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PROOF OF A MONOTONICITY CONJECTURE

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Abstract. Bennett gave a generalization of Schur's theorem to study various moment-preserving transformations. In this paper, we confirm a monotonicity conjecture of Bennett which is related to the generalized Schur's theorem and Haber's inequality.

1. Introduction

Bennett [1] gave a generalization of Schur's theorem and utilized its special cases to study various moment-preserving transformations. See [6, p. 164] for the original form of Schur's theorem and [1] for the application of the generalized Schur's theorem to the study of moment sequences. Note that various moment sequences arise naturally in many branches of mathematics and have been extensively studied. The reader is referred to [10, 16] for the broad background of moment sequences and [7, 15] for the latest work on some moment sequences.

While considering one special case of the generalized Schur's theorem, Bennett proposed the following conjecture [1, p. 31]. To formulate Bennett's conjecture, define the generalized binomial coefficients by $\binom{\lambda}{k} := \lambda(\lambda-1)\cdots(\lambda-k+1)/k!$, where λ is a complex number and k is a nonnegative integer (see [4, p. 8]). Let n be a fixed nonnegative integer and x, y be fixed nonnegative real numbers. Define a univariate function by

$$F(a) = \frac{1}{\binom{n+2a-1}{n}} \sum_{k=0}^{n} \binom{k+a-1}{k} x^k \binom{n-k+a-1}{n-k} y^{n-k},$$

where $a \in (0, +\infty)$.

CONJECTURE 1. (proposed by Bennett [1, p. 31]) *The function $F(a)$ decreases.*

The object of this paper is to confirm Conjecture 1 by showing the following result.

THEOREM 1. *The function $F(a)$ strictly decreases for $n \geqslant 2$, $x \neq y$ and $x, y > 0$. Otherwise, $F(a)$ is a constant function.*

In the next section, we give the proof of Theorem 1 by means of the symmetry and the unimodality of some binomial sequences. We end this paper with an interesting open problem and the connection between Theorem 1 and Haber's inequality.

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2. The proof of Theorem 1

We begin with preliminary terminologies. Let $a_0, a_1, \dots, a_n$ be a sequence of real numbers. It is *unimodal* if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{m-1} \leqslant a_m \geqslant a_{m+1} \geqslant \cdots \geqslant a_n$ for some m. In other words, the sequence $\{a_k\}_{0\leqslant k\leqslant n}$ is unimodal if it does not increase strictly after a strict decrease (see [17]). The sequence $\{a_k\}_{0\leqslant k\leqslant n}$ is *symmetric* (or *palindromic*) with center of symmetry at $n/2$ if $a_k = a_{n-k}$ for all k.

A real polynomial $f(t)$ is *unimodal* (resp. *symmetric*) if the sequence of its coefficients has the corresponding property. Symmetric and unimodal sequences or polynomials arise often in combinatorics, algebra, geometry, analysis and have a number of applications in computer science, probability and statistics. See [3, 11] for a broad overview of the subject.

The sequence of binomial coefficients $\{\binom{n}{k}\}_{k=0}^{n}$ are probably the best-known example of symmetric and unimodal sequences. More precisely, the binomial sequence $\{\binom{n}{k}\}_{k=0}^{n}$ is symmetric since $\binom{n}{k} = \binom{n}{n-k}$, and is (strictly) unimodal since $\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2\rfloor}$, where $\lfloor n/2\rfloor$ stands for the largest integer not exceeding the center $n/2$. See [9, 12, 13, 14] for the results on the unimodality of various binomial sequences.

LEMMA 1. *Suppose that $f(t) = \sum_{i=0}^{n} a_i t^i$ is a symmetric polynomial of degree n and that $f(t) = (t-1)^2 h(t)$. Then $h(t)$ is also symmetric. In addition, if $\sum_{i=0}^{k} a_i > 0$ for $0 \leqslant k \leqslant \lfloor (n-2)/2\rfloor$, then $h(t)$ is strictly unimodal and the coefficients of $h(t)$ are all positive.*

Proof. It is known that if $f(t)$ and $g(t)$ are two symmetric polynomials and $f(t) = g(t)h(t)$, then $h(t)$ is also symmetric ([14, Corollary 2.3(ii)]). Thus, the symmetry of $h(t)$ in the lemma follows from that of $(t-1)^2$.

Next we will prove the second part of the lemma. It is trivially true when $n = 2$. For $n = 3$, suppose that $h(t) = b_0 + b_1 t$. Then $b_0 = b_1 = a_0 = a_1 > 0$, which yields the lemma.

For $n \geqslant 4$, assume that $h(t) = \sum_{k=0}^{n-2} b_k t^k$. In order to show the strict unimodality of $h(t)$ and the positivity of b_ks when $\sum_{i=0}^{k} a_i > 0$ for $0 \leqslant k \leqslant \lfloor (n-2)/2\rfloor$, it suffices to prove that $0 < b_0 < b_1 < \cdots < b_{\lfloor (n-2)/2\rfloor}$ due to the symmetry of $h(t)$. Indeed, note that $b_k = \sum_{i=0}^{k} (k+1-i) a_i$ since

$$b_k = [t^k]h(t) = [t^k]f(t)(1-t)^{-2} = [t^k]f(t)\sum_{j\geqslant 0}\binom{1+j}{j}t^j = \sum_{i=0}^{k}(k+1-i)a_i.$$

Then $b_k - b_{k-1} = \sum_{i=0}^{k} a_i > 0$ for $1 \leqslant k \leqslant \lfloor (n-2)/2\rfloor$ and $b_0 = a_0 > 0$, as desired. □

LEMMA 2. *Let a be a real number and n an integer ($n \geqslant 2$). Then*

$$\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{k+a-1}{k}\binom{n-k+a-1}{n-k} = \begin{cases} \frac{1}{2}\binom{n+2a-1}{n} - \frac{\lfloor n/2\rfloor + a}{\lfloor n/2\rfloor+1}\binom{\lfloor n/2\rfloor+a-1}{\lfloor n/2\rfloor}^2, & n \text{ is odd};\\[2ex] \frac{1}{2}\left(\binom{n+2a-1}{n} - \binom{n/2+a-1}{n/2}^2\right), & n \text{ is even}. \end{cases}$$

Proof. Note that both sides of the desired identities can be viewed as polynomials in a of fixed degree if n is fixed. Thus, it suffices to show that the identities in the lemma hold under the assumption that a is any positive integer.

Suppose that a is a positive integer. It is known that the number of ways of placing n indistinguishable balls into $2a$ distinguishable boxes equals $\binom{n+2a-1}{n}$ (see [4, p. 15]). Now we divide the boxes into two classes: one class contains a boxes and each box is colored red, the other class consists of the remaining boxes which are colored blue. Then one distribution of n indistinguishable balls into $2a$ distinguishable boxes can be obtained by placing k balls into one class of boxes and $n-k$ balls into the other class of boxes. It follows that

$$\sum_{k=0}^{n}\binom{k+a-1}{k}\binom{n-k+a-1}{n-k}=\binom{n+2a-1}{n}. \tag{1}$$

Since the roles of the two colors are completely symmetric, when n is odd, we obtain the half sum

$$\begin{aligned}\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{k+a-1}{k}\binom{n-k+a-1}{n-k}&=\frac{1}{2}\left(\binom{n+2a-1}{n}-2\binom{\lfloor\frac{n}{2}\rfloor+a-1}{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor+a}{\lfloor\frac{n}{2}\rfloor+1}\right)\\&=\frac{1}{2}\binom{n+2a-1}{n}-\frac{\lfloor\frac{n}{2}\rfloor+a}{\lfloor\frac{n}{2}\rfloor+1}\binom{\lfloor\frac{n}{2}\rfloor+a-1}{\lfloor\frac{n}{2}\rfloor}^{2}.\end{aligned}$$

When n is even, the second required identity can be obtained in a similar way. □

We are now in a position to prove Theorem 1.

The proof of Theorem 1. Obviously, $F(a)\equiv 0$ if x or y equals to 0.

Now assume that x and y are both positive. Set $t=\frac{x}{y}$. Then

$$F(a)=\frac{y^n}{\binom{n+2a-1}{n}}\sum_{k=0}^{n}\binom{k+a-1}{k}\binom{n-k+a-1}{n-k}t^k.$$

For $n=0,1$, it is clear that $F(a)$ is a constant function.

For $n\geqslant 2$, assume that a_1,a_2 be two positive numbers and $a_2>a_1$. We will prove that $F(a_1)>F(a_2)$ if $t\neq 1$ and $F(a_1)=F(a_2)$ if $t=1$. Note that

$$F(a_1)-F(a_2)=\frac{y^n}{\binom{n+2a_1-1}{n}\binom{n+2a_2-1}{n}}G(t),$$

where we set $G(t)=\sum_{k=0}^{n}A_kt^k$ with

$$\begin{aligned}A_k=&\binom{n+2a_2-1}{n}\binom{k+a_1-1}{k}\binom{n-k+a_1-1}{n-k}\\&-\binom{n+2a_1-1}{n}\binom{k+a_2-1}{k}\binom{n-k+a_2-1}{n-k}.\end{aligned}$$

In the rest of the proof, we regard t as an indeterminate. So $F(a_1)$, $F(a_2)$ and $G(t)$ are three polynomials in t. Note that $F(a_1)$ and $F(a_2)$ are both symmetric with the same center due to the symmetry of the binomial sequence $\left\{\binom{k+a-1}{k}\binom{n-k+a-1}{n-k}\right\}_{k=0}^{n}$. Then $G(t)$ is symmetric since it is a linear combination of $F(a_1)$ and $F(a_2)$.

In order to prove Theorem 1, it suffices to show respectively that $(t-1)^2|G(t)$, i.e., $G(t)=(t-1)^2H(t)$, and that the coefficients of $H(t)$ are all positive.

To show that $(t-1)^2|G(t)$, it suffices to prove $G(1)=0$ and $G'(1)=0$ respectively.

Indeed, it follows from (1) that

$$\begin{aligned}
G(1) &= \sum_{k=0}^{n} A_k \\
&= \binom{n+2a_2-1}{n}\sum_{k=0}^{n}\binom{k+a_1-1}{k}\binom{n-k+a_1-1}{n-k} \\
&\quad -\binom{n+2a_1-1}{n}\sum_{k=0}^{n}\binom{k+a_2-1}{k}\binom{n-k+a_2-1}{n-k} \\
&= \binom{n+2a_2-1}{n}\binom{n+2a_1-1}{n}-\binom{n+2a_1-1}{n}\binom{n+2a_2-1}{n} \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
G'(1) &= \sum_{k=1}^{n} kA_k \\
&= \binom{n+2a_2-1}{n}\sum_{k=1}^{n}k\binom{k+a_1-1}{k}\binom{n-k+a_1-1}{n-k} \\
&\quad -\binom{n+2a_1-1}{n}\sum_{k=1}^{n}k\binom{k+a_2-1}{k}\binom{n-k+a_2-1}{n-k} \\
&= \binom{n+2a_2-1}{n}\sum_{k=1}^{n}a_1\binom{k+a_1-1}{k-1}\binom{n-k+a_1-1}{n-k} \\
&\quad -\binom{n+2a_1-1}{n}\sum_{k=1}^{n}a_2\binom{k+a_2-1}{k-1}\binom{n-k+a_2-1}{n-k} \\
&= a_1\binom{n+2a_2-1}{n}\sum_{k=0}^{n-1}\binom{k+a_1}{k}\binom{n-k-1+a_1-1}{n-1-k} \\
&\quad -a_2\binom{n+2a_1-1}{n}\sum_{k=0}^{n-1}\binom{k+a_2}{k}\binom{n-1-k+a_2-1}{n-1-k} \\
&= a_1\binom{n+2a_2-1}{n}\binom{n+2a_1-1}{n-1}-a_2\binom{n+2a_1-1}{n}\binom{n+2a_2-1}{n-1} \\
&= 0.
\end{aligned}$$

Thus, $G(t)=(t-1)^2H(t)$. Moreover, since $G(t)$ is symmetric, $H(t)$ is also symmetric by Lemma 1.

For the positivity of the coefficients of $H(t)$, by Lemma 1, it suffices to prove $\sum_{i=0}^{k} A_i > 0$ for $0 \leqslant k \leqslant \lfloor (n-2)/2 \rfloor$, which also yields the strict unimodality of $H(t)$.

Denote $B_k = \sum_{i=0}^{k} A_i$ for brevity. We will prove respectively that $B_0, B_{\lfloor (n-2)/2 \rfloor} > 0$ and that the sequence $\{B_k\}_{1 \leqslant k \leqslant \lfloor (n-2)/2 \rfloor}$ is unimodal. Then each $B_k > 0$.

Note that

$$\begin{aligned} A_0 &= \binom{n+2a_2-1}{n}\binom{n+a_1-1}{n} - \binom{n+2a_1-1}{n}\binom{n+a_2-1}{n} \\ &= \frac{1}{n!^2}\left(\prod_{i=1}^{n}(n+2a_2-i)(n+a_1-i) - \prod_{i=1}^{n}(n+2a_1-i)(n+a_2-i)\right) \end{aligned}$$

and

$$(n+2a_2-i)(n+a_1-i)-(n+2a_1-i)(n+a_2-i) = (n-i)(a_2-a_1) \geqslant 0,$$

where the equality holds if and only if $n=i$. This implies that $B_0 = A_0 > 0$.

Now consider the positivity of $B_{\lfloor (n-2)/2 \rfloor}$. When n is even, by Lemma 2,

$$\begin{aligned} B_{\lfloor (n-2)/2 \rfloor} &= \binom{n+2a_2-1}{n} \cdot \frac{1}{2}\left(\binom{n+2a_1-1}{n} - \binom{\frac{n}{2}+a_1-1}{\frac{n}{2}}^2\right) \\ &\quad - \binom{n+2a_1-1}{n} \cdot \frac{1}{2}\left(\binom{n+2a_2-1}{n} - \binom{\frac{n}{2}+a_2-1}{\frac{n}{2}}^2\right) \\ &= \frac{1}{2}\left(\binom{n+2a_1-1}{n}\binom{\frac{n}{2}+a_2-1}{\frac{n}{2}}^2 - \binom{n+2a_2-1}{n}\binom{\frac{n}{2}+a_1-1}{\frac{n}{2}}^2\right). \end{aligned}$$

Note that

$$\begin{aligned} &\binom{n+2a_1-1}{n}\binom{\frac{n}{2}+a_2-1}{\frac{n}{2}}^2 - \binom{n+2a_2-1}{n}\binom{\frac{n}{2}+a_1-1}{\frac{n}{2}}^2 \\ &= \frac{1}{n!(\frac{n}{2})!^2}\left[\prod_{i=1}^{n}(n+2a_1-i)\prod_{i=1}^{n/2}\left(\frac{n}{2}+a_2-i\right)^2 - \prod_{i=1}^{n}(n+2a_2-i)\prod_{i=1}^{n/2}\left(\frac{n}{2}+a_1-i\right)^2\right] \\ &= \frac{2^{\frac{n}{2}}}{n!(\frac{n}{2})!^2}\prod_{i=1}^{n/2}\left(\frac{n}{2}+a_1-i\right)\left(\frac{n}{2}+a_2-i\right) \\ &\quad \times \left[\prod_{i=1}^{n/2}(n+2a_1-2i+1)\left(\frac{n}{2}+a_2-i\right) - \prod_{i=1}^{n/2}(n+2a_2-2i+1)\left(\frac{n}{2}+a_1-i\right)\right] \\ &> 0, \end{aligned}$$

which follows from that

$$(n+2a_1-2i+1)\left(\frac{n}{2}+a_2-i\right) - (n+2a_2-2i+1)\left(\frac{n}{2}+a_1-i\right) = a_2-a_1 > 0.$$

Hence $B_{\lfloor (n-2)/2\rfloor} > 0$.

For odd n, by Lemma 2 again, we get

$$\begin{aligned}
B_{\lfloor (n-2)/2\rfloor} &= \binom{n+2a_2-1}{n}\left(\frac{1}{2}\binom{n+2a_1-1}{n} - \frac{\lfloor\frac{n}{2}\rfloor+a_1}{\lfloor\frac{n}{2}\rfloor+1}\binom{\lfloor\frac{n}{2}\rfloor+a_1-1}{\lfloor\frac{n}{2}\rfloor}^2\right)\\
&\quad -\binom{n+2a_1-1}{n}\left(\frac{1}{2}\binom{n+2a_2-1}{n} - \frac{\lfloor\frac{n}{2}\rfloor+a_2}{\lfloor\frac{n}{2}\rfloor+1}\binom{\lfloor\frac{n}{2}\rfloor+a_2-1}{\lfloor\frac{n}{2}\rfloor}^2\right)\\
&= \frac{1}{\lfloor\frac{n}{2}\rfloor+1}\left[\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2\right)\binom{n+2a_1-1}{n}\binom{\lfloor\frac{n}{2}\rfloor+a_2-1}{\lfloor\frac{n}{2}\rfloor}^2\right.\\
&\quad \left. -\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2\right)\binom{n+2a_2-1}{n}\binom{\lfloor\frac{n}{2}\rfloor+a_1-1}{\lfloor\frac{n}{2}\rfloor}^2\right]\\
&= \frac{1}{(\lfloor\frac{n}{2}\rfloor+1)n!\,\lfloor\frac{n}{2}\rfloor!^2}\left[\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2\right)\prod_{i=1}^{n}(n+2a_1-i)\prod_{i=1}^{\lfloor n/2\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2-i\right)^2\right.\\
&\quad \left. -\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1\right)\prod_{i=1}^{n}(n+2a_2-i)\prod_{i=1}^{\lfloor n/2\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1-i\right)^2\right]\\
&= \frac{1}{(\lfloor\frac{n}{2}\rfloor+1)n!\,\lfloor\frac{n}{2}\rfloor!^2}\cdot 2^{\lfloor n/2\rfloor}\prod_{i=1}^{\lfloor n/2\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1-i\right)\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2-i\right)\\
&\quad \times\left[\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2\right)(n+2a_1-1)\prod_{i=1}^{\lfloor n/2\rfloor}(n+2a_1-2i)\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2-i\right)\right.\\
&\quad \left. -\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1\right)(n+2a_2-1)\prod_{i=1}^{\lfloor n/2\rfloor}(n+2a_2-2i)\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1-i\right)\right]\\
&> 0,
\end{aligned}$$

where the strict inequality follows from that

$$\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2\right)(n+2a_1-1)-\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1\right)(n+2a_2-1) = 2(a_2-a_1) > 0,$$

$$(n+2a_1-2i)\left(\left\lfloor\frac{n}{2}\right\rfloor+a_2-i\right)-(n+2a_2-2i)\left(\left\lfloor\frac{n}{2}\right\rfloor+a_1-i\right) = a_2-a_1 > 0.$$

Now we will prove that $\{B_k\}_{0\leqslant k\leqslant\lfloor (n-2)/2\rfloor}$ is unimodal. It is clearly true when $n=2,3$. Consider the cases when $n\geqslant 4$. Note that $A_k = B_k - B_{k-1}$. By the definition of unimodality, we will prove the unimodality of $\{B_k\}_{0\leqslant k\leqslant\lfloor (n-2)/2\rfloor}$ by showing that the sequence $\{A_k\}_{1\leqslant k\leqslant\lfloor (n-2)/2\rfloor}$ will not be positive afterwards once it is negative.

Set $A_k = \alpha_k - \beta_k$, where

$$\alpha_k = \binom{n+2a_2-1}{n}\binom{k+a_1-1}{k}\binom{n-k+a_1-1}{n-k},$$

$$\beta_k = \binom{n+2a_1-1}{n}\binom{k+a_2-1}{k}\binom{n-k+a_2-1}{n-k}.$$

Clearly, $\beta_k > 0$ since $n \geqslant k \geqslant 1$ and $a_1, a_2 > 0$.

To prove the desired change of the signs of $\{A_k\}$, it suffices to show a chain of inequalities: for $1 \leqslant k \leqslant \lfloor \frac{n-2}{2} \rfloor - 1$,

$$\frac{A_{k+1}}{\beta_{k+1}} = \frac{\alpha_{k+1}-\beta_{k+1}}{\beta_{k+1}} < \frac{\alpha_k - \beta_k}{\beta_k} = \frac{A_k}{\beta_k}. \tag{2}$$

Indeed, the following equivalent form of (2)

$$\frac{\alpha_{k+1}}{\beta_{k+1}} < \frac{\alpha_k}{\beta_k}$$

is valid because

$$\frac{\alpha_{k+1}\beta_k}{\alpha_k\beta_{k+1}} = \frac{(k+a_1)(n-k+a_2-1)}{(k+a_2)(n-k+a_1-1)}$$

and

$$(k+a_1)(n-k+a_2-1)-(k+a_2)(n-k+a_1-1) = (n-2k+1)(a_1-a_2) < 0.$$

Thus, the sequence $\{B_k\}_{0\leqslant k\leqslant \lfloor (n-2)/2 \rfloor}$ is unimodal. The proof is completed. □

REMARK 1. In the proof of Theorem 1, we get a positive binomial sequence $\{B_k\}_{0\leqslant k\leqslant n-2}$, where $B_k = \sum_{i=0}^{k} A_i$. Furthermore, by Lemma 1, there is another positive binomial sequence $\{\sum_{i=0}^{k}(k+1-i)A_i\}_{k=0}^{n}$ which follows from the positivity of the coefficients of $H(t)$. It would be interesting to give combinatorial interpretations for these two positive binomial sequences.

REMARK 2. Bennett[1] revealed the connection between Schur's theorem and Haber's inequality. The original version of Haber's inequality [5] states that, if $x, y \geqslant 0$ and n is a positive integer, then

$$\left(\frac{x+y}{2}\right)^n \leqslant \frac{x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n}{n+1},$$

which is a special case of Schur's theorem with $N = 2$. Haber's inequality has been extensively developed. Its convexity version has been shown to connect with moment sequences (see [2]). As mentioned in [1], Conjecture 1 implies that when $a \to \infty$, Haber's inequality follows from the monotonicity of $F(a)$. When $a \to 0$,

$$\frac{x^n + x^{n-1}y + \cdots + y^n}{n+1} \leqslant \frac{x^n + y^n}{2},$$

which is an inequality complementary to Haber's inequality. So the extension of Haber's inequality by Theorem 1 would be expected to be developed further.

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