Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 329–344 doi:10.7153/mia-2018-21-24

POLYNOMIAL AND MULTILINEAR HARDY—LITTLEWOOD INEQUALITIES: ANALYTICAL AND NUMERICAL APPROACHES

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(*Communicated by C. P. Niculescu*)

Abstract. We investigate the constants of the polynomial and multilinear Hardy–Littlewood inequalities. Among other results, we show that a simple application of the best known constants of the Clarkson inequality improves a recent result of Araújo et al. In a final section, as an independent appendix, we present some computer-aided estimates for the lower bounds of the multilinear Hardy–Littlewood inequalities.

1. Introduction

Let $\mathbb{K}$ be the real or complex scalar field, and $m \geqslant 1$ be a positive integer. In 1930 Littlewood proved his well-known $4/3$ inequality to solve a problem posed by P.J. Daniell (see [20]). The Littlewood's $4/3$ inequality asserts that

$$\left(\sum_{i,j=1}^{\infty} \left| T(e_i, e_j) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leqslant \sqrt{2} \|T\|$$

for every continuous bilinear form $T : c_0 \times c_0 \to \mathbb{K}$, where

$$\|T\| := \sup_{z^{(1)}, z^{(2)} \in B_{c_0}} |T(z^{(1)}, z^{(2)})|.$$

The exponent $4/3$ is optimal and in the case $\mathbb{K} = \mathbb{R}$ the optimality of the constant $\sqrt{2}$ is also known (see [17]). Soon afterwards this inequality was generalized by Hardy and Littlewood ([19], 1934) for bilinear forms on ℓ_p and, in 1982 Praciano-Pereira ([32]) extended the result of Hardy and Littlewood to m-linear forms on ℓ_p.

The Hardy–Littlewood inequalities for m-linear forms is the following result:

Mathematics subject classification (2010): 46G25, 47L22, 47H60.

Keywords and phrases: Absolutely summing operators, Hardy–Littlewood inequality, Bohnenblust–Hille inequality.

J. R. Campos was supported by a Capes Postdoctoral scholarship. V. V. Fávaro was supported by FAPEMIG Grant APQ-03181-16 and CNPq Grant 307517/2014-4. W. Cavalcante was supported by Capes. D. M. Serrano-Rodríguez was supported by Capes Grant 000786/2015-02. D. Pellegrino was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq – Brasil, Grant 302834/2013-3.

THEOREM 1.1. (Multilinear Hardy–Littlewood/Praciano-Pereira) *Let $m \geqslant 2$ be a positive integer. For $p \geqslant 2m$, there is a constant $C_{\mathbb{K},m,p} \geqslant 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} |T(e_{i_1},\ldots,e_{i_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant C_{\mathbb{K},m,p}\,\|T\|,$$

for all m-linear forms $T:\ell_p\times\cdots\times\ell_p\to\mathbb{K}$.

The exponent $\frac{2mp}{mp+p-2m}$ is optimal and $\|T\| := \sup_{z^{(1)},\ldots,z^{(m)}\in B_{\ell_p}} |T(z^{(1)},\ldots,z^{(m)})|$. In the limiting case ($p=\infty$, considering, of course $f(\infty) := \lim_{p\to\infty} f(p)$ regardless of the function f), we recover the classical multilinear Bohnenblust–Hille inequality (see [9]). For another extensions of Hardy-Littlewood inequalities to m-linear forms see [6, 16].

The original upper estimate for $C_{\mathbb{K},m,p}$ is $2^{\frac{m-1}{2}}$. Recently, in some papers (see [4] and [5]), this estimate was improved for all m and p with the only exception of the case $C_{\mathbb{R},m,2m}$. The precise behavior of the growth of the optimal constants $C_{\mathbb{K},m,p}$ is still unknown (some partial results can be found in [3, 4, 5]). Up to now, the best known lower estimates for $C_{\mathbb{R},m,p}$ are always smaller than 2 and again the more critical situation is when $p=2m$, where the lower estimates presented in [3] are more difficult to obtain and not explicitly stated for the case $p=2m$.

As a consequence of the above inequality we have the Hardy–Littlewood inequalities for m-homogeneous polynomials:

THEOREM 1.2. (Polynomial Hardy–Littlewood inequality) *Let $m \geqslant 2$ be a positive integer. For $p \geqslant 2m$, there is a constant $D_{\mathbb{K},m,p} \geqslant 1$ such that*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant D_{\mathbb{K},m,p}\,\|P\|,$$

for all positive integers n and all m-homogeneous polynomials $P:\ell_p^n\to\mathbb{K}$ given by

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha.$$

Again, the exponent $\frac{2mp}{mp+p-2m}$ is optimal and $\|P\| := \sup\{|P(x)| : \|x\|=1\}$.

In recent works (see [2, 15, 25]), have appeared estimates for the constants in the polynomial case. However, as in the multilinear case, the optimal constants $D_{\mathbb{K},m,p}$, and the precise behavior of the growth of this constants are still unknown.

The aim of the present paper is to improve the estimates for the constants $C_{\mathbb{K},m,p}$ and $D_{\mathbb{K},m,p}$, and it's organized as follows: In Section 2 we improve previous results of Araujo et al. by using the Clarkson's inequality. More precisely, we relate the Clarkson inequality to the task of obtaining lower estimates for the constants of the multilinear Hardy–Littlewood inequality, and using the optimal constants of the Clarkson inequality we present a new closed formula for the lower estimates of the Hardy–Littlewood inequality, improving results from [3]. In Section 3 we investigate the polynomial Hardy–Littlewood inequality. Finally, in the last section, which can be regarded as an Appendix, we investigate the case $p = 2m$ using a computer-aided approach. Our approach has two novelties: a new class of multilinear forms, not investigated before in similar context, and a new numerical approach in this framework. As it will be clear along the paper the new family of multilinear forms introduced in this paper is more effective to obtain good lower estimates for the Hardy–Littlewood inequality.

The approaches of Section 3 is entirely analytic and do not depend on computation assistance.

2. The multilinear Hardy–Littlewood inequality

From now on, if $p \in [1,\infty)$, p^* is the extended real number such that $\frac{1}{p} + \frac{1}{p^*} = 1$. Also, E' denotes the topological dual of a Banach space E. By $\mathcal{L}(^mE;F)$ we denote the Banach space of all (bounded) m-linear operators $U : E \times \cdots \times E \to F$, with E, F Banach spaces over $\mathbb{K}$. For $1 \leqslant s \leqslant r < \infty$, $U \in \mathcal{L}(^mE;F)$ is called *multiple* (r,s)-*summing* if there exists a constant $C > 0$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \|U(x_{i_1},\ldots,x_{i_m})\|_F^r\right)^{\frac{1}{r}} \leqslant C\|U\| \prod_{k=1}^{m} \left\| (x_{i_k})_{i_k=1}^{n} \right\|_{w,s}$$

for all finite choice of vectors $x_{i_k} \in E$, $1 \leqslant i_k \leqslant n$, $1 \leqslant k \leqslant m$, where

$$\| (x_i)_{i=1}^n \|_{w,s} := \sup_{\|\varphi\|_{E'} \leqslant 1} \left(\sum_{i=1}^{n} |\varphi(x_i)|^s\right)^{\frac{1}{s}}.$$

The vector space of all multiple (r,s)-summing operators in $\mathcal{L}(^mE;F)$ is denoted by $\Pi_{(r,s)}(^mE;F)$. For more details of the theory of multiple summing operators theory see [22, 28, 29].

In the terminology of the multiple summing operators, it is well known (see, for instance, [1]) that the Hardy–Littlewood/Praciano-Pereira inequality is equivalent to the equality

$$\Pi_{\left(\frac{2mp}{mp+p-2m};p^*\right)}(^mE;\mathbb{K}) = \mathcal{L}(^mE;\mathbb{K}).$$

In other words, if $m \geqslant 2$ and $p \geqslant 2m$, then there is a constant $C_{\mathbb{K},m,p} \geqslant 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} |T(x_{i_1},\ldots,x_{i_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant C_{\mathbb{K},m,p}\|T\| \prod_{k=1}^{m} \left\| (x_{i_k})_{i_k=1}^{n} \right\|_{w,p^*}$$

for all m-linear forms $T : E \times \cdots \times E \to \mathbb{K}$, for all finite choice of vectors $x_{i_k} \in E$, $1 \leqslant i_k \leqslant n$, $1 \leqslant k \leqslant m$. For more coincidence results for multiple summing multilinear operators see [10, 30, 31].

As mentioned in the introduction, the case $p = 2m$ in the Hardy–Littlewood inequality is specially interesting. In this case we have very few information on the constants involved, and moreover, this case is a kind of dual version of the Bohnenblust–Hille inequality, in the sense that in the pair of parameters $\left(\frac{2mp}{mp+p-2m}; p^*\right)$, each case has a coordinate which is kept constant (in reverse location). More specifically, in the terminology of the multiple summing operators, the Bohnenblust–Hille inequality asserts that

$$\Pi_{\left(\frac{2m}{m+1};1\right)}(^m E;\mathbb{K}) = \mathscr{L}(^m E;\mathbb{K})$$

for all Banach spaces E. On the other hand, when $p = 2m$, the Hardy–Littlewood inequality is equivalent to

$$\Pi_{\left(2;\frac{2m}{2m-1}\right)}(^m E;\mathbb{K}) = \mathscr{L}(^m E;\mathbb{K})$$

for all Banach spaces E.

Up to now the best known upper estimates for the constants $\left(C_{\mathbb{R},m,p}\right)_{m=1}^{\infty}$ can be found in [5, page 1887] and [4]. The updated results on the lower bounds for these constants are:

- $C_{\mathbb{R},m,p} \geqslant 2^{\frac{mp+2m-2m^2-p}{mp}}$ for $p > 2m$ and $C_{\mathbb{R},m,p} > 1$ for $p = 2m$ (see [3]);

Of course, the search of optimal lower estimates for the multilinear Hardy– Littlewood (here, for real scalars), shall be done by choosing suitable operators $T : \ell_p \times \cdots \times \ell_p \to \mathbb{R}$ such that the quotient

$$\frac{\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{\|T\|}$$

is maximized. However, as it will be clear in this paper, we shall work with $m-$linear forms in finitely many variables: $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{R}$. The reader may wonder why it is relevant to work with finitely many variables in a problem of infinite-dimensional nature. One of the answers is the intrinsic difficulty of the problem, since the estimate of $\|T\| := \sup_{z^{(1)},\ldots,z^{(m)} \in B_{\ell_p}} |T(z^{(1)},\ldots,z^{(m)})|$ seems to be a quite hardwork for most of the operators $T : \ell_p \times \cdots \times \ell_p \to \mathbb{R}$. Another reason, and maybe the more important, is that it has been shown in previous works that just by working in finitely many variables sometimes we can achieve the optimal estimates (see, for instance [17, 26, 27]).

In this section we find an overlooked (and simple) connection between the Clarkson's inequalities and the Hardy–Littlewood's constants which helps to find analytical lower estimates (without the use of a computational aid) for these constants.

THEOREM 2.1. *Let $m \geqslant 2$ and $p \geqslant 2m$. The optimal constants of the Hardy-Littlewood inequalities satisfies*

$$C_{\mathbb{R},m,p} \geqslant \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{\sup_{x\in[0,1]} \frac{\left((1+x)^{p^*}+(1-x)^{p^*}\right)^{\frac{1}{p^*}}}{(1+x^p)^{1/p}}}.$$

Proof. For a given Banach space E we know that $\Psi : \mathscr{L}\left({}^2E;\mathbb{R}\right) \to \mathscr{L}\left(E;E^*\right)$ given by $\Psi(T)(x)(y) = T(x,y)$ is an isometric isomorphism. For $E = \ell_p^2$ and using the characterization of the dual of ℓ_p^2, we conclude that for the bilinear form

$$\begin{array}{rccl} T_{2,p}: & \ell_p^2 \times \ell_p^2 & \to & \mathbb{R} \\ & ((x_i^{(1)}),(x_i^{(2)})) & \mapsto & x_1^{(1)}x_1^{(2)} + x_1^{(1)}x_2^{(2)} + x_2^{(1)}x_1^{(2)} - x_2^{(1)}x_2^{(2)}, \end{array}$$

we have

$$\begin{array}{rccl} \Psi(T_{2,p}): & \ell_p^2 & \to & \ell_{p^*}^2 \\ & (x_i) & \mapsto & (x_1+x_2, x_1-x_2). \end{array}$$

Since $p \geqslant 2m$ and $m \geqslant 2$, using the best constants from the Clarkson's inequality in the real case (see [21, Theorem 2.1]) we know the norm of the linear operator $\Psi(T_{2,p})$ (and consequently the norm of the bilinear form $T_{2,p}$), i.e.,

$$\|T_{2,p}\| = \left\|\Psi(T_{2,p})\right\| = \sup_{x\in[0,1]} \frac{((1+x)^{p^*} + (1-x)^{p^*})^{\frac{1}{p^*}}}{(1+x^p)^{1/p}}.$$

Now, as in [3], we define inductively

$$\begin{array}{rcl} T_{m,p}: \ell_p^{2^{m-1}} \times \cdots \times \ell_p^{2^{m-1}} & \to & \mathbb{R} \\ (x^{(1)},\ldots,x^{(m)}) & \mapsto & (x_1^{(m)} + x_2^{(m)})T_{m-1,p}(x^{(1)},\ldots,x^{(m)}) \\ & & +(x_1^{(m)} - x_2^{(m)})T_{m-1,p}(B^{2^{m-1}}(x^{(1)}),\ldots,B^2(x^{(m-1)})), \end{array}$$

where $x^{(k)} = (x_j^{(k)})_{j=1}^{2^{m-1}} \in \ell_p^{2^{m-1}}$, $1 \leqslant k \leqslant m$, and B is the backward shift operator in $\ell_p^{2^{m-1}}$ and, again as in [3], we conclude that

$$\begin{aligned} |T_{m,p}(x^{(1)},\ldots,x^{(m)})| &\leqslant |x_1^{(m)} + x_2^{(m)}||T_{m-1,p}(x^{(1)},\ldots,x^{(m)})| \\ &\quad + |x_1^{(m)} - x_2^{(m)}||T_{m-1,p}(B^{2^{m-1}}(x^{(1)}),B^{2^{m-2}}(x^{(2)}),\ldots,B^2(x^{(m-1)}))| \\ &\leqslant \|T_{m-1,p}\|(|x_1^{(m)} + x_2^{(m)}| + |x_1^{(m)} - x_2^{(m)}|) \\ &\leqslant 2\|T_{m-1,p}\|\|x^{(m)}\|_p, \end{aligned}$$

i.e.,

$$\|T_{m,p}\| \leqslant 2^{m-2}\|T_{2,p}\|.$$

Now we have

$$(4^{m-1})^{\frac{mp+p-2m}{2mp}} = \left(\sum_{j_1,\ldots,j_m=1}^{2^{m-1}} \left|T_{m,p}(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant C_{\mathbb{R},m,p}2^{m-2}\|T_{2,p}\| \tag{2.1}$$

and thus

$$C_{\mathbb{R},m,p} \geqslant \frac{(4^{m-1})^{\frac{mp+p-2m}{2mp}}}{2^{m-2}\|T_{2,p}\|} = \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{\sup_{x\in[0,1]} \frac{\left((1+x)^{p^*}+(1-x)^{p^*}\right)^{1/p^*}}{(1+x^p)^{1/p}}}. \quad \square$$

When $m=2,$ using estimates of [21, page 1369], note that

$$C_{\mathbb{R},2,4} \geqslant \frac{2}{\sqrt{3}} > 1.1546$$

$$C_{\mathbb{R},2,8} \geqslant \frac{2^{\frac{5}{4}}}{1.892} > 1.2570$$

$$C_{\mathbb{R},2,p} \geqslant \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{1.9836} > 1.3591 \quad \text{for} \quad p = 1+\log_{9/10} 1/19$$

$$C_{\mathbb{R},2,p} \geqslant \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{1.9999} > 1.4105 \quad \text{for} \quad p = 1+\log_{99/100} 1/199.$$

Using the old estimates of [3] for $p > 2m$ (i.e., $C_{\mathbb{R},m,p} \geqslant 2^{\frac{mp+2m-2m^2-p}{mp}}$) we can easily verify that the old estimates are worse. Also, in the old estimates we have no closed formula for the case $p = 2m$.

REMARK 2.2. One may try to use the complex Clarkson's inequalities to obtain nontrivial lower bounds for the constants of the complex Hardy-Littlewood inequality. But, this is not effective, since we just get trivial lower bounds, i.e., 1.

REMARK 2.3. (The case $m < p < 2m$) There is also a version of Hardy– Littlewood's inequality for $m < p < 2m$, due to Dimant and Sevilla-Peris ([16] and the forthcoming Section 6). In this case, the optimal exponent is $\frac{p}{p-m}$ and we still denote the optimal constant for this inequality by $C_{\mathbb{K},m,p}$. The best information we have so far for the lower estimates for the constant $C_{\mathbb{R},m,p}$ are trivial, that is,

$$1 \leqslant C_{\mathbb{R},m,p} \leqslant (\sqrt{2})^{m-1}.$$

Similarly to the argument used in the proof of the Theorem 2.1, we can also provide a closed formula (which depends on p) for the lower bounds of $C_{\mathbb{R},m,p}$, but in this case, we do not always have nontrivial information. More precisely, we prove that

$$C_{\mathbb{R},m,p} \geqslant \frac{2^{\frac{mp+2m-2m^2}{p}}}{\sup_{x\in[0,1]} \frac{((1+x)^{p^*}+(1-x)^{p^*})^{\frac{1}{p^*}}}{(1+x^p)^{\frac{1}{p}}}}.$$

It is important to mention this case because, for suitable choices of p, we get nontrivial lower estimates for $C_{\mathbb{R},m,p}$. For instance,

$$C_{\mathbb{R},2,7/2} \geqslant 1.104, \quad C_{\mathbb{R},3,28/5} \geqslant 1.025, \quad \text{and} \quad C_{\mathbb{R},100,199999/1000} \geqslant 1.003.$$

This leads us to question the following: Would also be the optimal constants of the Hardy–Littlewood inequality for $m < p < 2m$ strictly greater than 1?

3. The polynomial Hardy–Littlewood inequality

Let E be a real or complex Banach space and m be a positive integer and let $\mathbb{K}$ be the real or complex scalar field. A map $P : E \to \mathbb{K}$ is a homogeneous polynomial on E of degree m if there exists a symmetric m-linear form L on E^m such that $P(x) = L(x, \ldots, x)$ for all $x \in E$. We denote by $\mathscr{P}(^m E)$ the space of continuous m-homogeneous polynomials on E endowed with the usual norm

$$\|P\| := \sup\{|P(x)| : \|x\| = 1\}.$$

Observe that an m-homogeneous polynomial in $\mathbb{K}^n$ can be written as

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We denote

$$|P|_p := \left(\sum_{|\alpha|=m} |a_\alpha|^p \right)^{1/p}$$

and

$$|P|_\infty := \max |a_\alpha|.$$

The polynomial Hardy–Littlewood inequality is:

THEOREM 3.1. (Polynomial Hardy–Littlewood inequality) *For $m < p \leqslant \infty$ there is a constant $D_{\mathbb{K},m,p} \geqslant 1$ such that*

$$\begin{aligned} \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} &\leqslant D_{\mathbb{K},m,p} \|P\|, \quad \text{if } p \geqslant 2m \\ \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} &\leqslant D_{\mathbb{K},m,p} \|P\|, \quad \text{if } m < p \leqslant 2m, \end{aligned} \tag{3.1}$$

for all positive integers n and all m-homogeneous polynomials $P : \ell_p^n \to \mathbb{K}$ given by

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha.$$

This is a consequence of the multilinear Hardy–Littlewood inequality, previously described, and the following inequality also known as Hardy–Littlewood inequality [16]:

THEOREM 3.2. (Hardy–Littlewood/Dimant–Sevilla-Peris) *For $m < p \leqslant 2m$, there is a constant $C_{\mathbb{K},m,p} \geqslant 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} |T(e_{i_1},\ldots,e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leqslant C_{\mathbb{K},m,p} \|T\|$$

for all positive integers n and all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$.

Above, the exponent $\frac{p}{p-m}$ is optimal and therefore in (3.1) both exponents $\frac{p}{p-m}$ and $\frac{2mp}{mp+p-2m}$ are optimal. The case $p = \infty$ in the appropriate inequality of (3.1), is the classical polynomial Bohnenblust–Hille inequality (see [9]).

From now on $D_{\mathbb{K},m,p}$ denotes the optimal constants satisfying (3.1). As in the multilinear case, the precise behavior of the growth of the constants $D_{\mathbb{K},m,p}$ is still unknown (partial results can be found in [2, 15, 25]). For instance, in [2, Theorem 3.1] it is proved that for $p \geqslant 2m$ we have

$$D_{\mathbb{R},m,p} \geqslant \left(\sqrt[16]{2} \right)^m .$$

When $p = \infty$ we know that (see [8, 14])

$$\limsup_m D_{\mathbb{R},m,\infty}^{1/m} = 2;$$

$$\limsup_m D_{\mathbb{C},m,\infty}^{1/m} = 1.$$

It will be convenient to define $H_1 = \{(p,m) \in \mathbb{R} \times \mathbb{N} : m < p < 2m\}$ and $H_2 = \{(p,m) \in \mathbb{R} \times \mathbb{N} : p \geqslant 2m\}$ with any total order. The main results of this section are the following:

LEMMA 3.3. *Let $j = 1,2$. Then*

$$\limsup_{H_j} D_{\mathbb{R},m,p}^{1/m} \geqslant 2.$$

Proof. Consider the sequence of norm-one j-homogeneous polynomials $Q_j : \ell_p \to \mathbb{R}$ defined recursively by

$$Q_2(x_1,x_2) = x_1^2 - x_2^2,$$
$$Q_{2^m}(x_1,\ldots,x_{2^m}) = Q_{2^{m-1}}(x_1,\ldots,x_{2^{m-1}})^2 - Q_{2^{m-1}}(x_{2^{m-1}+1},\ldots,x_{2^m})^2.$$

From the proof of [14, Theorem 3.1], we known that

$$|Q_{2^m}^n|_\infty \geqslant \left(\frac{2^n}{n+1} \right)^{2^m-1} \tag{3.2}$$

for every natural number n,m. Next, since for every homogeneous polynomial P we obviously have

$$|P|_p \geqslant |P|_\infty,$$

from (3.2) we conclude that

$$D_{\mathbb{R},n2^m,p} \geqslant \left(\frac{2^n}{n+1}\right)^{2^m-1}.$$

Note that

$$D_{\mathbb{R},n2^m,p}^{1/n2^m} \geqslant \left(\left(\frac{2^n}{n+1}\right)^{2^m-1}\right)^{\frac{1}{n2^m}} = \left(\frac{2^n}{n+1}\right)^{\frac{2^m-1}{n2^m}}$$

and making $m \to \infty$ we have

$$\left(\frac{2^n}{n+1}\right)^{\frac{2^m-1}{n2^m}} \to \frac{2}{(n+1)^{1/n}}$$

and now making $n \to \infty$ we have

$$\frac{2}{(n+1)^{1/n}} \to 2. \quad \square$$

From now on we write

$$\rho(p,m) = \frac{p}{p-m} \quad \text{if} \quad m < p \leqslant 2m,$$
$$\rho(p,m) = \frac{2mp}{mp+p-2m} \quad \text{if} \quad p \geqslant 2m.$$

Now we prove the theorem:

THEOREM 3.4. *Let $j = 1,2$. At least one of the following two sentences hold true:*

(a) $\limsup_{H_j} D_{\mathbb{R},m,p}^{1/m} = 2.$

(b) $\limsup_{H_j} D_{\mathbb{C},m,p}^{1/m} > 1.$

Proof. Suppose that (a) is not true for some j. So (using the previous result) we would have $\limsup_{H_j} D_{\mathbb{R},m,p}^{1/m} > (2+\varepsilon) > 2$. Therefore, for each $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$, $(p_k, m_k) \in H_j$ and a m_k-homogeneous polynomial $P_{m_k} : \ell_{p_k}^{n_k} \to \mathbb{R}$ such that

$$\left(\sum_{|\alpha|=m_k} |a_\alpha|^{\rho(p_k,m_k)}\right)^{\frac{1}{\rho(p_k,m_k)}} \leqslant D_{\mathbb{R},m_k,p_k} \left\|P_{m_k}\right\|,$$

with

$$D_{\mathbb{R},m_k,p_k} > (2+\varepsilon)^{m_k}.$$

Considering the complexification of P_{m_k} we know that

$$\left\|(P_{m_k})_{\mathbb{C}}\right\| \leqslant 2^{m_k-1} \left\|P_{m_k}\right\|$$

and now looking for the complex polynomials $(P_{m_k})_{\mathbb{C}}$ we would have

$$\left(\sum_{|\alpha|=m_k} |a_\alpha|^{\rho(p_k,m_k)}\right)^{\frac{1}{\rho(p_k,m_k)}} \leqslant D_{\mathbb{C},m_k,p_k} \left\| (P_{m_k})_{\mathbb{C}} \right\| \leqslant D_{\mathbb{C},m_k,p_k} 2^{m_k-1} \left\| P_{m_k} \right\|$$

and thus

$$D_{\mathbb{R},m_k,p_k} \leqslant D_{\mathbb{C},m_k,p_k} 2^{m_k-1},$$

i.e.,

$$D_{\mathbb{R},m_k,p_k}^{1/m_k} \leqslant D_{\mathbb{C},m_k,p_k}^{1/m_k} 2^{\frac{m_k-1}{m_k}} \leqslant 2D_{\mathbb{C},m_k,p_k}^{1/m_k}.$$

Now, since

$$D_{\mathbb{R},m_k,p_k}^{1/m_k} > 2+\varepsilon$$

we conclude that

$$D_{\mathbb{C},m_k,p_k}^{1/m_k} > 1+\frac{\varepsilon}{2} > 1$$

for all k, and thus

$$\limsup_{H_j} D_{\mathbb{C},m,p}^{1/m} > 1.$$

Reciprocally, if (b) is not true for some j, then $\limsup_{H_j} D_{\mathbb{C},m,p}^{1/m} = 1$ and thus

$$\limsup_{H_j} D_{\mathbb{R},m,p}^{1/m} \leqslant 2$$

and from the previous lemma we conclude that

$$\limsup_{H_j} D_{\mathbb{R},m,p}^{1/m} = 2. \quad \square$$

A. Appendix

A.1. Numerical estimates

In this section we use a computer-aided approach to obtain new lower bounds for the Hardy–Littlewood inequality for real scalars. Computer-aided arguments are essential in some parts of modern mathematics. For instance, some significant advances related to the Grothendieck constant are based in these arguments (see [11, 24]). In the case of Hardy–Littlewood inequality for real scalars, when p is taken to be infinity (i.e., the Bohnenblust–Hille inequality), estimating the constants involved is crucial for applications (see, for instance [23], for applications in Quantum Information Theory).

A.1.1. First estimates (using well-known multilinear forms)

Since the publication of [17], the family of m-linear forms $T_m : \ell_\infty \times \cdots \times \ell_\infty \to \mathbb{R}$ defined inductively by

$$T_2(x,y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2,$$

$$\begin{aligned} T_3(x,y,z) &= (z_1+z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) \\ &\quad + (z_1-z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4), \end{aligned}$$

$$\begin{aligned} T_4(x,y,z,w) &= (w_1+w_2)\begin{pmatrix} (z_1+z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) \\ +(z_1-z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4) \end{pmatrix} \\ &\quad + (w_1-w_2)\begin{pmatrix} (z_3+z_4)(x_5y_5 + x_5y_6 + x_6y_5 - x_6y_6) \\ +(z_3-z_4)(x_7y_7 + x_7y_8 + x_8y_7 - x_8y_8) \end{pmatrix}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} T_m(x_1,\ldots,x_m) &= (x_m^1 + x_m^2)T_{m-1}(x_1,\ldots,x_{m-1}) \\ &\quad + (x_m^1 - x_m^2)T_{m-1}(B^{2^{m-2}}(x_1), B^{2^{m-2}}(x_2), B^{2^{m-3}}(x_3)\ldots, B^2(x_{m-1})), \end{aligned} \tag{A.1}$$

where $B : \ell_\infty \to \ell_\infty$ is the backward shift operator, have been used to find lower estimates for Bohnenblust–Hille and related inequalities (se also [27]). In the context of the Hardy–Littlewood inequalities we also have good results, but in the next subsection we invent different multilinear forms that, in our context, provide better estimates.

The numerical issue involved to obtain our estimates is the calculus of $\|T_m\|$ when ℓ_∞ is replaced by ℓ_p (in this case we write $T_{m,p}$ instead of T_m). This task refers to a typical nonlinear optimization problem subject to restrictions.

To perform this computer-aided calculus we use a multi-paradigm numerical computing environment called MATLAB (MATrix LABoratory) (see [18]) to specify and solve our optimization problem. The MATLAB software has a toolbox called *Optimization* that provides a robust large-scale nonlinear optimization method called Interior Point Algorithm. Mathematical details of the interior point algorithm can be found in several publications (see for instance [12, 13, 33]).

The source codes were built to maximize the function $f(x) = |T_{m,2m}(x)|$ (denoted in the code simply by Tm2m) subject to the restriction $g(x) = \|x\|_{2m} - 1 = 0$. As the MATLAB offers only the minimization feature we placed the equivalent problem of minimize $-f(x)$ subject to the same restriction. Furthermore, as the absolute value function have problems with differentiability, which compromises the functionality of the algorithm, we finally calculate the problem by following the steps:

(i) Calculating the global minimum of $T_{m,2m}(x)$ with the restriction $g(x) = 0$;

(ii) Calculating the global minimum of $-T_{m,2m}(x)$ with the restriction $g(x) = 0$ and

(iii) Taking the greatest absolute value of numbers obtained in (i) and (ii).
In Appendix A.2 we present, as an example, the program source code which calculates (i).

So, performing these calculations for $T_{m,2m}$, we obtain

$C_{\mathbb{R},2,4} >$	$\frac{2}{1.74}$	> 1.149
$C_{\mathbb{R},3,6} >$	$\frac{4}{3.29}$	> 1.215
$C_{\mathbb{R},4,8} >$	$\frac{8}{6.40}$	> 1.250
$C_{\mathbb{R},5,10} >$	$\frac{16}{12.60}$	> 1.269
$C_{\mathbb{R},6,12} >$	$\frac{32}{25.00}$	> 1.280
$C_{\mathbb{R},7,14} >$	$\frac{64}{49.47}$	> 1.293
$C_{\mathbb{R},8,16} >$	$\frac{128}{98.36}$	> 1.301
$C_{\mathbb{R},9,18} >$	$\frac{256}{195.81}$	$> 1.307.$

(A.2)

A.1.2. New multilinear forms and better estimates

Up to now the best known multilinear forms by use in order to find lower bounds for the Bohnenblust–Hille and Hardy–Littlewood inequalities were defined in (A.1). Now we show that for $m = 4, 8, 16, \ldots$ we get better estimates using slightly different multilinear forms and numerical computation. Define by

$$\widetilde{T}_2(x,y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2,$$

$$\begin{aligned}\widetilde{T}_4(x,y,z,w) = {} & (x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2)(z_1w_1 + z_1w_2 + z_2w_1 - z_2w_2) \\ & + (x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2)(z_3w_3 + z_3w_4 + z_4w_3 - z_4w_4) \\ & + (x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)(z_1w_1 + z_1w_2 + z_2w_1 - z_2w_2) \\ & - (x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)(z_3w_3 + z_3w_4 + z_4w_3 - z_4w_4),\end{aligned}$$

$$\begin{aligned}\widetilde{T}_8(x,y,z,w,r,s,t,u) = {} & \widetilde{T}_4(x,y,z,w)\widetilde{T}_4(r,s,t,u) \\ & + \widetilde{T}_4(x,y,z,w)\widetilde{T}_4(B^4(r), B^4(s), B^4(t), B^4(u)) \\ & + \widetilde{T}_4(B^4(x), B^4(y), B^4(z), B^4(w))\widetilde{T}_4(r,s,t,u) \\ & - \widetilde{T}_4(B^4(x), B^4(y), B^4(z), B^4(w))\widetilde{T}_4(B^4(r), B^4(s), B^4(t), B^4(u)),\end{aligned}$$

and so on (recall that B is the shift operator, as defined before). When ℓ_∞ is replaced by ℓ_p we write $\widetilde{T}_{2^k,p}$ instead of $\widetilde{T}_{2^k}$. Using the same computational apparatus to $\widetilde{T}_4, \widetilde{T}_8$, etc, we obtain

$C_{\mathbb{R},4,8} >$	$\frac{2^3}{6.20}$	> 1.290
$C_{\mathbb{R},8,16} >$	$\frac{2^7}{91.48}$	> 1.399
$C_{\mathbb{R},16,32} >$	$\frac{2^{15}}{22137.70}$	$> 1.480,$

(A.3)

and this procedure seems clearly better than the former.

In order to deal with values of m that are not power of 2 we cannot define $\widetilde{T}_m$ (at least with the same rule) and, as we know, $\widetilde{T}_m$ provides better lower estimates for $C_{\mathbb{R},4,8}, C_{\mathbb{R},8,16}, C_{\mathbb{R},16,32}$. However, we can use the operators $\widetilde{T}_2, \widetilde{T}_4, \widetilde{T}_8, \widetilde{T}_{16}$ and, inductively, define, for $m = 3, 5, 6, 7, 9, 10\ldots$

$$\begin{aligned} \widetilde{T}_{m,p} : \ell_p^{2^{m-1}} \times \cdots \times \ell_p^{2^{m-1}} &\to \mathbb{R} \\ (x^{(1)}, \ldots, x^{(m)}) &\mapsto (x_1^{(m)} + x_2^{(m)}) \widetilde{T}_{m-1,p}(x^{(1)}, \ldots, x^{(m)}) \\ &\quad + (x_1^{(m)} - x_2^{(m)}) \widetilde{T}_{m-1,p}(B^{2^{m-1}}(x^{(1)}), \ldots, B^2(x^{(m-1)})), \end{aligned}$$

Hence, again using MATLAB, we can improve the estimates of (A.2) to:

	New Estimates	Estimates from (A.2)
$C_{\mathbb{R},2,4} >$	$\frac{2}{1.74} \approx 1.149$	$= 1.149$
$C_{\mathbb{R},3,6} >$	$\frac{4}{3.29} \approx 1.215$	$= 1.215$
$C_{\mathbb{R},4,8} >$	$\frac{8}{6.20} \approx 1.290$	> 1.250
$C_{\mathbb{R},5,10} >$	$\frac{16}{12.38} \approx 1.292$	> 1.269
$C_{\mathbb{R},6,12} >$	$\frac{32}{24.67} \approx 1.297$	> 1.280
$C_{\mathbb{R},7,14} >$	$\frac{64}{49.13} \approx 1.302$	> 1.293
$C_{\mathbb{R},8,16} >$	$\frac{128}{98.36} \approx 1.399$	> 1.301
$C_{\mathbb{R},9,18} >$	$\frac{256}{185.29} \approx 1.381$	$> 1.307.$

A.2. Source code

```
% functions

function [f] = T24(x)
% x: vector with four real coordinates
% T24: returns the value of the function in x
f = x(1)*x(3)+x(1)*x(4)+x(2)*x(3)-x(2)*x(4);
end

function [gdes,g] = restrict(x)
% gdes: inequality restriction (required by MATLAB syntax and
      empty for our purposes)
% g:  equality restriction
% restrict: returns the values of equality and inequality
      restrictions on the vector x
gdes = [ ];
g = [x(1)\symbol{94} 4+x(2)\symbol{94} 4-1;x(3)\symbol{94} 4+
x(4)\symbol{94}4-1];
end

% main routine

% x0: start point (vector with four coordinates) that must be
       provided by the user
options = optimset('Algorithm','interior-point');
problem = createOptimProblem('fmincon','x0',x0,'objective',
@(x)T24(x)\ldots,'Aeq',[ ],'beq',[ ],'options',[ ],'lb',[ ],'ub',
[],'nonlcon',@(x)restrict(x));
gs = GlobalSearch;
[xmin,fmin] = run(gs,problem);
```

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(Received April 4, 2017)

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