)\,dzdydx \\
&\qquad = \delta^{-2n+\alpha} |Q|^{-\frac{\alpha}{n}} \sum_j a_j \int_{\mathbb{R}^n} h_j(x)[b,T]_1(f_j,g_j)(x)\,dx \\
&\qquad \leqslant \delta^{-2n+\alpha} |Q|^{-\frac{\alpha}{n}} \sum_j |a_j| \int_{\mathbb{R}^n} |h_j(x)||[b,T]_1(f_j,g_j)(x)|\,dx \\
&\qquad \leqslant \delta^{-2n+\alpha} |Q|^{-\frac{\alpha}{n}} \sum_j |a_j| \|h_j\|_{Y'} \|[b,T]_1(f_j,g_j)\|_Y \\
&\qquad \leqslant \delta^{-2n+\alpha} |Q|^{-\frac{\alpha}{n}} \sum_j |a_j| \|h_j\|_{Y'} \|[b,T]_1\|_{X_1\times X_2\to Y} \|f_j\|_{X_1} \|g_j\|_{X_2} \\
&\qquad = \delta^{-2n+\alpha} \|[b,T]\|_{X_1\times X_2\to Y} \sum_j |a_j| \|\chi_Q\|_{Y'} \|\chi_{Q'}\|_{X_1} \|\chi_{Q''}\|_{X_2} |Q|^{-\frac{\alpha}{n}}.
\end{aligned}$$

Let $P = 2\sqrt{n}(1+\frac{8}{\delta})Q$. By inequality (11) we have that Q, Q', Q" $\subset P$, and $|P| \approx |Q|$. Therefore, by (2),

$$\begin{aligned}
&\int_Q |b(x)-b_{Q'}|\,dx \\
&\qquad \lesssim \|[b,T]\|_{X_1\times X_2\to Y} \sum_j |a_j| \|\chi_P\|_{Y'} \|\chi_P\|_{X_1} \|\chi_P\|_{X_2} |P|^{-\frac{\alpha}{n}} \\
&\qquad \lesssim |P| \|[b,T]_1\|_{X_1\times X_2\to Y} \sum_j |a_j|, \\
&\qquad \lesssim |Q|.
\end{aligned}$$

Since this is true for every cube Q, $b \in BMO$ and the proof is complete. □

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(Received January 26, 2017)

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