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THE NORMALIZED L_p INTERSECTION BODIES

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Abstract. In this paper, we define the normalized L_p intersection body and prove that the normalized L_p intersection body operator is GL(n) contravariant of weight 0. We show that the polar body operator can be obtained as a limit of the normalized L_p intersection body operator. And we establish a dual Brunn-Minkowski type inequality for normalized L_p intersection bodies. Furthermore, the normalized L_p-Busemann-Petty problem is shown.

1. Introduction

The notion of intersection bodies was introduced by Lutwak [21]. The intersection body, IK, of K is the star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^{\perp}$, the hyperplane orthogonal to u, i.e., for all $u \in S^{n-1}$,

$$\rho_{IK}(u) = \mathrm{vol}_{n-1}(K \cap u^{\perp}),$$

where vol_{n-1} denotes $(n-1)$-dimensional volume.

Intersection bodies have attracted increased interest during past two decades (see [2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 25, 28, 29]). In particular, intersection bodies turned out to be critical for the solution of the Busemann-Petty problem (see [3, 4, 6, 11, 12, 13, 14, 15, 16, 29]).

Haberl and Ludwig [10] extended the classical intersection bodies to L_p space. Let K be a star body in $\mathbb{R}^n$, $p < 1$, $p \neq 0$. The L_p intersection body, I_pK, of K is a centered star body, whose radial function is defined by, for all $u \in S^{n-1}$,

$$\rho^p_{I_pK}(u) = \frac{1}{(n-p)} \int_{S^{n-1}} \rho_K^{n-p}(v) |\langle v, u\rangle|^{-p} dv. \tag{1.1}$$

Haberl and Ludwig [10] pointed out that the intersection body IK is obtained as a limit of L_p intersection body I_pK, that is for all $u \in S^{n-1}$,

$$\rho_{IK}(u) = \lim_{p \to 1^-} \frac{1-p}{2} \rho^p_{I_pK}(u).$$

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Haberl and Ludwig [10] also established a characterization of L_p intersection bodies. Berck [1] proved that the L_p intersection body of a centered convex body is also convex. Haberl [9] studied the Busemann-Petty type problem for L_p intersection bodies (also see Yuan and Cheung [27]). More results on the L_p intersection body can be found in [26, 27].

In this paper, we define the normalized L_p intersection body as follows. Let K be a star body in $\mathbb{R}^n$, $p<1$, $p\neq 0$. The normalized L_p intersection body, $\bar{I}_pK$, of K is a centered star body, whose radial function is defined by, for all $u\in S^{n-1}$,

$$\rho^p_{\bar{I}_pK}(u)=\frac{1}{(n-p)V(K)}\int_{S^{n-1}}\rho_K^{n-p}(v)|\langle v,u\rangle|^{-p}dv. \tag{1.2}$$

From (1.1) and (1.2), one can obtain that

$$\bar{I}_pK=V(K)^{-\frac{1}{p}}I_pK.$$

One purpose of this paper is to establish the dual Brunn-Minkowski type inequality for normalized L_p intersection bodies.

THEOREM 1.1. *Let K and L be two star bodies in $\mathbb{R}^n$, and $\lambda,\mu\geqslant 0$ (not both zero). If $0<p<1$, then*

$$V(\bar{I}_p(\lambda\cdot K\widehat{+}_{-p}\mu\cdot L))^{\frac{p}{n}}\leqslant\lambda V(\bar{I}_pK)^{\frac{p}{n}}+\mu V(\bar{I}_pL)^{\frac{p}{n}}, \tag{1.3}$$

with equality holds if K and L are dilates of each other. If $p<0$, then the inequality (1.3) is reversed. Here $\widehat{+}_{-p}$ denotes the L_{-p} harmonic Blaschke radial sum (see Section 3 for a precise definition).

The other aim of this paper is to study the normalized L_p-Busemann-Petty problem. Our main results can be stated as follows.

THEOREM 1.2. *Let K be a normalized L_p intersection body and L be a centered star body in $\mathbb{R}^n$, and $0<p<1$ or $p<0$. If*

$$\bar{I}_pK\subset\bar{I}_pL,$$

then

$$V(K)\geqslant V(L),$$

with equality holds if and only if $K=L$.

This paper is organized as follows: In Section 2 we introduce above interrelated notations and their background materials. Section 3 contains the proofs and some applications of our main results.

2. Notation and background material

For general reference for the theory of convex (star) bodies the reader may wish to consult the books of Gardner [5] and Schneider [24].

The unit ball and its surface in $\mathbb{R}^n$ are denoted by B and S^{n-1}, respectively. We write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$. The radial function $\rho_K : S^{n-1} \to [0,\infty)$ of a compact star-shaped set about the origin, $K \in \mathbb{R}^n$, is defined, for $u \in S^{n-1}$, by

$$\rho_K(u) = \max\{\lambda \geqslant 0 : \ \lambda u \in K\}. \tag{2.1}$$

If $\rho_K(\cdot)$ is positive and continuous, then K is called a star body about the origin. The set of star bodies about the origin in $\mathbb{R}^n$ is denoted by $\mathscr{S}^n$. The subset of $\mathscr{S}^n$ containing centered star bodies will be denoted by $\mathscr{S}_e^n$. From the definition of the radial function, we have that, for $K \in \mathscr{S}^n$,

$$\rho_K(-u) = \rho_{-K}(u), \ \ \forall\, u \in S^{n-1}. \tag{2.2}$$

And for $K,L \in \mathscr{S}^n$,

$$K \subseteq L \Leftrightarrow \rho_K(u) \leqslant \rho_L(u), \ \ \forall\, u \in S^{n-1}. \tag{2.3}$$

If $\dfrac{\rho_K(u)}{\rho_L(u)}$ is independent of $u \in S^{n-1}$, then we say star bodies K and L are dilates of each other. If $s > 0$, we have

$$\rho_{sK}(u) = s\rho_K(u), \ \ \text{for all } u \in S^{n-1}. \tag{2.4}$$

If $\phi \in GL(n)$, we have

$$\rho_{\phi K}(u) = \rho_K(\phi^{-1}u), \ \ \text{for all } u \in S^{n-1}. \tag{2.5}$$

The radial Hausdorff metric between the star bodies K and L is

$$\widetilde{\delta}(K,L) = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\widetilde{\delta}(K_i,K) \to 0, \ \ \text{as} \ \ i \to \infty.$$

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of radial function $\rho_{K_i}(\cdot)$ converges uniformly to $\rho_K(\cdot)$.

Let K and L be two star bodies in $\mathbb{R}^n$ and $\lambda,\mu \geqslant 0$ (not both zero), then the L_p radial sum, $\lambda \cdot K \widetilde{+}_p \mu \cdot L$ $(p \neq 0)$, is defined by

$$\rho^p_{\lambda \cdot K \widetilde{+}_p \mu \cdot L}(u) = \lambda \rho^p_K(u) + \mu \rho^p_L(u), \ \ \forall u \in S^{n-1}. \tag{2.6}$$

By using Minkowski's integral inequality, we have the following L_p dual Brunn-Minkowski inequality. For $K,L \in \mathscr{S}^n$, and $\lambda,\mu \geqslant 0$ (not both zero). If $0 < p < n$, then

$$V(\lambda \cdot K \widetilde{+}_p \mu \cdot L)^{\frac{p}{n}} \leqslant \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}}, \tag{2.7}$$

with equality holds if and only if K and L are dilates of each other. If $p<0$ or $p>n$, then the inequality (2.7) is reversed.

The L_p dual mixed volume $\widetilde{V}_p(K,L)$ is defined by

$$\frac{n}{p}\widetilde{V}_p(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}_p\varepsilon\cdot L)-V(K)}{\varepsilon}.$$

In fact, the L_p dual mixed volume $\widetilde{V}_p(K,L)$ has the following integral representation:

$$\widetilde{V}_p(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-p}(u)\rho_L^p(u)du. \tag{2.8}$$

In particular, $\widetilde{V}_p(K,K)=V(K)$.

From an application of Hölder inequality, one can get the following L_p dual Minkowski inequality. For $K,L\in\mathscr{S}^n$. If $0<p<n$, then

$$\widetilde{V}_p(K,L)\leqslant V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.9}$$

with equality holds if and only if K and L are dilates of each other. If $p<0$ or $p>n$, then the inequality (2.9) is reversed.

The set of real-valued, continuous functions on S^{n-1} will be denoted by $C(S^{n-1})$. The subset of $C(S^{n-1})$ that contains the even functions will be denoted by $C_e(S^{n-1})$. The subset of $C_e(S^{n-1})$ that contains the nonnegative functions shall be denoted by $C_e^+(S^{n-1})$. If $f,g\in C(S^{n-1})$, then $\langle f,g\rangle$ is defined by

$$\langle f,g\rangle=\frac{1}{n}\int_{S^{n-1}}f(u)g(u)du. \tag{2.10}$$

For $f\in C(S^{n-1})$ and $p<1$, $p\neq 0$, the L_{-p} cosine transform, $\mathrm{C}_{-p}f$, of f is defined by (see [8])

$$(\mathrm{C}_{-p}f)(u)=\int_{S^{n-1}}|\langle u,v\rangle|^{-p}f(v)dv, \tag{2.11}$$

for $u\in S^{n-1}$.

It is well known that the linear transformation $\mathrm{C}_{-p}:C(S^{n-1})\to C(S^{n-1})$ is self-adjoint (see [23]), i.e., if $f,g\in C(S^{n-1})$, then

$$\langle \mathrm{C}_{-p}f,g\rangle=\langle f,\mathrm{C}_{-p}g\rangle. \tag{2.12}$$

Applying (1.2) and (2.11), we have that

$$\rho_{\bar{I}_pK}^p=\frac{1}{(n-p)V(K)}\mathrm{C}_{-p}\rho_K^{n-p}. \tag{2.13}$$

3. Main results

It is well known that the L_p intersection body operator I_p is GL(n) contravariant of weight $\frac{1}{p}$, i.e., for every $\phi \in GL(n)$ and every star body K, (see [10])

$$I_p(\phi K) = |\det \phi|^{\frac{1}{p}} \phi^{-t} I_p K.$$

We will show that the normalized L_p intersection body operator $\overline{I}_p$ is GL(n) contravariant of weight 0.

THEOREM 3.1. *Let $K \in \mathscr{S}^n$, $p < 1$, $p \neq 0$, and $\phi \in GL(n)$. Then*

$$\overline{I}_p(\phi K) = \phi^{-t} \overline{I}_p K.$$

Proof. By (1.2), (2.4) and (2.5), we obtain that

$$\begin{aligned}
\rho^p_{\overline{I}_p \phi K}(u) &= \frac{1}{(n-p)V(\phi K)} \int_{S^{n-1}} \rho^{n-p}_{\phi K}(v) |\langle v, u\rangle|^{-p} dv \\
&= \frac{1}{(n-p)|\det(\phi)|V(K)} \int_{S^{n-1}} \rho^{n-p}_{K}(\phi^{-1} v) |\langle v, u\rangle)|^{-p} dv \\
&= \frac{1}{(n-p)V(K)} \int_{S^{n-1}} \rho^{n-p}_{K}(v) |\langle v, \phi^t u\rangle|^{-p} dv \\
&= \rho^p_{\overline{I}_p K}(\phi^t u) \\
&= \rho^p_{\phi^{-t}\overline{I}_p K}(u). \quad \square
\end{aligned}$$

REMARK 1. (see [5]) Let K be a convex body which contains the origin in its interior in $\mathbb{R}^n$ and $\phi \in GL(n)$. Then

$$(\phi K)^* = \phi^{-t} K^*.$$

Let E_n denote the identity matrix of size n. If we take $\phi = cE_n$ in Theorem 3.1, then we can obtain the following result.

COROLLARY 3.2. *Let $K \in \mathscr{S}^n$, $p < 1$, $p \neq 0$ and $c \neq 0$. Then*

$$\overline{I}_p(cK) = \frac{1}{c} \overline{I}_p K.$$

Lutwak [22] introduced the harmonic Blaschke radial sum. Suppose $K, L \in \mathscr{S}^n$, and $\lambda, \mu \geqslant 0$ (not both zero), the harmonic Blaschke radial sum, $\lambda \cdot K \hat{+} \mu \cdot L$, is defined by, for $\forall u \in S^{n-1}$,

$$\frac{\rho^{n+1}_{\lambda \cdot K \hat{+} \mu \cdot L}(u)}{V(\lambda \cdot K \hat{+} \mu \cdot L)} = \frac{\lambda \rho^{n+1}_K(u)}{V(K)} + \frac{\mu \rho^{n+1}_L(u)}{V(L)}.$$

Similarly, we can define the L_p harmonic Blaschke radial sum. Suppose $K,L \in \mathscr{S}^n$, $p \neq -n$, and $\lambda,\mu \geqslant 0$ (not both zero), the L_p harmonic Blaschke radial sum, $\lambda \cdot K \hat{+}_p \mu \cdot L$, is defined by, for $\forall u \in S^{n-1}$,

$$\frac{\rho^{n+p}_{\lambda \cdot K \hat{+}_p \mu \cdot L}(u)}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)} = \frac{\lambda \rho_K^{n+p}(u)}{V(K)} + \frac{\mu \rho_L^{n+p}(u)}{V(L)}. \tag{3.1}$$

In particular, $\lambda \cdot K \hat{+}_1 \mu \cdot L$ is just the harmonic Blaschke radial sum $\lambda \cdot K \hat{+} \mu \cdot L$.

For $K \in \mathscr{S}^n$ and $0 \leqslant i \leqslant n$, we write $\widetilde{W}_i(K)$ for the dual mixed volume $\widetilde{V}(K,\ldots,K,B,\ldots,B)$, where K appears $n-i$ times and B appears i times, and is called the dual quermassintegral. It has the following integral representation (see [21]):

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)du. \tag{3.2}$$

If $i=0$, then $\widetilde{W}_0(K) = V(K)$.

In fact, we will prove the following L_p dual Brunn-Minkowski inequality which is more general than Theorem 1.1.

THEOREM 3.3. *Let $K,L \in \mathscr{S}^n$, $0 \leqslant i \leqslant n-1$, and $\lambda,\mu \geqslant 0$. If $0<p<1$, then*

$$\widetilde{W}_i(\bar{I}_p(\lambda \cdot K \widehat{+}_{-p} \mu \cdot L))^{\frac{p}{n-i}} \leqslant \lambda \widetilde{W}_i(\bar{I}_p K)^{\frac{p}{n-i}} + \mu \widetilde{W}(\bar{I}_p L)^{\frac{p}{n-i}}, \tag{3.3}$$

with equality holds if K and L are dilates of each other. If $p<0$, then the inequality (3.3) is reversed.

Proof. By (1.2) and (3.1), we have that, for $\forall u \in S^{n-1}$,

$$\begin{aligned} \rho^p_{\bar{I}_p(\lambda \cdot K \widehat{+}_{-p} \mu \cdot L)}(u) &= \frac{1}{(n-p)} \int_{S^{n-1}} \frac{\rho^{n-p}_{\lambda \cdot K \widehat{+}_{-p} \mu \cdot L}(v)}{V(\lambda \cdot K \widehat{+}_{-p} \mu \cdot L)} |\langle v,u\rangle|^{-p} dv \\ &= \frac{1}{(n-p)} \int_{S^{n-1}} (\lambda \frac{\rho_K^{n-p}(v)}{V(K)} + \mu \frac{\rho_L^{n-p}(v)}{V(L)}) |\langle v,u\rangle|^{-p} dv \\ &= \lambda \rho^p_{\bar{I}_p K}(u) + \mu \rho^p_{\bar{I}_p L}(u). \end{aligned} \tag{3.4}$$

If $0<p<1$, since $\dfrac{n-i}{p} > 1$, applying (3.2), (3.4) and Minkowski's integral inequality, we have that

$$
\begin{aligned}
&\widetilde{W}_i(\bar{I}_p(\lambda\cdot K\widehat{+}_{-p}\mu\cdot L))^{\frac{p}{n-i}}\\
&=\Big[\frac{1}{n}\int_{S^{n-1}}\rho^{n-i}_{\bar{I}_p(\lambda\cdot K\widehat{+}_{-p}\mu\cdot L)}(u)\Big]^{\frac{p}{n-i}}\\
&=\Big[\frac{1}{n}\int_{S^{n-1}}(\lambda\rho^{p}_{\bar{I}_pK}(u)+\mu\rho^{p}_{\bar{I}_pL}(u))^{\frac{n-i}{p}}du\Big]^{\frac{p}{n-i}} \qquad (3.5)\\
&\leqslant\lambda\Big[\frac{1}{n}\int_{S^{n-1}}\rho^{n-i}_{\bar{I}_pK}(u)du\Big]^{\frac{p}{n-i}}+\mu\Big[\frac{1}{n}\int_{S^{n-1}}\rho^{n-i}_{\bar{I}_pL}(u)du\Big]^{\frac{p}{n-i}}\\
&=\lambda\widetilde{W}_i(\bar{I}_pK)^{\frac{p}{n-i}}+\mu\widetilde{W}_i(\bar{I}_pL)^{\frac{p}{n-i}}.
\end{aligned}
$$

If K and L are dilates of each other, then there exists a constant c, such that $K=cL$. Using Corollary 3.2, we have that, for $\forall u\in S^{n-1}$,

$$\rho_{\bar{I}_pK}(u)=\rho_{\bar{I}_pcL}(u)=\rho_{\frac{1}{c}\bar{I}_pL}(u).$$

This means that $\bar{I}_pK$ and $\bar{I}_pL$ are dilates of each other. From the equality condition of Minkowski's integral inequality, equality in (3.5) holds.

If $p<0$, we have that $\dfrac{n-i}{p}<0$, then the inequality in (3.3) is reversed. This completes the proof. □

REMARK 2. The case $i=0$ of Theorem 3.3 is Theorem 1.1.

We denote $\frac{1}{2}\cdot K\widehat{+}_{-p}\frac{1}{2}\cdot(-K)$ by $\widehat{\nabla}_{-p}K$.

LEMMA 3.4. *Let $K\in\mathscr{S}^n$. If $0<p<1$ or $p<0$, then*

$$V(\widehat{\nabla}_{-p}K)\geqslant V(K), \tag{3.6}$$

with equality if and only if K is centered.

Proof. From (3.1), one can obtain

$$\frac{\rho^{n-p}_{\widehat{\nabla}_{-p}K}(u)}{V(\widehat{\nabla}_{-p}K)}=\frac{1}{2}\frac{\rho^{n-p}_{K}(u)}{V(K)}+\frac{1}{2}\frac{\rho^{n-p}_{-K}(u)}{V(-K)}, \tag{3.7}$$

equivalently,

$$\rho_{\widehat{\nabla}_{-p}K}(u)=\Big[\frac{V(\widehat{\nabla}_{-p}K)}{V(K)}\Big(\frac{1}{2}\rho^{n-p}_{K}(u)+\frac{1}{2}\rho^{n-p}_{-K}(u)\Big)\Big]^{\frac{1}{n-p}}.$$

Since $0<p<1$, applying (3.7) and Minkowski's integral inequality, we have that

$$\begin{aligned}
V\left(\widehat{\nabla}_{-p}K\right)^{\frac{n-p}{n}} &= \left(\frac{1}{n}\int_{S^{n-1}}\rho^{n}_{\widehat{\nabla}_{-p}K}(u)du\right)^{\frac{n-p}{n}} \\
&= \left\{\frac{1}{n}\int_{S^{n-1}}\left[\frac{V(\widehat{\nabla}_{-p}K)}{V(K)}\left(\frac{1}{2}\rho_K^{n-p}(u)+\frac{1}{2}\rho_{-K}^{n-p}(u)\right)\right]^{\frac{n}{n-p}}du\right\}^{\frac{n-p}{n}} \\
&\leqslant \frac{1}{2}\frac{V(\widehat{\nabla}_{-p}K)}{V(K)}\left[\left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n}(u)du\right)^{\frac{n-p}{n}}+\left(\frac{1}{n}\int_{S^{n-1}}\rho_{-K}^{n}(u)du\right)^{\frac{n-p}{n}}\right] \\
&= V(\widehat{\nabla}_{-p}K)V(K)^{-\frac{p}{n}}.
\end{aligned}$$

Note that $0<p<1$, we obtain that

$$V(\widehat{\nabla}_{-p}K)\geqslant V(K).$$

By the equality condition of Minkowski's integral inequality, equality holds in (3.6) if and only if K and $-K$ are dilates of each other. This means that K is centered.

Similarly, we can get the same result for $p<0$. □

LEMMA 3.5. *Let $K\in\mathscr{S}^n$, $p<1$, $p\neq 0$. Then,*

$$\bar{I}_p(\widehat{\nabla}_{-p}K)=\bar{I}_pK.$$

Proof. By (1.2), (3.7) and (2.2), it follows immediately that, for $\forall u\in S^{n-1}$,

$$\begin{aligned}
\rho^{p}_{\bar{I}_p(\widehat{\nabla}_{-p}K)}(u) &= \frac{1}{(n-p)V(\widehat{\nabla}_{-p}K)}\int_{S^{n-1}}\rho^{n-p}_{\widehat{\nabla}_{-p}K}(v)|u\cdot v|^{-p}dv \\
&= \frac{1}{2(n-p)V(K)}\int_{S^{n-1}}\rho_K^{n-p}(v)|u\cdot v|^{-p}dv \\
&\quad +\frac{1}{2(n-p)V(-K)}\int_{S^{n-1}}\rho_{-K}^{n-p}(v)|u\cdot v|^{-p}dv \\
&= \frac{1}{(n-p)V(K)}\int_{S^{n-1}}\rho_K^{n-p}(v)|u\cdot v|^{-p}dv \\
&= \rho^{p}_{\bar{I}_pK}(u).
\end{aligned}$$

Thus,

$$\bar{I}_p(\widehat{\nabla}_{-p}K)=\bar{I}_pK. \quad \square$$

LEMMA 3.6. *Let* $K \in \mathscr{S}^n$, $p<1$, $p \neq 0$. *Then, for* $\forall M \in \mathscr{S}_e^n$,

$$\frac{\widetilde{V}_p(\widehat{\nabla}_{-p}K,M)}{V(\widehat{\nabla}_{-p}K)} = \frac{\widetilde{V}_p(K,M)}{V(K)}.$$

Proof. By (2.8), (3.7) and (2.2), it follows that

$$\begin{aligned}\frac{\widetilde{V}_p(\widehat{\nabla}_{-p}K,M)}{V(\widehat{\nabla}_{-p}K)} &= \frac{1}{nV(\widehat{\nabla}_{-p}K)}\int_{S^{n-1}}\rho^{n-p}_{\widehat{\nabla}_{-p}K}(u)\rho_M^p(u)du\\ &= \frac{1}{n}\int_{S^{n-1}}\frac{1}{2}\frac{\rho_K^{n-p}(u)}{V(K)}\rho_M^p(u)du + \frac{1}{n}\int_{S^{n-1}}\frac{1}{2}\frac{\rho_{-K}^{n-p}(u)}{V(-K)}\rho_M^p(u)du\\ &= \frac{1}{n}\int_{S^{n-1}}\frac{1}{2}\frac{\rho_K^{n-p}(u)}{V(K)}\rho_M^p(u)du + \frac{1}{n}\int_{S^{n-1}}\frac{1}{2}\frac{\rho_K^{n-p}(u)}{V(K)}\rho_M^p(-u)du\\ &= \frac{1}{n}\int_{S^{n-1}}\frac{\rho_K^{n-p}(u)}{V(K)}\rho_M^p(u)du\\ &= \frac{\widetilde{V}_p(K,M)}{V(K)}. \quad \square\end{aligned}$$

In order to prove Theorem 1.2, the following theorem is required.

THEOREM 3.7. *Let* $K,L \in \mathscr{S}^n$, $p<1$, $p \neq 0$. *Then*

$$\frac{\widetilde{V}_p(K,\bar{I}_pL)}{V(K)} = \frac{\widetilde{V}_p(L,\bar{I}_pK)}{V(L)}.$$

Proof. By (2.8), (1.2) and Fubini's theorem, it follows that

$$\begin{aligned}\frac{\widetilde{V}_p(K,\bar{I}_pL)}{V(K)} &= \frac{1}{nV(K)}\int_{S^{n-1}}\rho_K^{n-p}(u)\rho_{\bar{I}_pL}^p(u)du\\ &= \frac{1}{nV(K)}\int_{S^{n-1}}\rho_K^n(u)\Big(\frac{1}{(n-p)V(L)}\int_{S^{n-1}}\rho_L^{n-p}(v)|u\cdot v|^{-p}dv\Big)du\\ &= \frac{1}{nV(L)}\int_{S^{n-1}}\rho_L^n(v)\Big(\frac{1}{(n-p)V(K)}\int_{S^{n-1}}\rho_K^{n-p}(u)|u\cdot v|^{-p}du\Big)dv\\ &= \frac{1}{nV(L)}\int_{S^{n-1}}\rho_L^{n-p}(v)\rho_{\bar{I}_pK}^p(v)dv\\ &= \frac{\widetilde{V}_p(L,\bar{I}_pK)}{V(L)}. \quad \square\end{aligned}$$

In this paper, we consider the following the normalized L_p-Busemann-Petty problem. Let $K, L \in \mathscr{S}^n$, $p < 1$, $p \neq 0$. If

$$\bar{I}_p K \subseteq \bar{I}_p L,$$

does it follow that

$$V(K) \geqslant V(L)?$$

Just as the classical Busemann-Petty problem, we will show that the normalized L_p-Busemann-Petty problem has an affirmative answer if K is a normalized L_p intersection body.

Proof of Theorem 1.2. For $0 < p < 1$, from the definition of the L_p dual mixed volume, if $L_1 \subseteq L_2$, then

$$\widetilde{V}_p(K, L_1) \leqslant \widetilde{V}_p(K, L_2).$$

Since K is a normalized L_p intersection body, there exists a star body M such that $K = \bar{I}_p M$. Using Theorem 3.7, we can conclude that

$$\begin{aligned}
\frac{\widetilde{V}_p(L,K)}{V(L)} &= \frac{\widetilde{V}_p(L, \bar{I}_p M)}{V(L)} \\
&= \frac{\widetilde{V}_p(M, \bar{I}_p L)}{V(M)} \\
&\geqslant \frac{\widetilde{V}_p(M, \bar{I}_p K)}{V(M)} \\
&= \frac{\widetilde{V}_p(K, \bar{I}_p M)}{V(K)} = 1.
\end{aligned}$$

Applying the L_p dual Minkowski inequality (2.9), we obtain that

$$V(K) \geqslant V(L),$$

with equality if and only if K and L are dilates of each other. Obviously, if $V(K) = V(L)$, then we must have $K = L$.

Similarly, we can obtain the same result for $p < 0$. $\square$

Next, we will characterizes the equality of normalized L_p intersection bodies in terms of normalized L_p dual mixed volumes.

THEOREM 3.8. *Let $K, L \in \mathscr{S}^n$, $p < 1$, $p \neq 0$. Then*

$$\bar{I}_p K = \bar{I}_p L$$

if and only if

$$\frac{\widetilde{V}_p(K,M)}{V(K)} = \frac{\widetilde{V}_p(L,M)}{V(L)} \tag{3.8}$$

for each centered star body $M \in \mathscr{S}_e^n$.

Proof. From Lemma 3.5 and Lemma 3.6, we may assume that $K, L \in \mathscr{S}_e^n$.

We first assume that (3.8) holds for all $M \in \mathscr{S}_e^n$. Let $f \in C_e^+(S^{n-1})$ and define $M \in \mathscr{S}_e^n$ by

$$\rho_M^p = C_{-p} f. \tag{3.9}$$

From (2.8), (2.10), (3.9), (2.12) and (2.13), one can obtain that

$$\begin{aligned} \frac{\widetilde{V}_p(K,M)}{V(K)} &= \left\langle \frac{\rho_K^{n-p}}{V(K)}, \rho_M^p \right\rangle \\ &= \left\langle \frac{\rho_K^{n-p}}{V(K)}, C_{-p} f \right\rangle \\ &= \left\langle C_{-p} \frac{\rho_K^{n-p}}{V(K)}, f \right\rangle \\ &= \langle (n-p)\rho_{\overline{I}_p K}^p, f \rangle. \end{aligned} \tag{3.10}$$

Similarly,

$$\frac{\widetilde{V}_p(L,M)}{V(L)} = \langle (n-p)\rho_{\overline{I}_p L}^p, f \rangle. \tag{3.11}$$

Thus, for all $f \in C_e^+(S^{n-1})$,

$$\langle \rho_{\overline{I}_p K}^p - \rho_{\overline{I}_p L}^p, f \rangle = 0.$$

But this must hold for all $f \in C_e(S^{n-1})$, since we can write an arbitrary function in $C_e(S^{n-1})$ as the difference of two functions in $C_e^+(S^{n-1})$. If we take $\rho_{\overline{I}_p K}^p - \rho_{\overline{I}_p L}^p$ for f, we obtain

$$\langle \rho_{\overline{I}_p K}^p - \rho_{\overline{I}_p L}^p, \rho_{\overline{I}_p K}^p - \rho_{\overline{I}_p L}^p \rangle = \frac{1}{n} \int_{S^{n-1}} (\rho_{\overline{I}_p K}^p - \rho_{\overline{I}_p L}^p)^2 du = 0.$$

Hence $\overline{I}_p K = \overline{I}_p L$.

On the other hand, let $\overline{I}_p K = \overline{I}_p L$. Suppose $M \in \mathscr{S}_e^n$ is such that $\rho_M \in C_{-p}(C_e(S^{n-1}))$ and hence there exists $f \in C_e(S^{n-1})$, such that

$$\rho_M^p = C_{-p} f.$$

From (3.10), (3.11) and note the fact $\overline{I}_p K = \overline{I}_p L$, we have

$$\frac{\widetilde{V}_p(K,M)}{V(K)} = \frac{\widetilde{V}_p(L,M)}{V(L)} \tag{3.12}$$

for all $M \in \mathscr{S}_e^n$ such that $\rho_M \in C_{-p}(C_e(S^{n-1}))$. Since $C_{-p}(C_e(S^{n-1}))$ is dense in $C_e(S^{n-1})$, and L_p dual mixed volumes are continuous, it follows that (3.12) must hold for all $M \in \mathscr{S}_e^n$. $\square$

Next, we will show that if we restrict to centered star bodies, then the operator $\overline{I}_p : \mathscr{S}_e^n \to \mathscr{S}_e^n$ is injective.

THEOREM 3.9. *Let $K \in \mathscr{S}_e^n$ and $L \in \mathscr{S}^n$, and $0 < p < 1$ or $p < 0$. If*

$$\bar{I}_p K = \bar{I}_p L,$$

then

$$V(K) \geqslant V(L),$$

with equality holds if and only if $K = L$.

Proof. Setting $M = K$ in Theorem 3.8, we obtain that

$$1 = \frac{\widetilde{V}_p(K,K)}{V(K)} = \frac{\widetilde{V}_p(L,K)}{V(L)}.$$

Since $0 < p < 1$, applying the L_p dual Minkowski inequality (2.9), one can obtain that

$$V(K) \geqslant V(L),$$

with equality holds if and only if $K = L$.

Similarly, we can obtain the same result when $p < 0$. □

If K and L are two star bodies in $\mathbb{R}^n$, by the definition of the radial function, then,

$$\rho_{K\cap L}(u) = \min\{\rho_K(u), \rho_L(u)\}. \tag{3.13}$$

If K is a convex body in $\mathbb{R}^n$, let h_K denote the support function of K, then, for $\forall u \in S^{n-1}, h_K(u) = \max\{x \cdot u,\ x \in K\}$. By the definition of the support function, we have that

$$h_K(-u) = h_{-K}(u), \tag{3.14}$$

where $-K = \{-x : x \in K\}$. If K is a centered convex body in $\mathbb{R}^n$, then $-K = K$.

Let K be a convex body which contains the origin in its interior in $\mathbb{R}^n$. The polar K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n \mid x \cdot y \leqslant 1, \forall y \in K\}.$$

It is easy to check that, for $\forall u \in S^{n-1}$, (see [5, 24])

$$\rho_{K^*}(u) = \frac{1}{h_K(u)}. \tag{3.15}$$

Let K be a convex body which contains the origin in its interior in $\mathbb{R}^n$. Note that: for $\forall v \in S^{n-1}$, the point $\rho_K(v)v$ lies to the boundary of K. Then, $\rho_K(v)v \cdot u \leqslant h_K(u)$ for $\forall v \in S^{n-1}$, and there exists a point $\rho_K(v_0)v_0 \in \partial K$, such that $h_K(u) = \rho_K(v_0)v_0 \cdot u$. Thus,

$$\max_{v \in S^{n-1}} \{|\rho_K(v)v \cdot u|\} = \max\{h_K(u), h_K(-u)\}. \tag{3.16}$$

Since $\dfrac{1}{nV(K)} \displaystyle\int_{S^{n-1}} \rho_K^n(v)dv = 1$, we write the dual cone-volume probability measure of K on S^{n-1} by

$$d\widetilde{v}_K(v) = \frac{\rho_K^n(v)}{nV(K)} dv. \tag{3.17}$$

By (1.2), (3.17), Jessen's inequality, (3.16), (3.14), (3.16) and (3.13), we have that

$$\begin{aligned}
\lim_{p\to-\infty}\rho_{\bar{I}_pK}(u) &= \lim_{p\to-\infty}\left[\frac{1}{(n-p)V(K)}\int_{S^{n-1}}\rho_K^{n-p}(v)|\langle v,u\rangle|^{-p}dv\right]^{\frac{1}{p}}\\
&= \lim_{p\to-\infty}\left[\frac{n}{n-p}\frac{1}{nV(K)}\int_{S^{n-1}}\rho_K^{n}(v)\Big(\frac{1}{|\rho_K(v)v\cdot u|}\Big)^{p}dv\right]^{\frac{1}{p}}\\
&= \lim_{p\to-\infty}\left[\frac{n}{n-p}\int_{S^{n-1}}\Big(\frac{1}{|\rho_K(v)v\cdot u|}\Big)^{p}d\widetilde{v}_K(v)\right]^{\frac{1}{p}}\\
&= \lim_{p\to-\infty}\Big(\frac{n}{n-p}\Big)^{\frac{1}{p}} = \min_{v\in S^{n-1}}\frac{1}{|\rho_K(v)v\cdot u|}\\
&= \frac{1}{\max\{h_K(u),h_K(-u)\}} = \min\Big\{\frac{1}{h_K(u)},\frac{1}{h_{-K}(u)}\Big\}\\
&= \min\{\rho_{K^*}(u),\rho_{(-K)^*}(u)\}\rho_{K^*\cap(-K)^*}(u).
\end{aligned} \tag{3.18}$$

Let K be a convex body which contains the origin in its interior in $\mathbb{R}^n$. For $\forall u\in S^{n-1}$, we define $\bar{I}_{-\infty}K$ by

$$\rho_{\bar{I}_{-\infty}K}(u) = \lim_{p\to-\infty}\rho_{\bar{I}_pK}(u). \tag{3.19}$$

From (3.18) and (3.19), we know that if K is a convex body which contains the origin in its interior in $\mathbb{R}^n$, then

$$\bar{I}_{-\infty}K = K^*\cap(-K)^*.$$

In particular, if K is a centered convex body in $\mathbb{R}^n$, then $\bar{I}_{-\infty}K = K^*$.

REMARK 3. Let K and L be two convex bodies which contain the origin in its interior in $\mathbb{R}^n$. If

$$K^*\subset L^*,$$

then

$$V(K)\geqslant V(L),$$

with equality if and only if $K=L$.

If K is not centered, then the answer to the normalized L_p-Busemann-Petty problem is negative.

THEOREM 3.10. *Let $K\in\mathscr{S}^n$ be a star body which is not centered. If $0<p<1$ or $p<0$, then there exists a centered star body L, such that*

$$\bar{I}_pK\subseteq\bar{I}_pL,$$

but

$$V(K) < V(L).$$

Proof. Since K is not centered, we know from Lemma 3.4 that

$$V(\widehat{\nabla}_{-p}K) > V(K). \tag{3.20}$$

Now set

$$L = \varepsilon\widehat{\nabla}_{-p}K, \tag{3.21}$$

where

$$2\varepsilon^n = 1 + \frac{V(K)}{V(\widehat{\nabla}_{-p}K)}. \tag{3.22}$$

By (3.20) and (3.22), it follows that

$$0 < \varepsilon < 1. \tag{3.23}$$

Then, from (3.21), Corollary 3.2, Lemma 3.5 and (3.22), we have that

$$\overline{I}_pL = \overline{I}_p(\varepsilon\widehat{\nabla}_{-p}K) = \frac{1}{\varepsilon}\overline{I}_p(\widehat{\nabla}_{-p}K) = \frac{1}{\varepsilon}\overline{I}_pK \supset \overline{I}_pK.$$

But, from (3.21), (3.22) and (3.20), we have that

$$V(L) = V(\varepsilon\widehat{\nabla}_{-p}K) = \varepsilon^n V(\widehat{\nabla}_{-p}K) = \frac{V(\widehat{\nabla}_{-p}K)}{2}\left(1 + \frac{V(K)}{V(\widehat{\nabla}_{-p}K)}\right)$$

$$= \frac{1}{2}[V(\widehat{\nabla}_{-p}K) + V(K)] > V(K). \quad \square$$

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