Mathematical
Inequalities
& Applications
Volume 21, Number 1 (2018), 179–200 doi:10.7153/mia-2018-21-15

LYAPUNOV–TYPE INEQUALITY FOR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH HILFER DERIVATIVE

NIMISHA PATHAK

(*Communicated by D. Hinton*)

Abstract. In this work, we establish Lyapunov-type inequalities for fractional boundary value problems containing Hilfer derivative of order α, $1 < \alpha \leqslant 2$ and type $0 \leqslant \beta \leqslant 1$. We consider the boundary value problems with the Dirichlet, and a mixed set of Dirichlet and Neumann boundary conditions. We consider both integer and fractional order eigenvalue problems, determine a lower bound for the smallest eigenvalue using Lyapunov-type inequalities, and improve these bounds using semi maximum norm and Cauchy-Schwarz inequalities. We use the improved lower bounds to obtain intervals where a certain Mittag-Leffler functions have no real zeros. Further, we discuss the particular cases for the type $\beta = 0$ and $\beta = 1$, which give the results respectively for Riemann-Liouville and Caputo fractional boundary value as well as eigenvalue problems. For both the fractional and the integer order eigenvalue problems, we give a comparison between the smallest eigenvalue and its lower bounds obtained from the Lyapunov-type, semi maximum norm and Cauchy-Schwarz inequalities. Results show that the Lyapunov-type inequality gives the worse and semi maximum norm and Cauchy-Schwarz inequalities give better lower bound estimates for the smallest eigenvalues.

1. Introduction

The Lyapunov inequality [12] has proved to be very useful in the study of spectral properties of ordinary differential equations (see [1], [13]). This inequality can be stated as follows:

THEOREM 1. (See [12]) *A necessary condition for the Boundary Value Problem (BVP)*

$$y''(t) + q(t)y(t) = 0, \quad a < t < b,$$
$$y(a) = 0, \quad y(b) = 0, \tag{1}$$

to have nontrivial solutions is that

$$\int_a^b |q(s)|ds > \frac{4}{b-a}, \tag{2}$$

Mathematics subject classification (2010): 34A08, 34A40, 26D10, 33E12.

Keywords and phrases: Lyapunov inequality, Hilfer fractional derivative, Green's function, Laplace transforms, Mittag-Leffler function.

where q is a real and continuous function. The constant 4 in equation (2) sharp so that it cannot be replaced by a larger number.

Recently, research on Lyapunov Type Inequalities (LTIs) for Fractional Boundary Value Problems (FBVPs) has begun. Ferreira [3], [4], Jleli and Samet [10], Rong and Bai [16] have established LTIs for FBVPs of order α, $\alpha \in (1,2]$ and different boundary conditions. In [14], we obtained the LTI for FBVP of order $2 < \alpha \leqslant 3$. We also improved the lower bound of the smallest eigenvalue of the eigenvalue problem using the semimaximum norm and Cauchy-Schwarz inequalities. In these work the authors considered the FBVPs with either Riemann-Liouville or Caputo derivatives. Motivated by the above work, in this paper we consider a FBVP involving Hilfer derivative of order $\alpha, \alpha \in (1,2]$ and type $\beta \in [0,1]$, and obtain a Lyapunov-type inequality for it. Specifically, we consider the following FBVP:

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad a < t < b \tag{3}$$

with the boundary conditions

$$y(a) = 0, \quad y(b) = 0 \tag{4}$$

or

$$y(a) = 0, \quad Dy(b) = 0; \quad D \equiv \frac{d}{dt} \tag{5}$$

where $q : [a,b] \to \mathbb{R}$ is a continuous function, $D_{a^+}^{\alpha,\beta}$ is the Hilfer derivative operator defined later. $I_{a^+}^{\alpha}$ is the standard Riemann-Liouville integral operator. We also find lower bound estimates for the smallest eigenvalue of Fractional Eigenvalue Problem (FEP) obtained from (3) with boundary conditions (4)–(5) respectively. The advantage of considering the FBVP and FEP with the Hilfer derivative is that the obtained results allow us to give results for Riemann-Liouville as well as Caputo derivative FBVPs and FEPs as its particular cases.

The outline of the paper is as follows. First, for the FBVP (3)–(4). we derive the Green's function and use it a) to reduce problem (3)–(4) to an equivalent Fredholm integral equation of the second kind and b) to establish a Lyapunov-type inequality. Second, we consider a FEP and determine a lower bound for the first eigenvalue from Lyapunov-type inequality. Third, we improve the lower bound for the first eigenvalue using a Semi Maximum Norm (SMN) and a Cauchy-Schwarz Inequality (CSI). We compare the smallest eigenvalues and their lower bounds obtained from the semi maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities for fractional as well as integer order systems. Finally, to demonstrate an application of the inequalities developed in the paper, we apply the improved bounds for the smallest eigenvalues to obtain intervals where a certain Mittag-Leffler function has no real zero. The same procedure we follow for the FBVPs (3) and (5).

2. Definitions and preliminaries

Applications of fractional calculus require fractional derivatives of different kinds [11], [17]. Integration of fractional order is traditionally defined by the Riemann-Liouville fractional integral operator $I_{a^+}^{\alpha}$, which is given by

$$I_{a^+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds, \quad \alpha > 0, \tag{6}$$

where $t > a$, $n = [\alpha]+1$, and $\Gamma(\alpha)$ denotes the Euler's gamma function. The operator in (7) is defined on the space $L(a,b)$ of Lebesgue measurable functions $y(t)$ on a finite interval $[a,b](b > a)$ of the real line $\mathbb{R}$:

$$\mathbf{L}(a,b) = \{y : ||y||_1 = \int_a^b |y(t)|dt < \infty\}.$$

Let $AC[a,b]$ be the space of real-valued functions $y(t)$ which are absolutely continuous on $[a,b]$. We denote by $AC^n[a,b]$ the space of real-valued functions $y(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $y^{(n-1)}(t) \in AC^n[a,b]$:

$$AC^n[a,b] = \left\{y : [a,b] \to \mathbb{R} : (D^{n-1}y)(t) \in AC[a,b]; D \equiv \frac{d}{dt}\right\}.$$

The Hilfer Fractional Derivative (HFD) or generalized Riemann-Liouville fractional derivative (GRLFD) of order $n-1 < \alpha \leqslant n, n \in \mathbb{N}$ and type $0 \leqslant \beta \leqslant 1$ with respect to t, is defined by Hilfer et al. [9], as follows:

$$\left(D_{a^+}^{\alpha,\beta}y\right)(t) = \left(I_{a^+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}\left(I_{a^+}^{(1-\beta)(n-\alpha)}y\right)\right)(t), \tag{7}$$

whenever the right-hand side exists. In the above definition, type β allows $D_{a^+}^{\alpha,\beta}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative [11], [17]. As in the case $\beta = 0$, (7) reduces to the classical Riemann-Liouville fractional derivative and for $\beta = 1$, (7) reduces to the Caputo fractional derivative.

The difference between fractional derivatives of different types becomes apparent from Laplace transformation. The Laplace transform formula of HFD (7) is defined as follows [18], [19]:

For $n-1 < \alpha \leqslant n$, $0 \leqslant \beta \leqslant 1$, $n \in \mathbb{N}$ the Laplace transform formula

$$\mathscr{L}\left\{\left(D_{0^+}^{\alpha,\beta}\right)y(t);s\right\} = s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{n-k-1-\beta(n-\alpha)}\frac{d^k}{dt^k}\left(I_{0^+}^{(1-\beta)(n-\alpha)}y\right)(0^+), \tag{8}$$

is valid. Where

$$\mathscr{L}\{y(t);s\} = \int_0^{\infty} e^{-st}y(t)dt = Y(s). \tag{9}$$

The Mittag-Leffler (M-L) functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad \alpha,z\in\mathbb{C},\quad \mathbf{R}(\alpha)>0. \tag{10}$$

and

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta,z\in\mathbb{C},\quad \mathbf{R}(\alpha),\mathbf{R}(\beta)>0. \tag{11}$$

Note that $E_{\alpha,1}(z)=E_\alpha(z)$ and $E_{1,1}(z)=E_1(z)=\exp(z)$. The Laplace transform of the function $\phi(t)=t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha)$ is given as

$$(\mathscr{L}\phi)(s)=\frac{s^{\alpha-\beta}}{s^\alpha\mp\lambda},\quad \mathbb{R}(s)>0,\quad \lambda\in\mathbb{C},\quad |\lambda s^{-\alpha}|<1,$$

and its inverse relationship is given as

$$\mathscr{L}^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha\mp\lambda}\right]=t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha),\quad \mathbb{R}(s)>0,\quad \lambda\in\mathbb{C},\quad |\lambda s^{-\alpha}|<1, \tag{12}$$

where $\mathscr{L}^{-1}$ is the inverse Laplace transform operator. Further, function $t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha)$ satisfies the following property [8], [11]:

$$\frac{d^n}{dt^n}[t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha)]=t^{\beta-n-1}E_{\alpha,\beta-n}(\pm\lambda t^\alpha),\quad \lambda\in\mathbb{C},\quad \mathbf{R}(\beta-n)>0,\quad n\in\mathbb{N}. \tag{13}$$

In [19], the compositional property of Riemann-Liouville fractional integral operator with the HFD operator is obtained.

LEMMA 1. [19] *Let $y\in\mathbf{L}[a,b]$, $n-1<\alpha\leqslant n$, $n\in\mathbb{N}$, $0\leqslant\beta\leqslant 1$, $I_{a^+}^{(n-\alpha)(1-\beta)}y\in AC^k[a,b]$. Then the Riemann-Liouville fractional integral $I_{a^+}^{\alpha}$ and the HFD operator $D_{a^+}^{\alpha,\beta}$ are connected by the relation*

$$\left(I_{a^+}^{\alpha}D_{a^+}^{\alpha,\beta}y\right)(t)=y(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)}\lim_{t\to a^+}\frac{d^k}{dt^k}\left(I_{a^+}^{(n-\alpha)(1-\beta)}y\right)(t). \tag{14}$$

In this paper we use the following result.

PROPOSITION 1. Let $\alpha\in(1,2]$, $\beta\in[0,1]$. We consider the FBVP
Problem P1:

$$\left(D_{a^+}^{\alpha,\beta}y\right)(t)+q(t)y(t)=0,\quad a<t<b, \tag{15}$$

where q is a real valued continuous function in interval $[a,b]$ and boundary conditions are:

$$\begin{aligned} a_1y(a)+a_2Dy(a)&=0\ ,\\ b_1y(b)+b_2Dy(b)&=0\ , \end{aligned} \tag{16}$$

with $a_1^2+a_2^2 \neq 0$, $b_1^2+b_2^2 \neq 0$. Then the FBVP (15)–(16) can be written in its equivalent integral form as

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{17}$$

where $G(t,s)$ is a Green's function. Green's function depends on the BVPs which will be addressed latter in this paper.

From (17) it follows that if y is a nontrivial continuous solution of the FBVP (15)–(16) then

$$|y(t)| \leqslant \int_a^b |G(t,s)q(s)||y(s)|ds. \tag{18}$$

Let $B = \mathbb{C}[a,b]$ be a Banach space endowed a norm

$$||y||_\infty = \max_{a\leqslant t\leqslant b} |y(t)|, y \in B.$$

Hence, from (18) we get

$$||y||_\infty \leqslant \max_{a\leqslant t\leqslant b} \int_a^b |G(t,s)q(s)|ds||y||_\infty,$$

or equivalently,

$$1 \leqslant \max_{a\leqslant t\leqslant b} \int_a^b |G(t,s)q(s)|ds. \tag{19}$$

Using the properties of Green's function $G(t,s)$ particularly, $\max_{a\leqslant t\leqslant b}|G(t,s)| = G_{\max}$ in (19) gives the inequality

$$\int_a^b |q(s)|ds \geqslant \frac{1}{G_{\max}} \tag{20}$$

called the Lyapunov-type inequality for FBVP (15)–(16). Additionally from (17) and the Cauchy-Schwarz inequality we obtain that

$$y^2(t) \leqslant \left[\int_a^b |G(t,s)q(s)|^2 ds\right]\left[\int_a^b y^2(s)ds\right]. \tag{21}$$

Integrating this inequality over $[a,b]$ and then dividing the result by $||y||_2$, we get

$$1 \leqslant \left[\int_a^b \int_a^b |G(t,s)q(s)|^2 dsdt\right], \tag{22}$$

we call (22) the CSI for FBVP (15)–(16). Now, consider the following linear Fractional Differential Equation (FDE) and the boundary conditions

Problem P2:

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) + \lambda y(t) = 0, \quad a < t < b, \tag{23}$$

$$\begin{aligned} a_1y(a) + a_2Dy(a) &= 0,\\ b_1y(b) + b_2Dy(b) &= 0, \end{aligned} \tag{24}$$

with $a_1^2+a_2^2\neq 0$, $b_1^2+b_2^2\neq 0$ where the function $y(t)$ and the number λ are unknown. A function $y(t)$ that satisfies equations (23) and (24) is known as an eigenfunction, the corresponding λ the eigenvalue associated with $y(t)$, and the problem a FEP. By setting $\alpha=2$ and $\beta=0$ or $\beta=1$ in equation (23), we obtain an Integer Order Eigenvalue Problem (IOEP).

Next, we give three methods to estimate the lower bound for the smallest eigenvalue of problem P2. Note that FBVP (15)–(16) and problem P2 are the same except that q(t) in equation (15) has been replaced with λ to obtain equation (23). Thus, the LTI equation (20) and the CSI equation (22) for FBVP (15)–(16) can be used to find a lower bound for the smallest eigenvalue of problem P2. These are called two methods LTI and CSI methods. In the discussion to follow, we will use the following definition for a Lyapunov inequality lower bound.

DEFINITION 1. A Lyapunov Inequality Lower Bound (LILB) is defined as a lower estimate for the smallest eigenvalue obtained from Lyapunov and Lyapunov-type inequalities given in equations (2) and (20).

If we replace $q(t)=\lambda$ in (22), then we obtain a lower bound for the smallest eigenvalue of problem P2

$$\lambda \geqslant \left[\int_a^b \int_a^b G^2(t,s)dsdt\right]^{-\frac{1}{2}}. \tag{25}$$

In the discussion to follow, we define a Cauchy-Schwarz Inequality Lower Bound as follows:

DEFINITION 2. A Cauchy-Schwarz Inequality Lower Bound (CSILB) is defined as an estimate of the lower bound for the smallest eigenvalue obtained from the Cauchy-Schwarz inequality of type given in equation (22).

To describe the Semi Maximum Norm method, note that a linear FBVP P1 reduces to

$$1 \leqslant \max_{a\leqslant t\leqslant b}\int_a^b |G(t,s)q(s)|ds$$

(see (19)), and for a FEP P2, $q(s)$ in the above equation is replaced with λ to obtain

$$\lambda \geqslant \frac{1}{\max_{a\leqslant t\leqslant b}\int_a^b |G(t,s)|ds}. \tag{26}$$

The above inequality gives a lower bound estimate for the smallest eigenvalue. In this case, we do not take the maximum norm of $|G(t,s)|$ but only the maximum norm of the integral $\int_a^b |G(t,s)|ds$ over $[a,b]$, and for this reason, we call this method of obtaining a lower bound for λ the Semi Maximum Norm method. Also note that

$$\max_{a\leqslant t\leqslant b}\int_a^b |G(t,s)|ds \leqslant (b-a)\max_{[a,b]\times[a,b]}|G(t,s)|$$

and therefore the Semi Maximum Norm method provides a better estimate for the smallest eigenvalue than that provided by the Lyapunov-type inequalities. In the sequel we define a Semi Maximum Norm Lower Bound as follows:

DEFINITION 3. A Semi Maximum Norm Lower Bound (SMNLB) is defined as the lower estimate for the smallest eigenvalue obtained from the Semi Maximum Norm inequality of type given in (26).

3. Lyapunov-type inequalities for the FBVPs and eigenvalue estimates for FEPs

In this section we establish Lyapunov-type inequalities for the FBVPs with the Dirichlet and a mixed set of Dirichlet and Neumann boundary conditions. We also obtain the eigenvalue estimates for the smallest eigenvalue of FEPs. We apply these estimates to obtain the interval in which Mittag-Leffler functions have no real zeros.

3.1. Lyapunov-type inequality for FVBP (3)–(4)

Replacing $a_1 = b_1 = 1$, $a_2 = b_2 = 0$ in equation (16) we obtain the FBVP Problem P3:

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad a < t < b$$
$$y(a) = y(b) = 0.$$

LEMMA 2. *Problem P3 can be written as (17) where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)} (b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leqslant s \leqslant t \leqslant b, \\ \left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)} (b-s)^{\alpha-1}, & a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{27}$$

is the Green's function for the problem.

Proof. Taking $I_{a^+}^{\alpha}$ on the first equation of P3 and using Lemma 1 with $n = 2$, we obtain

$$y(t) = c_1 \frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_2 \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)y(s)ds, \tag{28}$$

where c_1 and c_2 are the real constants given by

$$c_1 = \left(I_{a^+}^{(2-\alpha)(1-\beta)} y\right)(a^+), c_2 = \frac{d}{dt}\left(I_{a^+}^{(2-\alpha)(1-\beta)} y\right)(a^+).$$

Since $y(a) = 0$, we get $c_1 = 0$. Now $y(b) = 0$ gives

$$c_2 = \frac{\Gamma(2-(2-\alpha)(1-\beta))}{\Gamma(\alpha)(b-a)^{1-(2-\alpha)(1-\beta)}} \int_a^b (b-s)^{\alpha-1} q(s)y(s)ds.$$

Hence, equality (28) becomes

$$y(t)=\frac{1}{\Gamma(\alpha)}\left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}\int_a^b (b-s)^{\alpha-1}q(s)y(s)ds-\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)y(s)ds,$$

which can be written as equation (17) with $G(t,s)$ given by (27). This concludes the proof. □

In [5], the author determines the maximum of the Green's function. We use the similar approach to prove Lemma 3. The Green's function G defined by (27) satisfies the following property:

LEMMA 3. *For all* $(t,s)\in[a,b]\times[a,b]$,

$$|G(t,s)|=\frac{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}. \tag{29}$$

Proof. Let us define two functions

$$G_1(t,s):=(b-s)^{\alpha-1}\left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}-(t-s)^{\alpha-1},\quad a\leqslant s\leqslant t\leqslant b,$$

and

$$G_2(t,s):=(b-s)^{\alpha-1}\left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)},\quad a\leqslant t\leqslant s\leqslant b.$$

Here, G_2 is an increasing function of t and $0\leqslant G_2(t,s)\leqslant G_2(s,s)$, where

$$G_2(s,s)=(b-s)^{\alpha-1}\left(\frac{s-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}:=f(s);\quad s\in[a,b].$$

To find $\max_{s\in[a,b]}f(s)$

$$f'(s)=\frac{(s-a)^{\alpha-2+\beta(2-\alpha)}(b-s)^{\alpha-2}[(\alpha-1+\beta(2-\alpha))(b-s)-(\alpha-1)(s-a)]}{(b-a)^{\alpha-1+\beta(2-\alpha)}}$$

$$\begin{aligned}f'(s)=0&\Rightarrow[\alpha-1+\beta(2-\alpha)]b-s[2\alpha-2+\beta(2-\alpha)]+a(\alpha-1)=0\\&\Rightarrow s=\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)},\end{aligned}$$

and $f'(s)>0$ for $s<\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}$, and $f'(s)<0$ for $s>\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}$.

$$\max_{(t,s)\in[a,b]\times[a,b]}|G_2(t,s)|=f\left[\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}\right],$$

where

$$\begin{aligned}&f\left[\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}\right]\\&=\frac{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}{[2\alpha-2+\beta(2-\alpha)]^{2\alpha-2+\beta(2-\alpha)}}.\end{aligned}$$

We use the Fritz John theorem and take advantage of some kind of symmetry of the partial derivatives of our problem in order to find the candidates to maxima of the function $G_1(t,s)$ with $a<s<t<b$. We have

$$\frac{\partial G_1}{\partial t}=\frac{[\alpha-1+\beta(2-\alpha)](t-a)^{\alpha-2+\beta(2-\alpha)}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1+\beta(2-\alpha)}}-(\alpha-1)(t-s)^{\alpha-2},$$

$$\frac{\partial G_1}{\partial s}=\frac{-(\alpha-1)(b-s)^{\alpha-2}(t-a)^{\alpha-1+\beta(2-\alpha)}}{(b-a)^{\alpha-1+\beta(2-\alpha)}}+(\alpha-1)(t-s)^{\alpha-2}.$$

Now, $\frac{\partial G_1}{\partial t}=0$ and $\frac{\partial G_1}{\partial s}=0$ gives,

$$\begin{aligned}&\frac{-(\alpha-1)(b-s)^{\alpha-2}(t-a)^{\alpha-1+\beta(2-\alpha)}}{(b-a)^{\alpha-1+\beta(2-\alpha)}}\\&+\frac{[\alpha-1+\beta(2-\alpha)](t-a)^{\alpha-2+\beta(2-\alpha)}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1+\beta(2-\alpha)}}=0\\&\frac{(b-s)^{\alpha-2}(t-a)^{\alpha-2+\beta(2-\alpha)}}{(b-a)^{\alpha-1+\beta(2-\alpha)}}[(b-s)(\alpha-1+\beta(2-\alpha))-(\alpha-1)(t-a)]=0\\\Rightarrow\ &t=t^*=a+\frac{[\alpha-1+\beta(2-\alpha)](b-s)}{\alpha-1},\end{aligned}$$

provided $s<t<b$.

$$\begin{aligned}s<t\Leftrightarrow\ &s<a+\frac{[\alpha-1+\beta(2-\alpha)](b-s)}{\alpha-1}\\\Leftrightarrow\ &s(\alpha-1)<a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]-s[\alpha-1+\beta(2-\alpha)]\\\Leftrightarrow\ &s<\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}.\end{aligned}$$

On the other hand,

$$t<b\Leftrightarrow a+\frac{[\alpha-1+\beta(2-\alpha)](b-s)}{\alpha-1}<b, \tag{30}$$

which gives

$$\begin{aligned}&a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]-s[\alpha-1+\beta(2-\alpha)]<b(\alpha-1)\\\Leftrightarrow\ &-s[\alpha-1+\beta(2-\alpha)]<-[b\beta(2-\alpha)+a(\alpha-1)]\\\Leftrightarrow\ &s>\frac{b\beta(2-\alpha)+a(\alpha-1)}{\alpha-1+\beta(2-\alpha)}\end{aligned}$$

i.e.

$$\frac{b\beta(2-\alpha)+a(\alpha-1)}{\alpha-1+\beta(2-\alpha)}<s<\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}.$$

Using $t = t^*$ in $G_1(t,s)$, which after a simplification gives $F(s)$. Consider $G_1(t^*,s) := F(s)$ then

$$F(s) = \frac{(b-s)^{2\alpha-2+\beta(2-\alpha)}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{(\alpha-1)^{\alpha-1+\beta(2-\alpha)}(b-a)^{\alpha-1+\beta(2-\alpha)}} - \left[a + \frac{[\alpha-1+\beta(2-\alpha)](b-s)}{\alpha-1} - s\right]^{\alpha-1}.$$

Now

$$F'(s) = [2\alpha-2+\beta(2-\alpha)](b-s)^{\alpha-2} \times \left[1 - \frac{(b-s)^{\alpha-1+\beta(2-\alpha)}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{(\alpha-1)^{\alpha-1+\beta(2-\alpha)}(b-a)^{\alpha-1+\beta(2-\alpha)}}\right],$$

from equation (30), and using $\alpha - 2 < 0$ and $2\alpha - 2 + \beta(2-\alpha) > 0$ we get

$$F'(s) > 0.$$

This gives

$$\max_{s \in \left[\frac{b\beta(2-\alpha)+a(\alpha-1)}{\alpha-1+\beta(2-\alpha)}, \frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}\right]} |F(s)| = F\left[\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}\right],$$

where

$$F\left[\frac{a(\alpha-1)+b[\alpha-1+\beta(2-\alpha)]}{2\alpha-2+\beta(2-\alpha)}\right] = \frac{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}{[2\alpha-2+\beta(2-\alpha)]^{2\alpha-2+\beta(2-\alpha)}}.$$

Here

$$F\left[\frac{b\beta(2-\alpha)+a(\alpha-1)}{\alpha-1+\beta(2-\alpha)}\right] = 0.$$

This proves Lemma. □

THEOREM 2. *If a nontrivial continuous solution of the problem P3 exists, then for P3 the LTI is*

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}, \tag{31}$$

and in particular, for $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ in P3 gives the standard Lyapunov inequality for BVP (1) as (2).

Proof. Using (29) in LTI equation (20) proves the inequality (31). Replacing $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ in (31) we obtain (2). □

Setting $a_1 = b_1 = 1$, $a_2 = b_2 = 0$ in equation (23) we obtain the FEP

Problem P4:

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) + \lambda y(t) = 0, \quad a < t < b$$
$$y(a) = y(b) = 0, \tag{32}$$

In this work we consider the positive real eigenvalues.

COROLLARY 1. *Let λ be the smallest eigenvalue of FEP P4 for $\alpha \in (1,2]$ and $\beta \in [0,1]$, the smallest eigenvalue estimates of FEP P4 are given by*

1. the LILB

$$\lambda \geqslant \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}, \tag{33}$$

and in particular, for IOEP P4, i.e. $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ this bound is

$$\lambda \geqslant \frac{4}{b-a} \tag{34}$$

2. the SMNLB

$$\lambda \geqslant \frac{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{1-\beta(2-\alpha)}}}{(b-a)^{\alpha}[\alpha-1+\beta(2-\alpha)]^{\frac{\alpha-1+\beta(2-\alpha)}{1-\beta(2-\alpha)}}[1-\beta(2-\alpha)]} \tag{35}$$

and in particular, for IOEP P4, this bound is

$$\lambda \geqslant \frac{8}{(b-a)^2} \tag{36}$$

3. and CSILB

$$\lambda \geqslant \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{4\alpha-1+2\beta(2-\alpha)}{2\alpha(2\alpha-1)[4\alpha-1+2\beta(2-\alpha)]} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}, \tag{37}$$

where $C_1(\alpha) = \int_0^1 t^{\alpha-(2-\alpha)(1-\beta)+1} {}_2F_1(1-\alpha,1;\alpha+1;t)dt$ and ${}_2F_1(a,b;c;t)$ is a hypergeometric function and in particular, for IOEP P4, CSILB is

$$\lambda \geqslant \frac{3\sqrt{10}}{(b-a)^2}. \tag{38}$$

Proof. Setting $q(t) = \lambda$ in equations (31) and (2), the inequalities in the first part follow. Using the Green's function from equation (27) we get

$$\int_a^b G(t,s)ds = \frac{(b-a)^{1-\beta(2-\alpha)}}{\Gamma(\alpha+1)}\left[(t-a)^{\alpha-1+\beta(2-\alpha)} - (b-a)^{\beta(2-\alpha)-1}(t-a)^{\alpha}\right].$$

After some calculations we obtain

$$\max_{a\leqslant t\leqslant b}\int_a^b |G(t,s)|ds = \frac{(b-a)^\alpha[\alpha-1+\beta(2-\alpha)]^{\frac{\alpha-1+\beta(2-\alpha)}{1-\beta(2-\alpha)}}[1-\beta(2-\alpha)]}{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{1-\beta(2-\alpha)}}}.$$

In view of (25), we get the inequality in equation (35). Similarly, substituting the Green's function from equation (27), in (25) and simplifying the result, we obtain the inequality in (37). Setting $\alpha = 2$ in equations (33),(35) and (37) we get the inequalities (34),(36) and (38). □

We first consider the integer order case, i.e. $\alpha = 2$, $a = 0$ and $b = 1$. For this case, the LILB, SMNLB and CSILB for the smallest λ of FEP P4 are given as 4, 8 and $3\sqrt{10} \simeq 9.48683$, respectively (see equations (34), (36) and (38)) . For $\alpha = 2$, the FEP P4 with $a = 0$ and $b = 1$ can be solved in closed form using the tools from integer order calculus. Results show, that the smallest eigenvalue of FEP P4 for $\alpha = 2$ is the root of $\sin(\sqrt{\lambda}) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 9.86960$. Comparing this λ with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue.

The FEP P4 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (1,2]$ as a root of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$. This is explained in the following theorem and its proof. We note that for $1 < \alpha \leqslant 2$, $\beta \in [0,1]$, equation (39) has an infinite number of eigenvalues [7].

THEOREM 3. *For* $1 < \alpha \leqslant 2$, $\beta \in [0,1]$, $a = 0$ *and* $b = 1$ *the FEP P4 has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function* $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$, *i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda) = 0. \tag{39}$$

Proof. To prove this, we take Laplace transform of the first equation in P4 with $a = 0$ and $b = 1$, using (8) for $n = 2$ which after some manipulations leads to

$$Y(s) = \frac{a_0 s^{1-\beta(2-\alpha)}}{s^\alpha+\lambda} + \frac{a_1 s^{-\beta(2-\alpha)}}{s^\alpha+\lambda}, \tag{40}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $a_i = D^i\left[I_{0^+}^{(1-\beta)(2-\alpha)}y\right](0^+)$, $i = 0,1$. Taking inverse Laplace transform of equation (39) and using equation (12), we obtain

$$y(t) = a_0 t^{(\alpha-1+\beta(2-\alpha))-1}E_{\alpha,\alpha-1+\beta(2-\alpha)}(-\lambda t^\alpha) + a_1 t^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda t^\alpha). \tag{41}$$

Due to the singular behavior of the term $t^{(\alpha-1+\beta(2-\alpha))-1}$ at $t = 0$, we get $a_0 = 0$. Using the second boundary condition of P4, we obtain (39). □

We compute the smallest eigenvalues for FEP P4 from equation (39) and its LILB, SMNLB and CSILB for different α, $\alpha \in (1,2]$ and $\beta = 0,1$ from equations (33), (35) and (37). Notice that according the definition of Hilfer derivative in (7) for $\beta = 0$ and

$\beta = 1$, equation (7) reduces to respectively the classical Reimann-Liouville and Caputo derivatives. Hence, FBVP P3 and FEP P4 give the results for classical Reimann-Liouville and Caputo derivative FBVP as well as FEVP for $\beta = 0$ and $\beta = 1$ respectively. A few results reduce to the the work on LTI for FBVPs in [3] and [4]. Particularly, for $\beta = 0$ and $\beta = 1$ in FBVP P3 and FEP P4, reduce to the results in [3] and [4] respectively. The results are shown in the following tables 1 and 2.

Table 1: *Results for* $\alpha \in (1,2]$ *and* $\beta = 0$ *(FBVP P3 and FEP P4 with Riemann-Liouville derivative)*

LTI	LILB	SMNLB	CSILB
$\int_a^b \lvert q(s)\rvert ds \geqslant \frac{\Gamma(\alpha)4^{\alpha-1}}{(b-a)^{\alpha-1}}$ [3]	$\lambda \geqslant \frac{\Gamma(\alpha)4^{\alpha-1}}{(b-a)^{\alpha}}$ [3]	$\lambda \geqslant \frac{\Gamma(\alpha+1)\alpha^{\alpha}}{(b-a)^{\alpha}(\alpha-1)^{\alpha-1}}$	$\lambda \geqslant \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \cdot \left[\frac{4\alpha-1}{2\alpha(2\alpha-1)^2} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}$; $C_1(\alpha) = \int_0^1 t^{2\alpha-1} {}_2F_1(1-\alpha, 1; \alpha+1; t)dt$

Table 2: *Results for* $\alpha \in (1,2]$ *and* $\beta = 1$ *(FBVP P3 and FEP P4 with Caputo derivative)*

LTI	LILB	SMNLB	CSILB
$\int_a^b \lvert q(s)\rvert ds \geqslant \frac{\Gamma(\alpha)\alpha^{\alpha}}{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}}$ [4]	$\lambda \geqslant \frac{\Gamma(\alpha)\alpha^{\alpha}}{(b-a)^{\alpha}(\alpha-1)^{\alpha-1}}$ [4]	$\lambda \geqslant \frac{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{\alpha-1}}}{(b-a)^{\alpha}(\alpha-1)}$	$\lambda \geqslant \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \cdot \left[\frac{2\alpha+3}{6\alpha(2\alpha-1)} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}$; $C_1(\alpha) = \int_0^1 t^{\alpha+1} {}_2F_1(1-\alpha, 1; \alpha+1; t)dt$

For comparison purpose, we compute the smallest eigenvalues for FEP P4 with $a = 0$ and $b = 1$ for particular values of type $\beta = 0$ and $\beta = 1$ and its LILB, SMNLB and CSILB for different α, $\alpha \in (1,2]$ from tables 1 and 2. The results are shown in figures 1 and 2 respectively. These figures clearly demonstrate that among the three estimates considered here, the LILB provides the worse estimate and the CSILB and SMNLB provide better estimate for the smallest eigenvalues of FEP P4 for $\beta = 0$ and $\beta = 1$. We use MATHEMATICA and MATLAB code to find the smallest eigenvalue of the Mittag-Leffler functions. We note that the MATLAB code was contributed by Podlubny [15], and the algorithm is based on the paper of Gorenflo et al. [6]. By this code we can calculate the Mittag-Leffler function with desired accuracy. Throughout this work we calculate the Mittag-Leffler function with the accuracy 10^{-5}. Setting $\beta = 1$ in equation (39), it reduces to $E_{\alpha,2}(-\lambda) = 0$. We analyzed that $E_{\alpha,2}(z) = 0$ has no real solution for $\alpha = 1.1$ to $\alpha = 1.5991152$. Furthermore, for $\alpha = 1.5991152$, $E_{\alpha,2}(z)$ has no real zeros and an infinite number of complex zeros. Whereas for $\alpha = 1.5991153$, $E_{\alpha,2}(z)$ has two real zeros and an infinite number of complex zeros (see [2], [7]). We note that if $\alpha = 1.5991153$ to $\alpha = 2$, then FEP P4 with $a = 0$, $b = 1$ and $\beta = 1$ has zero solution. For $\alpha = 1.5991153, 1.6, 1.7, 1.8, 1.9, 2$, we calculate the eigenvalues. Which is shown in figure 2.

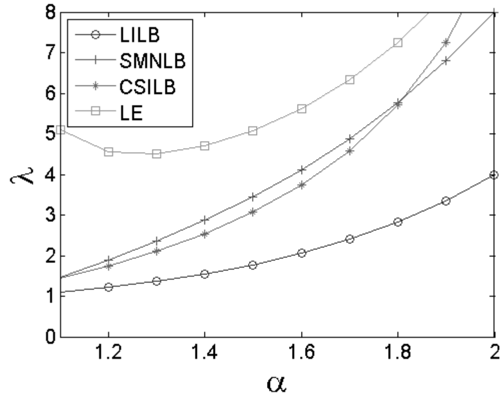

Figure 1: *Comparison of the lower bounds for* λ *obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. (* $-\circ-$*: LILB;* $-+-$ *: SMNLB;* $-*-$*: CSILB;* $-\square-$*: LE - the Lowest Eigenvalue* λ *) (* $a=0,\ b=1,\ \beta=0$ *Riemann-Liouville derivative FEP P4)*

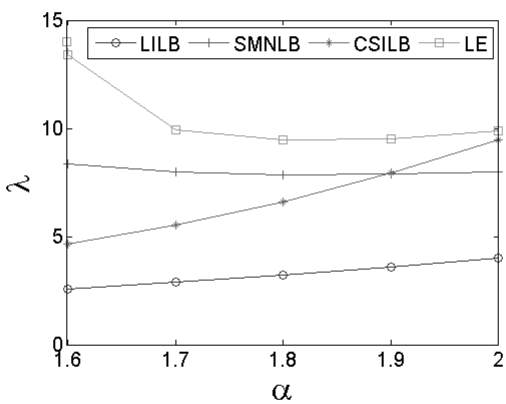

Figure 2: *Comparison of the lower bounds for* λ *obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. (* $-\circ-$*: LILB;* $-+-$ *: SMNLB;* $-*-$*: CSILB;* $-\square-$*: LE - the Lowest Eigenvalue* λ *) (* $a=0, b=1, \beta=1$ *Caputo derivative FEP P4)*

We now consider an application of the lower bounds for the smallest eigenvalues of FEP P4 found in Corollary 1 and Theorem 3. In [3], [4], [10] and [16], the authors have applied the LILB to the FEPs for $\alpha \in (1,2]$ to find the interval in which certain Mittag-Leffler functions have no real zeros. On the other hand, in [14], We applied the improved bounds to obtain these intervals for certain Mittag-Leffler functions for $\alpha \in (2,3]$. We follow a similar procedure, which is discussed in the following theorem.

THEOREM 4. *Let* $1 < \alpha \leqslant 2$ *if* $\beta = 0$, *and* $1.5991153 \leqslant \alpha \leqslant 2$ *if* $\beta \in (0,1]$. *Then based on the LILB, SMNLB and CSILB inequalities, the Mittag-Leffler function* $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ *has no real zeros in the following domains:*

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}, 0 \right], \tag{42}$$

SMNLB inequality:

$$z \in \left(-\frac{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{1-\beta(2-\alpha)}}}{[\alpha-1+\beta(2-\alpha)]^{\frac{\alpha-1+\beta(2-\alpha)}{1-\beta(2-\alpha)}}[1-\beta(2-\alpha)]}, 0 \right], \tag{43}$$

CSILB inequality:

$$z \in \left(-\Gamma(\alpha)\left[\frac{4\alpha-1+2\beta(2-\alpha)}{2\alpha(2\alpha-1)[2\alpha-1+2\beta(2-\alpha)]} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}, 0 \right]. \tag{44}$$

Proof. Let λ be the smallest eigenvalue of the FEP P4, then $z = \lambda$ is the smallest value of z for which $E_{\alpha,\alpha+\beta(2-\alpha)}(-z) = 0$. If there is another z smaller than λ for which $E_{\alpha,\alpha+\beta(2-\alpha)}(-z) = 0$, then it will contradict that λ is the smallest eigenvalue. Therefore, $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ has no real zero for $z \in (-\lambda, 0]$. Now, according to LILB,

$$\lambda \geqslant \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}$$

(see equation (33)). Thus, $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ has no real zero for

$$z \in \left(-\frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}, 0 \right].$$

This proves equation (42). Equations (43) and (44) are proved in a similar fashion. □

From figures 1 and 2, it is clear that among the three inequalities discussed in the paper, LILB provides the smallest interval, and CSILB and SMNLB provide the larger intervals in which the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)}$ has no real zero. Particularly, we discuss two cases, $\beta = 0$ and $\beta = 1$.

3.2. Lyapunov-type inequality for FVBP (3) and (5)

Setting $a_1 = b_2 = 1$, $a_2 = b_1 = 0$ in equation (16) we obtain the FBVP Problem P5:

$$\begin{gathered} \left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad a < t < b \\ y(a) = 0, Dy(b) = 0. \end{gathered} \tag{45}$$

LEMMA 4. *Problem P5 can be written as (17) where* $G(t,s) = \frac{H(t,s)}{\Gamma(\alpha)(b-s)^{2-\alpha}}$ *and* $H(t,s)$ *is given by*

$$H(t,s) = \begin{cases} \frac{(\alpha-1)(t-a)^{1-(2-\alpha)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{1-(2-\alpha)(1-\beta)} - (t-s)^{\alpha-1}(b-s)^{2-\alpha}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{(\alpha-1)(t-a)^{1-(2-\alpha)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{1-(2-\alpha)(1-\beta)}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{46}$$

Proof. Using Lemma 1, we get equation (28) as discussed in Lemma 2. Since, $y(a) = 0$, we obtain $c_1 = 0$. Thus we get

$$y(t) = c_2 \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) y(s) ds.$$

The time derivative of the above equation gives

$$Dy(t) = c_2[1-(2-\alpha)(1-\beta)] \frac{(t-a)^{(\alpha-2)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{\alpha-1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} q(s) y(s) ds.$$

Now $Dy(b) = 0$ gives

$$c_2 = \frac{\Gamma(2-(2-\alpha)(1-\beta))(\alpha-1)(b-a)^{(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)} - \int_a^b (b-s)^{\alpha-2} q(s) y(s) ds.$$

Hence, we get

$$y(t) = \frac{(\alpha-1)(t-a)^{1-(\alpha-2)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{\Gamma(\alpha)[1-(2-\alpha)(1-\beta)]} \int_a^b (b-s)^{\alpha-2} q(s) y(s) ds$$
$$- \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) y(s) ds.$$

This concludes the proof. □

LEMMA 5. *The function* H *defined in Lemma 4 satisfies the following property:*

$$|H(t,s)| \leqslant \frac{b-a}{\alpha-1+\beta(2-\alpha)} \max\left\{\alpha-1, \beta(2-\alpha)\right\},$$

$(t,s) \in [a,b] \times [a,b]$.

Proof. Here $H(t,s)$ is an increasing function of t for $a \leqslant t < s \leqslant b$. For $a \leqslant s < t \leqslant b$ and a fixed $s \in [a,b]$, since, $\left(\frac{b-a}{t-a}\right)^{(2-\alpha)(1-\beta)} < \left(\frac{b-a}{t-a}\right)^{2-\alpha} < \left(\frac{b-s}{t-s}\right)^{2-\alpha}$, we get

$$\frac{\partial H}{\partial t} = (\alpha-1)\left[\left(\frac{b-a}{t-a}\right)^{(2-\alpha)(1-\beta)} - \left(\frac{b-s}{t-s}\right)^{2-\alpha}\right] \leqslant 0.$$

So, in $a \leqslant s \leqslant t \leqslant b$ for a given s, $H(t,s)$ is a decreasing function of $t \in [s,b]$. Hence,

$$\max_{t\in[a,b]} H(t,s) \leqslant \max\{|H(s,s)|,|H(b,s)|\}.$$

After some calculations we obtain

$$|H(b,s)| \leqslant \frac{b-a}{\alpha-1+\beta(2-\alpha)} \max\left\{\alpha-1,\beta(2-\alpha)\right\}$$

and

$$|H(s,s)| \leqslant \frac{(\alpha-1)(s-a)^{1-(2-\alpha)(1-\beta)}}{\alpha-1+\beta(2-\alpha)(b-a)^{(\alpha-2)(1-\beta)}} \leqslant \frac{(\alpha-1)(b-a)}{\alpha-1+\beta(2-\alpha)},$$

which concludes the proof. □

THEOREM 5. *If a nontrivial continuous solution of the FBVP P5 exists, then the LTI is given by*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\Gamma(\alpha)[\alpha-1+\beta(2-\alpha)]}{(b-a)\max\left\{\alpha-1,\beta(2-\alpha)\right\}}. \tag{47}$$

Proof. Substituting $G(t,s)$ from Lemma 4 in equation (19) we get

$$1 \leqslant \frac{1}{\Gamma(\alpha)} \max_{a\leqslant t\leqslant b} \int_a^b (b-s)^{\alpha-2}|H(s,s)q(s)|ds.$$

Now an application of Lemma 5 proves the inequality (47). □

Setting $a_1 = b_2 = 1$, $a_2 = b_1 = 0$ in equation (24) we obtain the FEP Problem P6:

$$\begin{aligned} (D_{a^+}^{\alpha,\beta} y)(t) + \lambda y(t) &= 0, \quad a<t<b \\ y(a) = Dy(b) &= 0. \end{aligned} \tag{48}$$

The eigenvalue estimates for the smallest eigenvalue of FEP P6 can be obtained in the similar way as we discussed in Corollary 1.

COROLLARY 2. *For $\alpha \in (1,2]$ and $\beta \in [0,1]$ the eigenvalue estimates for the smallest eigenvalue of FEP P6 are given by*

1. *the LILB*

$$\lambda \geqslant \frac{\Gamma(\alpha)(\alpha-1)[\alpha-1+\beta(2-\alpha)]}{(b-a)^{\alpha}\max\left\{\alpha-1,\beta(2-\alpha)\right\}} \tag{49}$$

and in particular, for IOEP P6, i.e. $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ this bound is

$$\lambda \geqslant \frac{1}{b-a} \tag{50}$$

2. *the SMNLB*

$$\lambda \geqslant \frac{\Gamma(\alpha+1)[\alpha-1+\beta(2-\alpha)]^{\alpha}}{(b-a)^{\alpha}[2(\alpha-1)^{\alpha-1}(\alpha-1+\beta(2-\alpha))^{\alpha-1}(1-\beta(2-\alpha))]} \tag{51}$$

and in particular, for IOEP P6, SMNLB is

$$\lambda \geqslant \frac{2}{(b-a)^2} \tag{52}$$

3. *and CSILB*

$$\lambda \geqslant \left[\frac{2\alpha(\alpha-1)^2(2\alpha-1)+[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)}{2\alpha(2\alpha-1)[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)} - \frac{2(\alpha-1)C_2(\alpha)}{\alpha[1-(2-\alpha)(1-\beta)]}\right]^{-1/2} \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}, \tag{53}$$

where $C_2(\alpha)=\int_0^1 t^{\alpha-(2-\alpha)(1-\beta)+1}{}_2F_1(2-\alpha,1;\alpha+1;t)dt$, $\alpha>\frac{3}{2}$ *and in particular, for IOEP P6, CSILB is*

$$\lambda \geqslant \frac{\sqrt{6}}{(b-a)^2}. \tag{54}$$

Proof. Setting $q(t)=\lambda$ in equation (47) and evaluating the resulting integral, the first inequality in the first part follows. Substituting the Green's function from equation (46), in (25) and simplifying the result, we obtain the inequality in equation (53). Substituting $\alpha=2$ and $\beta=0$ or $\beta=1$, in inequalities (49) and (53), prove the inequalities (50) and (54) respectively. To prove (2), since the maximum of $\int_a^b |G(t,s)|ds$ occurs at $t=b$ for $s\in[a,t]$. From(46) we get

$$\frac{(b-a)(\alpha-1)}{1-(2-\alpha)(1-\beta)}-(b-s)=0$$

which is satisfied by

$$s=\frac{b\beta(2-\alpha)+a(\alpha-1)}{\alpha-1+\beta(2-\alpha)}.$$

Hence,

$$\max_{t\in[a,b]}\int_a^b |G(t,s)|ds=\int_a^s |G(b,s)|ds+\int_s^b |G(b,s)|ds. \tag{55}$$

Using $G(t,s)$ from (46) in (55) we obtain

$$\max_{t\in[a,b]}\int_a^b |G(t,s)|ds=\frac{2(\alpha-1)^{\alpha-1}-(\alpha-1+\beta(2-\alpha))^{\alpha-1}(1-\beta(2-\alpha))}{(b-a)^{-\alpha}\Gamma(\alpha+1)[\alpha-1+\beta(2-\alpha)]^{\alpha}}. \tag{56}$$

Substituting (56) in (26) completes the proof. □

For the integer order case, i.e. $\alpha = 2$, $a = 0$ and $b = 1$, the LILB, SMNLB and CSILB for the smallest λ of FEP P6 are given as 1, 2 and $\sqrt{6} \simeq 2.4495$, respectively (see equations (50), (52) and (54)) . For $\alpha = 2$, the smallest eigenvalue of FEP P6 with $a = 0$ and $b = 1$ is the root of $\cos(\sqrt{\lambda}) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 2.4674011$. Comparing this λ with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue.

The eigenvalues of the FEP P6 for $\alpha \in (1,2]$ are the roots of the Mittag-Leffler function given in the following theorem.

THEOREM 6. *The FEP P6 for $1 < \alpha \leqslant 2$, $\beta \in [0,1]$, $a = 0$ and $b = 1$ has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$, i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(2-\alpha)-1}(-\lambda) = 0 \tag{57}$$

Proof. The proof is similar to the proof of Theorem 3. □

We compute the smallest eigenvalues for FEP P6 from equation (57) and its LILB, SMNLB and CSILB for different α, $\alpha \in (1,2]$ and $\beta = 0$ and $\beta = 1$ from equations (49), (51) and (53). The results are shown in the following tables 3 and 4.

Table 3: *Results for $\alpha \in (1,2]$ and $\beta = 0$ (FBVP P5 and FEP P6 with Riemann-Liouville derivative)*

LTI	LILB	SMNLB	CSILB
$\int_a^b (b-s)^{\alpha-2}\lvert q(s)\rvert ds$ t $\geqslant \frac{\Gamma(\alpha)}{b-a}$	$\lambda \geqslant \frac{\Gamma(\alpha)(\alpha-1)}{(b-a)^\alpha}$	$\lambda \geqslant \frac{\Gamma(\alpha+1)(\alpha-1)^\alpha}{(b-a)^\alpha}$	$\lambda \geqslant \frac{\Gamma(\alpha)}{(b-a)^\alpha} \cdot \left[\frac{4\alpha-3}{2\alpha(2\alpha-1)(2\alpha-3)} - \frac{2C_2(\alpha)}{\alpha}\right]^{-1/2}$; $\alpha > \frac{3}{2}$, $C_2(\alpha) = \int_0^1 t^{2\alpha-1} {}_2F_1(2-\alpha,1;\alpha+1;t)dt$

Table 4: *Results for $\alpha \in (1,2]$ and $\beta = 1$ (FBVP P5 and FEP P6 with Caputo derivative)*

LTI	LILB	SMNLB	CSILB
$\int_a^b (b-s)^{\alpha-2}\lvert q(s)\rvert ds \geqslant \frac{\Gamma(\alpha)}{(b-a)} \cdot \frac{1}{\max\{\alpha-1,2-\alpha\}}$ [10]	$\lambda \geqslant \frac{\Gamma(\alpha)}{(b-a)^\alpha} \cdot \frac{\alpha-1}{\max\{\alpha-1,2-\alpha\}}$ [10]	$\lambda \geqslant \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \cdot \frac{(\alpha-1)^{-1}}{2(\alpha-1)^{\alpha-2}-1}$	$\lambda \geqslant \frac{\Gamma(\alpha)}{(b-a)^\alpha} \cdot \left[\frac{2(\alpha-1)^2\alpha(2\alpha-1)-3(2\alpha-3)}{6\alpha(2\alpha-1)(2\alpha-3)} - \frac{2(\alpha-1)C_2(\alpha)}{\alpha}\right]^{\frac{-1}{2}}$; $\alpha > \frac{3}{2}$, $C_2(\alpha) = \int_0^1 t^{\alpha+1} {}_2F_1(2-\alpha,1;\alpha+1;t)dt$

We compute the smallest eigenvalues for FEP P6 with $a=0$ and $b=1$ for particular values of type $\beta=0$ and $\beta=1$, and its LILB, SMNLB and CSILB for different α, $\alpha\in(1,2]$ from tables 3 and 4. The results are shown in figures 3 and 4 respectively. We note that a few results for the particular case $\beta=1$ in FBVP P5 and FEP P6, reduce to the results in [10] (page 447, 449). From the figures it is clear that the CSILB and SMNLB provide better estimate for the smallest eigenvalues than LILB of FEP P6 for $\beta=0,1$. We notice that in figure 3, the CSILB is valid for $\alpha\in(1.5,2]$.

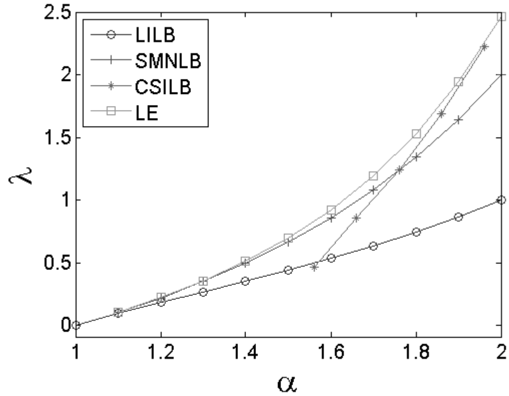

Figure 3: *Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($a=0$, $b=1$, $\beta=0$, Riemann-Liouville derivative FEP P6)*

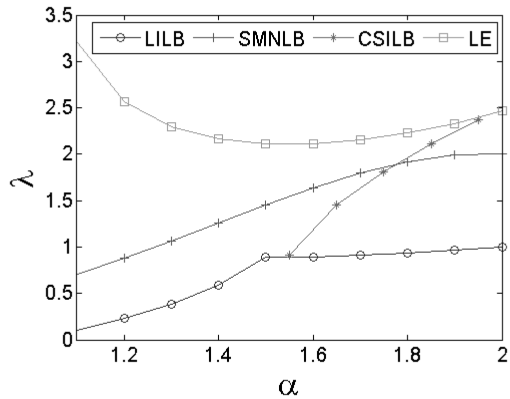

Figure 4: *Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($a=0$, $b=1$, $\beta=1$, Caputo derivative FEP P6)*

We apply the lower bounds for the smallest eigenvalues of FEP P6 with $a=0$ and $b=1$ found in Corollary 2 and Theorem 6 for $\alpha\in(1,2]$ to find the interval in which the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zeros. The proof is similar to the proof of Theorem 4, we omit it.

THEOREM 7. *Let $1<\alpha\leqslant 2$. The Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zeros in the following domains:*
LILB inequality:

$$z\in\left(-\frac{\Gamma(\alpha)(\alpha-1)[\alpha-1+\beta(2-\alpha)]}{\max\left\{\alpha-1,\beta(2-\alpha)\right\}},0\right], \tag{58}$$

SMNLB inequality:

$$z\in\Big(-\frac{\Gamma(\alpha+1)[\alpha-1+\beta(2-\alpha)]^{\alpha}}{[2(\alpha-1)^{\alpha-1}-(\alpha-1+\beta(2-\alpha))^{\alpha-1}(1-\beta(2-\alpha))]},0\Big], \tag{59}$$

CSILB inequality:

$$z\in\Bigg(-\Bigg[\frac{2\alpha(\alpha-1)^2(2\alpha-1)+[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)}{2\alpha(2\alpha-1)[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)} - \frac{2(\alpha-1)C_2(\alpha)}{\alpha[1-(2-\alpha)(1-\beta)]}\Bigg]^{-1/2}\Gamma(\alpha),0\Bigg]. \tag{60}$$

Proof. The proof is similar to the proof of Theorem 4. □

From figures 3 and 4, it is clear that among the three inequalities, LILB provides the smallest interval, and CSILB and SMNLB provide the larger intervals in which the Mittag-Leffler functions $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ for $\beta=0$ and $\beta=1$, have no real zero.

4. Conclusion

In this paper, we established a Lyapunov-type inequality for a fractional boundary value problem with Hilfer derivative of order α for $\alpha\in(1,2]$ and type $\beta\in[0,1]$. We considered the Dirichlet, and a mixed set of Dirichlet and Neumann boundary conditions. For the integer and fractional eigenvalue problems, we determined lower bounds for the first eigenvalue from Semi Maximum Norm and Lyapunov-type and Cauchy-Schwarz type inequalities. We showed that for FEPs P4 and P6 with $a=0$ and $b=1$, the Semi maximum and Cauchy-Schwarz type inequalities provide better estimate for the smallest eigenvalue than the Lyapunov type inequality. We used these bounds for the smallest eigenvalue to find the domain in which certain Mittag-Leffler functions have no zero. Results showed that the Semi maximum norm and Cauchy-Schwarz type inequalities provide the largest domain in which Mittag-Leffler functions have no zero.

Acknowledgement. The author is very grateful to the anonymous referees for their valuable suggestions.

REFERENCES

[1] R. C. BROWN AND D. B. HINTON, *Lyapunov inequalities and their applications, in Survey on classical inequalities*, Math. Appl. **517**, Kluwer Acad. Publ. Dordrecht. (2000), 1–25.

[2] J. S. DUAN, Z. WANGC AND Y. L. LIU, X. QIU, *Eigenvalue problems for fractional ordinary differential equations*, Chaos Solutions Fract. **46** (2013), 46–53.

[3] R. A. C. FERREIRA, *A Lyapunov-type inequality for a fractional boundary value problem*, Fract. Calc., Appl. Anal. **16**, 4 (2013), 978–984.

[4] R. A. C. FERREIRA, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl. **412**, 2 (2014), 1058–1063.

[5] R. A. C. FERREIRA, *Lyapunov-type inequalities for some sequential fractional boundary value problems*, Adv. Dyn. Syst. Appl. **11**, 1 (2016), 33–43.

[6] R. GORENFLO, J. LOUTCHKO AND Y. LUCHKO, *Computation of the MittagLeffler function $E_{a;b}(z)$ and its derivative*, Fract Calc Appl Anal. **5** (2002), 491–518.

[7] JOHN W. HANNEKEN, B. N. NARAHARI ACHAR AND DAVID M. VAUGHT, *An Alpha-Beta Phase Diagram Representation of the Zeros and Properties of the Mittag-Leffler Function*, Hindavi Publl. corporation, Advances in Mathematical Physics, Article ID 421685 (2013), 13 pages.

[8] H. J. HAUBOLD, A. M. MATHAI AND R.K. SAXENA, *Mittag-Leffler functions and their applications*, J. Appl. Math., Article ID 421685 (2013), 13 pages.

[9] R. HILFER, Y. LUCHKO AND Ž. TOMOVSKI, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **12**, 3 (2009), 299–318.

[10] M. JLELI AND B. SAMET, *Lyapunov-type inequatlities for a fractional differential equation with mixed boundary conditions*, Mathemaical Inequalities and applications **18**, 2 (2015), 443–451.

[11] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[12] A. M. LYAPUNOV, *Probleme général de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse. **2**, 9 (1907), 203–474.

[13] B. G. PACHPATTE, *Mathematical Inequalities*, North Holland Mathematical Library **1** (2005).

[14] N. PATHAK, *Lyapunov-type inequality and eigenvalue estimate for fractional problems of order $\alpha, \alpha \in (2,3]$*, Mathematical inequalities and applications, (to be appear) (2016).

[15] I. PODLUBNY, *Mittag-Leffler function: Calculates the Mittag-Leffler function with desired accuracy (File ID: No. 8738, mlf.m)*, MATLAB Central/File Exchange, 2005.

[16] JI. RONG AND C. BAI, *Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions*, Advances in Difference equations **82** (2015).

[17] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional integrals and derivatives, translated from the 1987 Russian original*, Gordon and Breach, Yverdon.

[18] Ž. TOMOVSKI AND R. HILFER, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions*, Integral Transforms and Special Functions **21**, 11 (2010), 797–814.

[19] Ž. TOMOVSKI, *Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator*, Nonlinear Analysis, DOI: 10.1016/j.na.2011.12.034, (2012).

(Received June 10, 2016)

Nimisha Pathak
Department of Mathematics
Southern Illinois University
Carbondale, IL-62901, USA
e-mail: nimishasp@siu.edu