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AN INFINITE SEQUENCE OF INEQUALITIES INVOLVING SPECIAL VALUES OF THE RIEMANN ZETA FUNCTION

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Dedicated to the 70th Anniversary
of Professor Constantin P. Niculescu

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Abstract. In this paper, we give an infinite sequence of inequalities involving the Riemann zeta function with even arguments $\zeta(2n)$ and the Chebyshev-Stirling numbers of the first kind. This result is based on a recent connection between the Riemann zeta function and the complete homogeneous symmetric functions [18]. An interesting asymptotic formula related to the nth complete homogeneous symmetric function is conjectured in this context:

$$h_n\left(1,\left(\frac{k}{k+1}\right)^2,\left(\frac{k}{k+2}\right)^2,\ldots\right)\sim\binom{2k}{k},\qquad n\to\infty.$$

1. Introduction

The main object of our investigation is the Riemann zeta function or Euler-Riemann zeta function

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$$

which is defined over the complex plane when the real part of s is greater than 1. Originally the Riemann zeta function was defined for real arguments by Euler as

$$\zeta(x)=\sum_{k=1}^{\infty}\frac{1}{k^x},\qquad x>1.$$

Moreover, for real $x>1$, we have

$$\zeta(x)>\zeta(x+1)\qquad\text{and}\qquad\lim_{x\to\infty}\zeta(x)=1.$$

In spite of its utter simplicity, this function plays a pivotal role in analytic number theory having applications in physics, probability theory, applied statistics and other fields of

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mathematics. The reader should consult the classical papers by Abramowitz and Stegun [1], Apostol [5], Berndt [6], Everest, Röttger and T. Ward [7], Ireland and Rosen [13], Murty and Reece [23], and Weil [24] for the full background on this function.

Being given an infinite set of variables $\{x_1,x_2,x_3,\ldots\}$, recall [14] that the nth complete homogeneous symmetric function h_n is the sum of all monomials of total degree n in these variables so that $h_0 = 1$ and for $n > 0$

$$h_n = h_n(x_1,x_2,x_3,\ldots) = \sum_{1\leqslant i_1\leqslant i_2\leqslant \ldots\leqslant i_n} x_{i_1}x_{i_2}\ldots x_{i_n}.$$

In a recent paper [18, Eq. (3.1)], the Riemann zeta function with even arguments, $\zeta(2n)$, was expressed in terms of the nth complete homogeneous symmetric function of the numbers $\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots$, as follows

$$h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right) = 2\left(1-\frac{2}{2^{2n}}\right)\zeta(2n), \qquad n \geqslant 0. \tag{1}$$

Considering the recurrence relation

$$h_n(x_1,x_2,x_3,\ldots) = x_1 h_{n-1}(x_1,x_2,x_3,\ldots) + h_n(x_2,x_3,x_4,\ldots), \tag{2}$$

we deduce the inequality

$$\left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) > 0. \tag{3}$$

This result seems more interesting if we consider the trivial inequality

$$\zeta(2n) - \zeta(2n-2) < 0.$$

Upon reflection, one expects that there might be an infinite family of such inequalities where (3) is the second entry, and the trivial inequality

$$\left(1-\frac{2}{2^{2n}}\right)\zeta(2n) > 0$$

is the first.

For all nonnegative integers n and k, we define $S_n(k)$ by

$$S_n(k) = \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \left(1-\frac{2}{2^{2n-2i}}\right)\zeta(2n-2i),$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{1/2}$ are the Chebyshev-Stirling numbers of the first kind. Recall that the Chebyshev-Stirling numbers of the first kind are known in the literature [9, 10, 17] as the case $\gamma = 1/2$ of the Jacobi-Stirling numbers of the first kind that can be given through the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\gamma} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\gamma} + (n-1)(n+2\gamma-2)\begin{bmatrix} n-1 \\ k \end{bmatrix}_{\gamma} \tag{4}$$

with the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\gamma = \delta_{0,n} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ k \end{bmatrix}_\gamma = \delta_{0,k},$$

where $\delta_{i,j}$ is the Kronecker delta. The Jacobi-Stirling numbers were discovered in 2007 as a result of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression. In the last decade, these numbers received considerable attention especially in combinatorics and graph theory, see, e.g., [2, 3, 4, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 22].

In this paper, we shall prove the following inequalities.

THEOREM 1. *For $n,k \geqslant 0$,*

1. $S_n(k) > 0$;
2. $\dfrac{S_n(k)}{k!^2} > \dfrac{S_n(k+1)}{(k+1)!^2}$.

EXAMPLE 1. Having

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}_{1/2}\right]_{n,k=\overline{1,5}} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 4 & 5 & 1 & & \\ 36 & 49 & 14 & 1 & \\ 576 & 820 & 273 & 30 & 1 \end{bmatrix},$$

we can write the following sequence of inequalities:

$$\begin{aligned}
&\left(1-\frac{2}{2^{2n}}\right)\zeta(2n) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \frac{5}{4}\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) + \frac{1}{4}\left(1-\frac{2}{2^{2n-4}}\right)\zeta(2n-4) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \frac{49}{36}\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) \\
&\quad + \frac{7}{18}\left(1-\frac{2}{2^{2n-4}}\right)\zeta(2n-4) - \frac{1}{36}\left(1-\frac{2}{2^{2n-6}}\right)\zeta(2n-6) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \frac{205}{144}\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) + \frac{91}{192}\left(1-\frac{2}{2^{2n-4}}\right)\zeta(2n-4) \\
&\quad - \frac{5}{96}\left(1-\frac{2}{2^{2n-6}}\right)\zeta(2n-6) + \frac{1}{576}\left(1-\frac{2}{2^{2n-8}}\right)\zeta(2n-8) \\
&> \ldots > 0.
\end{aligned}$$

Related to the first inequality of Theorem 1, we remark a well-known property of the Chebyshev-Stirling numbers of the first kind, that is,

$$\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} = 0.$$

2. Proof of Theorem 1

Firstly, we prove the following lemma in two ways.

LEMMA 1. *For $n,k \geqslant 0$,*

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right) = \sum_{i=0}^{k}\frac{(-1)^i}{k!^2}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right).$$

Proof 1. This proof invokes the generating functions for the complete and elementary symmetric functions. Being given a set of variables $\{x_1,x_2,\ldots,x_n\}$, recall [14] that the kth elementary symmetric function $e_k(x_1,x_2,\ldots,x_n)$ is given by

$$e_k(x_1,x_2,\ldots,x_n) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} x_{i_1}x_{i_2}\cdots x_{i_k}$$

for $k=1,2,\ldots,n$. We set $e_0(x_1,x_2,\ldots,x_n)=1$ by convention. For $k<0$ or $k>n$, we set $e_k(x_1,x_2,\ldots,x_n)=0$. In particular, according to Merca [18], we have

$$e_k\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots,\frac{1}{n^2}\right) = \frac{1}{n!^2}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1/2}.$$

The elementary symmetric functions are characterized by the following identity of formal power series in t:

$$\sum_{k=0}^{\infty} e_k(x_1,x_2,\ldots,x_n)t^k = \prod_{k=1}^{n}(1+x_kt).$$

For the complete homogeneous symmetric functions in infinitely many variables $x_1,x_2,\ldots$, we have

$$\sum_{k=0}^{\infty} h_k(x_1,x_2,\ldots)t^k = \prod_{k=1}^{\infty}(1-x_kt)^{-1}.$$

Thus, we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)t^n\\
&\quad=\prod_{i=k+1}^{\infty}\left(1-\frac{t}{i^2}\right)^{-1}=\prod_{i=1}^{k}\left(1-\frac{t}{i^2}\right)\times\prod_{i=1}^{\infty}\left(1-\frac{t}{i^2}\right)^{-1}\\
&\quad=\left(\sum_{i=0}^{\infty}\frac{(-1)^i}{k!^2}\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}t^i\right)\left(\sum_{i=0}^{\infty}h_i\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)t^i\right)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{k}\frac{(-1)^i}{k!^2}\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)\right)t^n,
\end{aligned}$$

where we have invoked the well known Cauchy multiplication of two power series. Equating coefficients of t^n give the result. □

Proof 2. We are going to prove this lemma by induction on k. For $k=0$, we have

$$h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)=\begin{bmatrix}1\\ 1\end{bmatrix}_{1/2}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right).$$

The base case of induction is finished. We suppose that the relation

$$h_n\left(\frac{1}{(k'+1)^2},\frac{1}{(k'+2)^2},\frac{1}{(k'+3)^2},\ldots\right)=\sum_{i=0}^{k'}\frac{(-1)^i}{k'!^2}\begin{bmatrix}k'+1\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right).$$

is true for any integer k', $0\leqslant k'<k$. Taking into account (2), we can write

$$\begin{aligned}
&h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)\\
&\quad=h_n\left(\frac{1}{k^2},\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\ldots\right)-\frac{1}{k^2}h_{n-1}\left(\frac{1}{k^2},\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\ldots\right)\\
&\quad=\sum_{i=0}^{k-1}\frac{(-1)^i}{(k-1)!^2}\begin{bmatrix}k\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)\\
&\qquad-\frac{1}{k^2}\sum_{i=0}^{k-1}\frac{(-1)^i}{(k-1)!^2}\begin{bmatrix}k\\ i+1\end{bmatrix}_{1/2}h_{n-1-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)\\
&\quad=\sum_{i=0}^{k-1}\frac{(-1)^i}{(k-1)!^2}\begin{bmatrix}k\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)\\
&\qquad-\frac{1}{k^2}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{(k-1)!^2}\begin{bmatrix}k\\ i\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)\\
&\quad=\sum_{i=0}^{k}\frac{(-1)^i}{k!^2}\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right),
\end{aligned}$$

where we have invoked the recurrence relation (4), with γ replaced by $1/2$. Thus, the proof of the lemma is finished. □

Theorem 1 follows considering this lemma, the equation (1) and the inequalities

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)>0$$

and

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)>h_n\left(\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\frac{1}{(k+4)^2},\ldots\right).$$

3. Concluding remarks

A formula for the nth complete homogeneous symmetric function

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)$$

in terms of the complete homogeneous symmetric functions

$$h_i\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right),\qquad i=n-k,\ldots,n$$

has been introduced in this paper. Using this result, we derived an infinite sequence of inequalities involving the Riemann zeta function with even arguments $\zeta(2n)$ and the Chebyshev-Stirling numbers of the first kind.

There is a substantial amount of numerical evidence to conjecture that the following inequality is true.

CONJECTURE 1. For $n,k\geqslant 0$,

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)<\frac{1}{(k+1)^{2n}}\binom{2k+2}{k+1}.$$

Moreover, we conjecture that the sequence

$$\left\{(k+1)^{2n}h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)\right\}_{n\geqslant 0}$$

converges to

$$\binom{2k+2}{k+1}.$$

CONJECTURE 2. For $k>0$,

$$\lim_{n\to\infty}h_n\left(1,\left(\frac{k}{k+1}\right)^2,\left(\frac{k}{k+2}\right)^2,\ldots\right)=\binom{2k}{k}.$$

Finally, assuming Conjecture 1, we can write the following inequality.

CONJECTURE 3. For $n,k \geqslant 0$,

$$S_n(k) < \frac{(2k+1)!}{(k+1)^{2n+1}}.$$

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] G. E. ANDREWS AND L. L. LITTLEJOHN, *A combinatorial interpretation of the Legendre-Stirling numbers*, Proceedings AMS, **137**, 8 (2009), 2581–2590.

[3] G. E. ANDREWS, W. GAWRONSKI AND L. L. LITTLEJOHN, *The Legendre-Stirling numbers*, Discrete Math., **311**, 14 (2011), 1255–1272.

[4] G. E. ANDREWS, E. S. EGGE, W. GAWRONSKI AND L. L. LITTLEJOHN, *The Jacobi-Stirling numbers*, J. Combin. Theory, Ser A., **120**, 1 (2013), 288–303.

[5] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer-Verlag, New York-Heidelberg-Berlin, 1976

[6] B. C. BERNDT, *Elementary evaluation of* $\zeta(2n)$, Math. Magazine, **48**, 3 (1975), 148–154.

[7] G. EVEREST, C. RÖTTGER AND T. WARD, *The continuing story of zeta*, The Math. Intelligencer, **31**, 3 (2009), 13–17.

[8] W. N. EVERITT, K. H. KWON, L. L. LITTLEJOHN, R. WELLMAN AND G. J. YOON, *Jacobi-Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression* J. Comput. Appl. Math., **208**, 1 (2007), 29–56.

[9] W. GAWRONSKI, L. L. LITTLEJOHN AND T. NEUSCHEL, *Asymptotics of Stirling and Chebyshev-Stirling numbers of the second kind*, Stud. Appl. Math., **133**, 1 (2014), 1–17.

[10] W. GAWRONSKI, L. L. LITTLEJOHN AND T. NEUSCHEL, *On the asymptotic normality of the Legendre-Stirling numbers of the second kind*, European J. Combin., **49** (2015), 218–231.

[11] Y. GELINEAU AND J. ZENG, *Combinatorial interpretations of the Jacobi-Stirling numbers* Electron. J. Combin. **17** (2010), R70.

[12] I. M. GESSEL, Z. LIN AND J. ZENG, *Jacobi-Stirling polynomials and P-partitions*, European J. Combin. **33**, 8 (2012), 1987–2000.

[13] K. IRELAND AND M. ROSEN, *A Classical Introduction to Modern Number Theory, 2nd ed.*, Springer, Berlin, 1990

[14] I. G. MACDONALD, *Symmetric Functions and Hall Polynomials, 2nd ed.*, Clarendon Press, Oxford, 1995

[15] M. MERCA, *A convolution for complete and elementary symmetric functions*, Aequat. Math., **86**, 3 (2013), 217–229.

[16] M. MERCA, *A note on the Jacobi-Stirling numbers*, Integral Transforms Spec. Funct., **25**, 3 (2014), 196–202.

[17] M. MERCA, *A connection between Jacobi-Stirling numbers and Bernoulli polynomials*, J. Number Theory., **151** (2015), 223–229.

[18] M. MERCA, *Asymptotics of the Chebyshev-Stirling numbers of the first kind*, Integral Transforms Spec. Funct., **27**, 4 (2016), 259–267.

[19] M. MERCA, *The cardinal sine function and the Chebyshev-Stirling numbers*, J. Number Theory. **160** (2016), 19–31.

[20] M. MERCA, *New convolution for complete and elementary symmetric functions*, Integral Transforms Spec. Funct., **27**, 12 (2016), 965–973.
[21] P. MONGELLI, *Total positivity properties of Jacobi-Stirling numbers*, Adv. Appl. Math., **48**, 2 (2012), 354–364.
[22] P. MONGELLI, *Combinatorial interpretations of particular evaluations of complete and elementary symmetric functions*, Electron. J. Combin., **19**, 1 (2012), P60.
[23] M. R. MURTY AND M. REECE, *A simple derivation of* $\zeta(1-K) = -B_K/K$, Funct. Approx. Comment. Math., **28** (2000), 141–154.
[24] A. WEIL, *Number Theory. An Approach Through History From Hammurapi to Legendre*, Birkhäuser, Boston, 1984.

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