Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 369–383 doi:10.7153/mia-2018-21-27

THE COMPLEX L_p LOOMIS–WHITNEY INEQUALITY

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(*Communicated by H. Martini*)

Abstract. The complex L_p Loomis-Whitney inequality for complex isotropic measures is established, which extends the real version of the L_p Loomis-Whitney inequality for isotropic measures due to the first two authors.

1. Introduction

A convex body K is a compact convex set in $\mathbb{R}^n$ which is assumed to contain the origin in its interior. Denote by $V(K)$ the corresponding dimensional volume. Each convex body K is uniquely determined by its support function $h(K,\cdot)$ defined by, for $x \in \mathbb{R}^n$, $h(K,x) = \max\{\langle x,y\rangle : y \in K\}$, where $\langle x,y\rangle = \sum_{k=1}^n x_k y_k$ denotes the scalar product of x and y in $\mathbb{R}^n$. If $x,y \in \mathbb{C}^n$, we denote their complex scalar product by $\langle x,y\rangle_c = \sum_{k=1}^n x_k \overline{y_k}$ and the modulus of x by $\|x\| = \sqrt{\langle x,x\rangle_c}$.

The classical Loomis-Whitney inequality states that for a convex body K in $\mathbb{R}^n$,

$$V(K)^{n-1} \leqslant \prod_{k=1}^{n} V(K|e_k^{\perp}), \tag{1}$$

where $K|e_k^{\perp}$ denotes the orthogonal projection of K onto the 1-codimensional subspace $e_k^{\perp}$ perpendicular to e_k and $\{e_1,\ldots,e_n\}$ is the canonical basis of $\mathbb{R}^n$. Moreover, equality in (1) holds if and only if K is a coordinate box; i.e., up to translations, there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$K = \sum_{k=1}^{n} \alpha_k[-e_k,e_k],$$

where $[-e_k,e_k]$ is the segment jointing $-e_k$ to e_k and the sum is the Minkowski addition of convex sets. This inequality was first proved by Loomis and Whitney [27] in

Mathematics subject classification (2010): 52A20, 52A40.

Keywords and phrases: Complex L_p Loomis-Whitney inequality, complex isotropic measures, generalized $\ell_p(\mathbb{C}^n)$-balls, complex L_p projection bodies, complex L_p zonoid.

The first author was supported by the National Natural Science Foundation of China (No. 11701219 and 11626115). The second author was supported by NSFC-Henan Joint Fund (No. U1204102) and Key Research Project for Higher Education in Henan Province (No. 17A110022). The third author was supported by the Natural Science Foundation of Hunan Province (No. 2017JJ3085).

1949 and has been widely studied in recent years (see e.g., [8, 9, 11, 12, 15, 16, 17, 18, 20, 24, 25, 26, 33]). It is well-known (see e.g., [32, (5.77)]) that

$$V(K|e_k^{\perp}) = \frac{n}{2}V(K,[n-1];[-e_k,e_k]),$$

where $V(K,[n-1];[-e_k,e_k])$ is the mixed volume of $(n-1)$-copies of K and one copy of $[-e_k,e_k]$. Thus, the Loomis-Whitney inequality (1) can be rewritten as

$$V(K)^{n-1} \leqslant \frac{n^n}{2^n}\prod_{k=1}^{n} V(K,[n-1];[-e_k,e_k]). \tag{2}$$

In order to define the volume in $\mathbb{C}^n$, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping from $x=(x_1,\dots,x_n)=(x_{11}+ix_{12},\dots,x_{n1}+ix_{n2})$ to $(x_{11},x_{12},\dots,x_{n1},x_{n2})$. A complex version of (2), as a special case of our main result, can be stated as follows: if K is a convex body in $\mathbb{R}^{2n}$, then

$$V(K)^{2n-1} \leqslant \frac{n^{2n}}{\pi^n}\prod_{k=1}^{n} V(K,[2n-1];D_k)^2, \tag{3}$$

where D_k is a unit disc in $\mathrm{span}\{e_{2k-1},e_{2k}\}$ and $\{e_1,\dots,e_{2n}\}$ denotes the canonical basis of $\mathbb{R}^{2n}$. Moreover, equality in (3) holds if and only if K is a polydisc; i.e., up to translations, there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$K=\sum_{k=1}^{n}\alpha_k D_k.$$

Motivated by the recent work of the first two authors [24] on the L_p Loomis-Whitney inequality for isotropic measures, this paper is devoted to the *complex L_p Loomis-Whitney inequality* for complex isotropic measures. The following two notions are essential to our main result.

The *complex isotropic measure*, recently introduced by the first author and He [19], is a Borel measure μ on the unit sphere S^{2n-1} of $\mathbb{C}^n$ satisfying

$$\int_{S^{2n-1}} |\langle x,v\rangle_c|^2 d\mu(v) = \|x\|^2, \tag{4}$$

for all $x\in\mathbb{C}^n$. Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, (4) can be written as

$$\int_{S^{2n-1}} [\langle x,v\rangle^2 + \langle x,v^{\dagger}\rangle^2] d\mu(v) = \|x\|^2, \tag{5}$$

where the operator $\dagger:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is defined as

$$v=(v_{11},v_{12},\dots,v_{n1},v_{n2}) \mapsto v^{\dagger}=(-v_{12},v_{11},\dots,-v_{n2},v_{n1}).$$

An important example of complex isotropic measures on S^{2n-1} is the *complex cross measure* introduced in [19], which is the R_θ-invariant complex isotropic measure μ such that

$$\mathrm{supp}\,\mu = \{\mathrm{span}\{v_1,v_1^{\dagger}\}\cap S^{2n-1},\dots,\mathrm{span}\{v_n,v_n^{\dagger}\}\cap S^{2n-1}\},$$

where $\{v_1, v_1^\dagger \ldots, v_n, v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$. Furthermore, a *generalized $\ell_p(\mathbb{C}^n)$-ball* $B_{p,\alpha}(\mathbb{C}^n) := B_{p,\alpha}(\mathbb{C}^n)(\mu)$ formed by the complex cross measure μ (concentrated on $\{\text{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \ldots, \text{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\}$) is defined as follows: there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$\begin{aligned} B_{p,\alpha}(\mathbb{C}^n) &= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^n \alpha_k |\langle x, v_k\rangle_c|^p\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\ &= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^n \alpha_k [\langle x, v_k\rangle^2 + \langle x, v_k^\dagger\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\}, \quad 1 \leqslant p < \infty, \end{aligned} \tag{6}$$

and for $p = \infty$,

$$\begin{aligned} B_{\infty,\alpha}(\mathbb{C}^n) &= \Big\{x \in \mathbb{R}^{2n} : \alpha_k |\langle x, v_k\rangle_c| \leqslant 1 \text{ for all } k = 1, \ldots, n\Big\} \\ &= \Big\{x \in \mathbb{R}^{2n} : \alpha_k [\langle x, v_k\rangle^2 + \langle x, v_k^\dagger\rangle^2]^{\frac{1}{2}} \leqslant 1 \text{ for all } k = 1, \ldots, n\Big\}. \end{aligned} \tag{7}$$

We shall mention that $B_{\infty,\alpha}(\mathbb{C}^n) = \sum_{k=1}^n \alpha_k^{-1}(B_2^{2n} \cap \text{span}\{v_k, v_k^\dagger\})$ is also called a polydisc formed by μ, where B_2^{2n} is the Euclidean unit ball in $\mathbb{R}^{2n}$.

For $p \geqslant 1$, we define the *R_θ-invariant L_p complex projection body* $\Pi_p^D K$ of a convex body K in $\mathbb{R}^{2n}$, in terms of its support function is given by, for $v \in S^{2n-1}$,

$$\begin{aligned} h(\Pi_p^D K, v) &= \Big(\frac{1}{2n}\int_{S^{2n-1}} |\langle v, u\rangle_c|^p dS_p(K, u)\Big)^{\frac{1}{p}} \\ &= \Big(\frac{1}{2n}\int_{S^{2n-1}} [\langle v, u\rangle^2 + \langle v, u^\dagger\rangle^2]^{\frac{p}{2}} dS_p(K, u)\Big)^{\frac{1}{p}}, \end{aligned}$$

where $dS_p(K, \cdot)$ is the L_p surface area measure of K. For $p = 1$, it reduces to the R_θ-invariant complex projection body introduced by Abardia and Bernig [3].

Thus, the complex Loomis-Whitney inequality for complex isotropic measures can be formulated as follows:

THEOREM 1. *Suppose $p \geqslant 1$ and K is a convex body in $\mathbb{R}^{2n}$. If μ is a complex isotropic measure on S^{2n-1}, then*

$$V(K)^{\frac{2n-p}{p}} \leqslant A_{n,p} \exp\Big\{\int_{S^{2n-1}} \log h(\Pi_p^D K, v)^2 d\mu(v)\Big\}, \tag{8}$$

where

$$A_{n,p} = \frac{n^{2n/p}\Gamma(2n + 1 - \frac{2n}{p})}{\pi^n \Gamma(3 - \frac{2}{p})^n}.$$

In addition, if μ is a complex cross measure on S^{2n-1}, then equality in (8) holds for $p > 1$ if and only if K is a generalized $\ell_{p^*}(\mathbb{C}^n)$-ball formed by μ and equality in (8) holds for $p = 1$ if and only if K is a polydisc formed by μ (up to translations).

Here p^* is the Hölder conjugate of p; i.e., $1/p+1/p^*=1$.

When $p=1$, together with (22), inequality (8) reduces to

$$V(K)^{2n-1} \leqslant \frac{n^{2n}}{\pi^n} \exp\Big(\int_{S^{2n-1}} \log V(K,[2n-1];D\cdot v)^2 d\mu(v)\Big), \tag{9}$$

where $D\cdot v := \{cv : c\in D\}$ and D is the unit disk in $\mathbb{C}$. Inequality (3) now follows from (9) by taking the *basic* complex cross measure μ, which is a complex cross measure such that $\operatorname{supp}\mu = \{\operatorname{span}\{e_1,e_2\}\cap S^{2n-1},\dots,\operatorname{span}\{e_{2n-1},e_{2n}\}\cap S^{2n-1}\}$. Note that a complex cross measure is just a rotation of the basic complex cross measure, since $\{v_1,v_1^\dagger,\dots,v_n,v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$.

2. Background materials

2.1. Elements of the L_p Brunn-Minkowski theory

We collect in this section some elements of the L_p Brunn-Minkowski theory, which has its origins in the work of Firey from the 1960s and has expanded rapidly over the last two decade since the remarkable works of Lutwak [28, 29]. For further details we refer the reader to [32, Chapter 9] and the references therein.

The Minkowski functional $\|\cdot\|_K$ of a convex body K in $\mathbb{R}^n$ is defined by $\|x\|_K = \min\{\lambda\geqslant 0 : x\in\lambda K\}$. In this case,

$$h(K,\cdot) = \|\cdot\|_{K^*}, \tag{10}$$

where the polar body K^* of K is defined by

$$K^* = \{x\in\mathbb{R}^n : \langle x,y\rangle \leqslant 1 \text{ for all } y\in K\}.$$

For $A\in \mathrm{GL}(\mathbb{R}^n)$, we have

$$(AK)^* = A^{-t}K^*, \tag{11}$$

where A^{-t} is the inverse of the transpose of A. Using the polar coordinate formula, it is easy to see that the volume of a convex body K in $\mathbb{R}^n$ is given by

$$V(K) = \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \exp(-\|x\|_K^p)dx, \tag{12}$$

where the integral is with respect to Lebesgue measure on $\mathbb{R}^n$.

For $p\geqslant 1$ and $\varepsilon>0$, the L_p Minkowski-Firey combination $K+_p\varepsilon\cdot L$ of convex bodied K,L is the convex body whose support function is given by

$$h(K+_p\varepsilon\cdot L,\cdot)^p = h(K,\cdot)^p+\varepsilon h(L,\cdot)^p.$$

The L_p mixed volume $V_p(K,L)$ of K,L was defined in [28] by

$$V_p(K,L) = \frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K+_p\varepsilon\cdot L)-V(K)}{\varepsilon}.$$

In particular, $V_p(K,K) = V(K)$. It was shown in [28] that for convex bodies K,L, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} so that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u), \tag{13}$$

where $dS_p(K,\cdot) = h(K,\cdot)^{1-p} dS(K,\cdot)$ is the L_p surface area measure of K and $dS(K,\cdot)$ is the classical surface area measure of K. Recall that for a Borel set $\omega \subset S^{n-1}$, $S(K,\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω.

The L_p Minkowski inequality [28] states that for convex bodies K,L,

$$V_p(K,L)^n \geqslant V(K)^{n-p} V(L)^p, \tag{14}$$

with equality if and only if K and L are dilates when $p > 1$, and if and only if K and L are homothetic (i.e., they coincide up to translations and dilatations) when $p = 1$.

2.2. Complex isotropic measures

The unit sphere $\{x \in \mathbb{C}^n : \|x\| = 1\}$ of $\mathbb{C}^n$ is denoted by S^{2n-1}. Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we can say that a convex body K is R_θ-invariant if for each $\theta \in [0,2\pi]$ and each $x = (x_{11}, x_{12}, \ldots x_{n1}, x_{n2}) \in \mathbb{R}^{2n}$,

$$\|x\|_K = \|R_\theta(x_{11},x_{12}),\ldots,R_\theta(x_{n1},x_{n2})\|_K,$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We say a measure (or a function) on S^{2n-1} is R_θ-invariant if it assumes the same value on a set (or a point) and its R_θ image for each $\theta \in [0,2\pi]$. For $\xi \in \mathbb{C}^n$ such that $\|\xi\| = 1$, denote by

$$H_\xi = \left\{ x \in \mathbb{C}^n : \langle x,\xi\rangle_c = \sum_{k=1}^{n} x_k \overline{\xi_k} = 0 \right\}$$

the complex hyperplane through the origin perpendicular to ξ. Under the mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ is a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11},\xi_{12},\ldots,\xi_{n1},\xi_{n2}) \quad \text{and} \quad \xi^\dagger = (-\xi_{12},\xi_{11},\ldots,-\xi_{n2},\xi_{n1}).$$

The complex isotropic measure μ defined in (4) has the following properties (see [19]):

- the complex isotropic measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$.
- $\mu(S^{2n-1}) = n$.

2.3. Generalized $\ell_p(\mathbb{C}^n)$-balls

Let $B_p(\mathbb{C}^n)$ denote the unit ball of $\ell_p(\mathbb{C}^n)$-space, understood as

$$B_p(\mathbb{C}^n) = \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^{n} [x_{k1}^2 + x_{k2}^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$,

$$B_\infty(\mathbb{C}^n) = \{x \in \mathbb{R}^{2n} : [x_{k1}^2 + x_{k2}^2]^{\frac{1}{2}} \leqslant 1, \text{ for all } k = 1, \dots, n\}.$$

For $n, p \in [1, \infty)$, denote by $\kappa_{2n}(p)$ the volume of the unit ball of $\ell_p(\mathbb{C}^n)$ (see [19, Proposition 6.1]), which equals to

$$\kappa_{2n}(p) = \frac{\pi^n (\Gamma(1 + \frac{2}{p}))^n}{\Gamma(1 + \frac{2n}{p})}.$$

Recall that a generalized $\ell_p(\mathbb{C}^n)$-ball $B_{p,\alpha}(\mathbb{C}^n)$ formed by the complex cross measure μ is defined in (6) and (7). Let $A = \operatorname{diag}\{\alpha_1^{1/p}, \alpha_1^{1/p}, \dots, \alpha_n^{1/p}, \alpha_n^{1/p}\}$. Since there exists $U \in \mathrm{O}(\mathbb{R}^{2n})$ such that $v_k = Ue_{2k-1}$ and $v_k^\dagger = Ue_{2k}$ for $k = 1, \dots, n$, we have

$$\begin{aligned}
B_{p,\alpha}(\mathbb{C}^n) &= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^{n} \alpha_k [\langle x, v_k\rangle^2 + \langle x, v_k^\dagger\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\
&= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^{n} \alpha_k [\langle x, Ue_{2k-1}\rangle^2 + \langle x, Ue_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\
&= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^{n} \alpha_k [\langle U^{-1}x, e_{2k-1}\rangle^2 + \langle U^{-1}x, e_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\
&= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^{n} [\langle AU^{-1}x, e_{2k-1}\rangle^2 + \langle AU^{-1}x, e_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\
&= \Big\{UA^{-1}x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^{n} [\langle x, e_{2k-1}\rangle^2 + \langle x, e_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\
&= UA^{-1}B_p(\mathbb{C}^n). \qquad (15)
\end{aligned}$$

Then we immediately get

$$V(B_{p,\alpha}(\mathbb{C}^n)) = V(UA^{-1}B_p(\mathbb{C}^n)) = V(B_p(\mathbb{C}^n))\Big(\prod_{k=1}^{n} \alpha_k\Big)^{-\frac{2}{p}}. \qquad (16)$$

It follows from (10) and (6) that

$$h((B_{p,\alpha}(\mathbb{C}^n))^*, x) = \Big(\sum_{k=1}^{n} \alpha_k |\langle x, v_k\rangle_c|^p\Big)^{\frac{1}{p}}. \qquad (17)$$

Moreover, for $p>1$, by (15), (11) and [19, Proposition 2.1], we have

$$
\begin{aligned}
(B_{p,\alpha}(\mathbb{C}^n))^* &= (UA^{-1}B_p(\mathbb{C}^n))^* = UA^t(B_p(\mathbb{C}^n))^* = UA^tB_{p^*}(\mathbb{C}^n)\\
&= \Big\{UA^tx\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n[\langle x,e_{2k-1}\rangle^2+\langle x,e_{2k}\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= \Big\{x\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n[\langle A^{-t}U^{-1}x,e_{2k-1}\rangle^2+\langle A^{-t}U^{-1}x,e_{2k}\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= \Big\{x\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n[\langle x,UA^{-1}e_{2k-1}\rangle^2+\langle x,UA^{-1}e_{2k-1}\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= \Big\{x\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n \alpha_k^{-p^*/p}[\langle x,v_k\rangle^2+\langle x,v_k^\dagger\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n).
\end{aligned}
\tag{18}
$$

For $p=1$, by the same way, we have

$$
(B_{1,\alpha}(\mathbb{C}^n))^* = B_{\infty,\alpha^{-1}}(\mathbb{C}^n). \tag{19}
$$

Then, from (16) we obtain, for $p>1$,

$$
V((B_{p,\alpha}(\mathbb{C}^n))^*) = V(B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)) = V(B_{p^*}(\mathbb{C}^n))\Big(\prod_{k=1}^n\alpha_k\Big)^{\frac{2}{p}}, \tag{20}
$$

and for $p=1$,

$$
V((B_{1,\alpha}(\mathbb{C}^n))^*) = V(B_{\infty,\alpha^{-1}}(\mathbb{C}^n)) = V(B_\infty(\mathbb{C}^n))\Big(\prod_{k=1}^n\alpha_k\Big)^2. \tag{21}
$$

2.4. Complex L_p projection bodies

In recent years, the study of varieties of convex bodies in $\mathbb{C}^n$ has received considerable attention; see, e.g., [1, 2, 3, 4, 5, 6, 7, 13, 14, 21, 22, 23, 31]. For example, the notion of the complex projection body was introduced by Abardia and Bernig [3] in 2011: for a convex body $K\subset\mathbb{C}^n$ and a convex body $C\subset\mathbb{C}$, the complex projection body Π^CK is the convex body whose support function is defined by

$$
h(\Pi^CK,v) = V(K,[2n-1];C\cdot v), \quad v\in\mathbb{C}^n,
$$

where $C\cdot v := \{cv : c\in C\}\subset\mathbb{C}^n$. Obviously, the R_θ-invariant complex projection body Π^DK can be defined by letting C be a unit disk D in $\mathbb{C}$; i.e.,

$$\begin{aligned} h(\Pi^D K, v) &= V(K,[2n-1];D\cdot v) \\ &= \frac{1}{2n}\int_{S^{2n-1}} h(D\cdot v,u)dS(K,u) \\ &= \frac{1}{2n}\int_{S^{2n-1}} \sup_{\theta\in[0,2\pi]} \{Re\langle e^{i\theta}v,u\rangle_c\}dS(K,u) \\ &= \frac{1}{2n}\int_{S^{2n-1}} \sup_{\theta\in[0,2\pi]} \{Re(e^{i\theta}\langle v,u\rangle_c)\}dS(K,u) \\ &= \frac{1}{2n}\int_{S^{2n-1}} |\langle v,u\rangle_c|dS(K,u), \end{aligned} \tag{22}$$

for every $v \in \mathbb{C}^n$. For $p \geqslant 1$, the R_θ-invariant L_p complex projection body $\Pi_p^D K$ of a convex body K in $\mathbb{C}^n$ can be defined by

$$h(\Pi_p^D K, v) = \Big(\frac{1}{2n}\int_{S^{2n-1}} |\langle v,u\rangle_c|^p dS_p(K,u)\Big)^{\frac{1}{p}}, \quad v\in\mathbb{C}^n. \tag{23}$$

The fact that $h(\Pi_p^D K, v)$ is the support function of a convex body in $\mathbb{R}^{2n}$ can be verified as in [19, Theorem 4.3].

3. Proof of the main result

Assume that the measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$. Let $\alpha : S^{2n-1} \to (0,+\infty)$ be a R_θ-invariant positive continuous function. For $p \geqslant 1$, we define the *complex L_p zonoid* $Z_{p,\alpha}(\mu)$ with generating measure $\alpha d\mu$ as the R_θ-invariant convex body in $\mathbb{C}^n$, in terms of its support function, for $u \in S^{2n-1}$,

$$\begin{aligned} h(Z_{p,\alpha}(\mu),u) &= \Big(\int_{S^{2n-1}} |\langle u,v\rangle_c|^p \alpha(v)d\mu(v)\Big)^{\frac{1}{p}} \\ &= \Big(\int_{S^{2n-1}} \big\|u|\mathrm{span}\{v,v^\dagger\}\big\|^p \alpha(v)d\mu(v)\Big)^{\frac{1}{p}}. \end{aligned} \tag{24}$$

Here $\big\|u|\mathrm{span}\{v,v^\dagger\}\big\|$ is the length of the orthogonal projection of u onto the 2-dimensional subspace $\mathrm{span}\{v,v^\dagger\}$. The fact that $h(Z_{p,\alpha}(\mu),u)$ is the support function of a convex body in $\mathbb{R}^{2n}$ can be verified as in [19, Theorem 4.3].

In particular, if μ is a complex cross measure, then we may assume that $\mathrm{supp}\mu = \{\mathrm{span}\{v_1,v_1^\dagger\}\cap S^{2n-1},\ldots,\mathrm{span}\{v_n,v_n^\dagger\}\cap S^{2n-1}\}$. It was shown in [19, Lemma 4.1] that $\mu(\mathrm{span}\{v_k,v_k^\dagger\}\cap S^{2n-1}) = 1$ for $1 \leqslant k \leqslant n$. Denote $\alpha(v_k) =: \alpha_k > 0$. By (24), (17), (18) and (19), we have, for $p > 1$,

$$\begin{aligned} h(Z_{p,\alpha}(\mu),x) &= \Big(\sum_{k=1}^n \alpha(v_k)|\langle x,v_k\rangle_c|^p\Big)^{\frac{1}{p}} \\ &= h((B_{p,\alpha}(\mathbb{C}^n))^*,x) = h(B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n),x), \end{aligned} \tag{25}$$

and

$$h(Z_{1,\alpha}(\mu),x) = h((B_{1,\alpha}(\mathbb{C}^n))^*,x) = h(B_{\infty,\alpha^{-1}}(\mathbb{C}^n),x), \tag{26}$$

for each $x \in \mathbb{R}^{2n}$. From (20) and (21), we get, for $p > 1$,

$$V(Z_{p,\alpha}(\mu)) = V(B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)) = V(B_{p^*}(\mathbb{C}^n))\Big(\prod_{k=1}^{n}\alpha_k\Big)^{\frac{2}{p}}, \tag{27}$$

and

$$V(Z_{1,\alpha}(\mu)) = V(B_{\infty,\alpha^{-1}}(\mathbb{C}^n)) = V(B_\infty(\mathbb{C}^n))\Big(\prod_{k=1}^{n}\alpha_k\Big)^{2}. \tag{28}$$

The following particular case of multidimensional reverse Brascamp-Lieb inequality [10] is needed.

LEMMA 1. *Suppose* $v_1,\dots,v_m \in S^{2n-1}$ *and* $c_1,\dots,c_m > 0$ *such that*

$$\sum_{k=1}^{m} c_k\big\|x|\mathrm{span}\{v_k,v_k^\dagger\}\big\|^2 = \|x\|^2 \quad \textit{for every } x \in \mathbb{R}^{2n}. \tag{29}$$

Then for all integrable functions $f_i : \mathrm{span}\{v_k,v_k^\dagger\} \to [0,\infty)$, $1 \leqslant k \leqslant m$,

$$\int_{\mathbb{R}^{2n}}^{*} \sup\Big\{\prod_{k=1}^{m} f_k(y_k)^{c_k} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \mathrm{span}\{v_k,v_k^\dagger\}\Big\}dx \geqslant \prod_{k=1}^{m}\Big(\int_{\mathrm{span}\{v_k,v_k^\dagger\}} f_k\Big)^{c_k}.$$

The following lemma extends Theorem 6.5 in [19].

LEMMA 2. *Suppose* $p \geqslant 1$ *and* α *is a* R_θ*-invariant continuous positive function on* S^{2n-1}. *If* μ *is a complex isotropic measure on* S^{2n-1}, *then*

$$V(Z_{p,\alpha}(\mu)) \geqslant V(B_{p^*}(\mathbb{C}^n))\Big(\exp\int_{S^{2n-1}}\log\alpha(v)d\mu(v)\Big)^{\frac{2}{p}}, \tag{30}$$

with equality if μ *is a complex cross measure on* S^{2n-1}.

Proof. Suppose the measure $\mu = \sum_{k=1}^{m} c_k\delta_{v_k}$ is a discrete complex isotropic measure on S^{2n-1}. Then the complex isotropic condition (5) is just the condition (29). Write $\alpha(v_k) =: \alpha_k > 0$.

Case $p = 1$: By (24) and the fact that $\big\|x|\mathrm{span}\{v_k,v_k^\dagger\}\big\| = h(B_2^{2n}|\mathrm{span}\{v_k,v_k^\dagger\},x)$, we have, for every $x \in \mathbb{R}^{2n}$,

$$\begin{aligned}
h(Z_{1,\alpha}(\mu),x) &= \sum_{k=1}^{m}\alpha_k c_k\big\|x|\mathrm{span}\{v_k,v_k^\dagger\}\big\| \\
&= \sum_{k=1}^{m}\alpha_k c_k h(B_2^{2n}|\mathrm{span}\{v_k,v_k^\dagger\},x) \\
&= h\Big(\sum_{k=1}^{m} c_k\alpha_k B_2^{2n}|\mathrm{span}\{v_k,v_k^\dagger\},x\Big).
\end{aligned}$$

Hence

$$Z_{1,\alpha}(\mu)=\Big\{x\in\mathbb{R}^{2n}:x=\sum_{k=1}^{m}c_ky_k,y_k\in\alpha_kB_2(\mathbb{R}^{2n})|\mathrm{span}\{v_k,v_k^{\dagger}\}\Big\}. \tag{31}$$

Define functions $f_k:\mathrm{span}\{v_k,v_k^{\dagger}\}\to[0,\infty),1\leqslant k\leqslant m$, by

$$f_k(y)=\mathbf{1}_{[0,\alpha_k]}(\|y\|).$$

From (31), Lemma 1 and the fact that $\mu(S^{2n-1})=\sum_{k=1}^{m}c_k=n$, we obtain

$$\begin{aligned}V(Z_{1,\alpha}(\mu))&=\int_{\mathbb{R}^{2n}}\sup\Big\{\prod_{k=1}^{m}\mathbf{1}_{[0,\alpha_k]}(\|y_k\|)^{c_k}:x=\sum_{k=1}^{m}c_ky_k,y_k\in\mathrm{span}\{v_k,v_k^{\dagger}\}\Big\}dx\\&=\int_{\mathbb{R}^{2n}}\sup\Big\{\prod_{k=1}^{m}f_k(y_k)^{c_k}:x=\sum_{k=1}^{m}c_ky_k,y_k\in\mathrm{span}\{v_k,v_k^{\dagger}\}\Big\}dx\\&\geqslant\prod_{k=1}^{m}\Big(\int_{\mathrm{span}\{v_k,v_k^{\dagger}\}}f_k\Big)^{c_k}=\prod_{k=1}^{m}\Big(\int_{\mathrm{span}\{v_k,v_k^{\dagger}\}}\mathbf{1}_{[0,\alpha_k]}(\|x\|)dx\Big)^{c_k}\\&=\pi^n\prod_{k=1}^{m}\alpha_k^{2c_k}.\end{aligned}$$

Case $p>1$: We claim that

$$\|x\|_{Z_{p,\alpha}(\mu)}^{p^*}\leqslant\inf\Big\{\sum_{k=1}^{m}\alpha_k^{1-p^*}c_k(r_{k1}^2+r_{k2}^2)^{p^*/2}:\sum_{k=1}^{m}c_k(r_{k1}v_k+r_{k2}v_k^{\dagger})=x\Big\}. \tag{32}$$

In fact, let $x=\sum_{k=1}^{m}c_k(r_{k1}v_k+r_{k2}v_k^{\dagger})$. By Hölder's inequality twice, (24) and (10), we have

$$\begin{aligned}\langle x,y\rangle&=\sum_{k=1}^{m}c_k(r_{k1}\langle y,v_k\rangle+r_{k2}\langle y,v_k^{\dagger}\rangle)\\&\leqslant\sum_{k=1}^{m}c_k(r_{k1}^2+r_{k2}^2)^{\frac12}(\langle y,v_k\rangle^2+\langle y,v_k^{\dagger}\rangle^2)^{\frac12}\\&\leqslant\Big(\sum_{k=1}^{m}\alpha_k^{1-p^*}c_k(r_{k1}^2+r_{k2}^2)^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\Big(\sum_{k=1}^{m}\alpha_kc_k(\langle y,v_k\rangle^2+\langle y,v_k^{\dagger}\rangle^2)^{\frac{p}{2}}\Big)^{\frac1p}\\&=\Big(\sum_{k=1}^{m}\alpha_k^{1-p^*}c_k(r_{k1}^2+r_{k2}^2)^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\|y\|_{(Z_{p,\alpha}(\mu))^*}.\end{aligned}$$

Let $m_x=(\sum_{k=1}^{m}\alpha_k^{1-p^*}c_k(r_{k1}^2+r_{k2}^2)^{p^*/2})^{1/p^*}$. Thus, the fact that $y/\|y\|_{(Z_{p,\alpha}(\mu))^*}$ lies on the boundary of the convex body $(Z_{p,\alpha}(\mu))^*$ implies

$$\frac{x}{m_x}\in Z_{p,\alpha}(\mu).$$

Hence

$$\Big\|\frac{x}{m_x}\Big\|_{Z_{p,\alpha}(\mu)} \leqslant 1.$$

That is,

$$\|x\|_{Z_{p,\alpha}(\mu)} \leqslant \Big(\sum_{k=1}^{m} \alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{p^*/2}\Big)^{1/p^*},$$

for $x = \sum_{k=1}^{m} c_k(r_{k1}v_k + r_{k2}v_k^{\dagger})$. Taking the infimum yields the claim.

Define functions $f_k : \operatorname{span}(v_k, v_k^{\dagger}) \to [0,\infty), 1 \leqslant k \leqslant m$, by

$$f_k(y) = \exp(-\alpha_k^{1-p^*}\|y\|^{p^*}).$$

From (12), (32), Lemma 1 and the fact that $\mu(S^{2n-1}) = \sum_{k=1}^{m} c_k = n$, we have

$$\begin{aligned}
&\Gamma\Big(1+\frac{2n}{p^*}\Big)V(Z_{p,\alpha}(\mu))\\
&= \int_{\mathbb{R}^{2n}} \exp(-\|x\|^{p^*}_{Z_{p,\alpha}(\mu)})dx\\
&\geqslant \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{k=1}^{m} \exp(-\alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{\frac{p^*}{2}}) : \sum_{k=1}^{m} c_k(r_{k1}v_k + r_{k2}v_k^{\dagger}) = x\Big\}dx\\
&= \int_{\mathbb{R}^{2n}} \sup\Big\{\prod_{k=1}^{m} f_k(y_k)^{c_k} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \operatorname{span}\{v_k, v_k^{\dagger}\}\Big\}dx\\
&\geqslant \prod_{k=1}^{m}\Big(\int_{\operatorname{span}\{v_k,v_k^{\dagger}\}} f_i\Big)^{c_k}\\
&= \prod_{k=1}^{m}\Big(\int_{\operatorname{span}\{v_k,v_k^{\dagger}\}} e^{-\alpha_k^{1-p^*}\|x\|^{p^*}}dx\Big)^{c_k} = \Big(\pi\Gamma\Big(1+\frac{2}{p^*}\Big)\Big)^n\Big(\prod_{k=1}^{m}\alpha_k^{c_k}\Big)^{\frac{2}{p}}.
\end{aligned}$$

Therefore, $V(Z_{p,\alpha}(\mu)) \geqslant \kappa_{2n}(p^*)\Big(\prod_{k=1}^{m}\alpha_k^{c_k}\Big)^{\frac{2}{p}}$.

Now let μ be an arbitrary complex isotropic measure on S^{2n-1}. As shown in [19, Theorem 3.2], there exists a sequence $\mu_l, l \in \mathbb{N}$, of discrete complex isotropic measures such that μ_l converges weakly to μ as $l \to \infty$. Thus,

$$\lim_{l\to\infty} h(Z_{p,\alpha}(\mu_l),u) = h(Z_{p,\alpha}(\mu),u), \quad u \in S^{2n-1}.$$

Note that the pointwise convergence of support functions implies the convergence of the corresponding convex bodies in the Hausdorff metric (see e.g., [32]). Then the continuity of volume and the fact that

$$\Big(\prod_{k=1}^{m}\alpha_k^{c_k}\Big)^{\frac{2}{p}} = \Big(\exp\Big(\sum_{k=1}^{n} c_k \log\alpha_k\Big)\Big)^{\frac{2}{p}}$$

give inequality (30).

If μ is a complex cross measure on S^{2n-1} such that $\operatorname{supp}\mu = \{\operatorname{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \ldots, \operatorname{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\}$, then the equality of (30) follows from (27) and (28). $\square$

Finally, we complete the proof of Theorem 1.

THEOREM 2. *Suppose $p \geqslant 1$ and K is a convex body in $\mathbb{R}^{2n}$. If μ is a complex isotropic measure on S^{2n-1}, then*

$$V(K)^{2n-p} \leqslant n^{2n}\kappa_{2n}(p^*)^{-p}\exp\Big(\int_{S^{2n-1}} \log h(\Pi_p^D K, v)^{2p} d\mu(v)\Big). \tag{33}$$

In addition, if μ is a complex cross measure on S^{2n-1}, then equality in (33) holds for $p > 1$ if and only if K is a generalized $\ell_{p^*}(\mathbb{C}^n)$-ball formed by μ, and equality in (33) holds for $p = 1$ if and only if K is a polydisc formed by μ (up to translations).

Proof. Let

$$\alpha(v) = h(\Pi_p^D K, v)^{-p} = \Big(\frac{1}{2n}\int_{S^{2n-1}} |\langle v, u\rangle_c|^p dS_p(K,u)\Big)^{-1}, \tag{34}$$

for $v \in \operatorname{supp}\mu$. From (14), (13), the definition of $Z_{p,\alpha}(\mu)$ (24), Fubini's theorem, (34) and the fact that $\mu(S^{2n-1}) = n$, we have

$$\begin{aligned}
V(K)^{2n-p} &\leqslant V(Z_{p,\alpha}(\mu))^{-p} V_p(K, Z_{p,\alpha}(\mu))^{2n} \\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\frac{1}{2n}\int_{S^{2n-1}} h(Z_{p,\alpha}(\mu), u)^p dS_p(K,u)\Big)^{2n} \\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\frac{1}{2n}\int_{S^{2n-1}} \Big(\int_{S^{2n-1}} |\langle u, v\rangle_c|^p \alpha(v) d\mu(v)\Big) dS_p(K,u)\Big)^{2n} \\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}} |\langle u, v\rangle_c|^p dS_p(K,u)\alpha(v) d\mu(v)\Big)^{2n} \\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\int_{S^{2n-1}} h(\Pi_p^D K, v)^p \alpha(v) d\mu(v)\Big)^{2n} \\
&= n^{2n} V(Z_{p,\alpha}(\mu))^{-p}.
\end{aligned}$$

By Lemma 2,

$$\begin{aligned}
V(K)^{2n-p} \leqslant n^{2n} V(Z_{p,\alpha}(\mu))^{-p} &\leqslant n^{2n}\kappa_{2n}(p^*)^{-p}\exp\Big(\int_{S^{2n-1}} \log \alpha^{-2}(v) d\mu(v)\Big) \\
&= n^{2n}\kappa_{2n}(p^*)^{-p}\exp\Big(\int_{S^{2n-1}} \log h(\Pi_p^D K, v)^{2p} d\mu(v)\Big),
\end{aligned} \tag{35}$$

which is the desired inequality.

For the equality conditions of (35), by the L_p Minkowski inequality (14), equality of the first inequality in (35) holds if and only if K and $Z_{p,\alpha}(\mu)$ are dilates when $p > 1$ (K and $Z_{p,\alpha}(\mu)$ are homothetic when $p = 1$). If μ is a complex cross measure on S^{n-1}, Lemma 2 implies that equality of the second inequality in (35) holds and $Z_{p,\alpha}(\mu)$ is the generalized $\ell_{p^*}(\mathbb{C}^n)$-ball $B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)$ formed by μ. Hence K is a dilation of the generalized $\ell^n_{p^*}$-ball formed by the cross measure μ, which is still the generalized $\ell^n_{p^*}$-ball formed by μ when $p>1$ (K is a polydisc formed by μ up to translations when $p=1$).

Conversely, we will show that, when $p>1$, equality in (35) holds if K is the generalized $\ell^n_{p^*}(\mathbb{C}^n)$-ball formed by μ; i.e., there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$K=\Big\{x\in\mathbb{R}^n:\Big(\sum_{k=1}^n \alpha_k|\langle x,v_k\rangle_c|^{p^*}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}, \tag{36}$$

where $\operatorname{supp}\mu=\{\operatorname{span}\{v_1,v_1^\dagger\}\cap S^{2n-1},\dots,\operatorname{span}\{v_n,v_n^\dagger\}\cap S^{2n-1}\}$ and $\{v_1,v_1^\dagger\dots,v_n,v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$. From (35), it is sufficient to verify that K and $Z_{p,\alpha}(\mu)$ are dilates. From (15), we have

$$K=B_{p^*,\alpha}(\mathbb{C}^n)=UA^{-1}B_{p^*}(\mathbb{C}^n),$$

where $A=\operatorname{diag}\{\alpha_1^{1/p^*},\alpha_1^{1/p^*},\dots,\alpha_n^{1/p^*},\alpha_n^{1/p^*}\}$ and $U\in\mathrm{O}(\mathbb{R}^{2n})$ such that $v_k=Ue_{2k-1}$, $v_k^\dagger=Ue_{2k}$ for $k=1,\dots,m$. From (34) and [30, Proposition 1.2], we get

$$\begin{aligned}
\alpha(v_k)&=h(\Pi_p^DK,v_k)^{-p}=h(\Pi_p^D(UA^{-1}B_{p^*}(\mathbb{C}^n)),v_k)^{-p}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}|\langle v_k,u\rangle_c|^p dS_p(UA^{-1}B_{p^*}(\mathbb{C}^n),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}[\langle v_k,u\rangle^2+\langle v_k^\dagger,u\rangle^2]^{\frac{p}{2}} dS_p(UA^{-1}B_{p^*}(\mathbb{C}^n),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}[\langle v_k,UA^tu\rangle^2+\langle v_k^\dagger,UA^tu\rangle^2]^{\frac{p}{2}} dS_p(B_{p^*}(\mathbb{C}^n)),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}[\langle AU^tv_k,u\rangle^2+\langle AU^tv_k^\dagger,u\rangle^2]^{\frac{p}{2}} dS_p(B_{p^*}(\mathbb{C}^n)),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}\alpha_k^{p/p^*}[\langle e_{2k-1},u\rangle^2+\langle e_{2k},u\rangle^2]^{\frac{p}{2}} dS_p(B_{p^*}(\mathbb{C}^n)),u)\Big)^{-1}\\
&=h(\Pi_p^D(B_{p^*}(\mathbb{C}^n)),e_{2k-1})^{-p}\alpha_k^{-p/p^*}
\end{aligned}$$

for every $k=1,\dots,n$. Notice that $h(\Pi_p^D(B_{p^*}(\mathbb{C}^n)),e_{2k-1})^{-p}$ is a constant for all $k=1,\dots,n$. Thus, there exists a constant $c>0$ such that $\alpha(v_k)=c\alpha_k^{-p/p^*}$ for every $k=1,\dots,n$. Now, it follows from (25) and (36) that

$$\begin{aligned}
Z_{p,\alpha}(\mu)&=B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)\\
&=\Big\{x\in\mathbb{R}^n:\Big(\sum_{k=1}^n\alpha(v_k)^{-p^*/p}|\langle x,v_k\rangle_c|^{p^*}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&=\Big\{x\in\mathbb{R}^n:\Big(\sum_{k=1}^n c^{-p^*/p}\alpha_k|\langle x,v_k\rangle_c|^{p^*}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}=c^{\frac{1}{p}}K.
\end{aligned}$$

That is, K and $Z_{p,\alpha}(\mu)$ are dilates when $p>1$. When $p=1$, the proof, together with the observation that $\Pi^D(K+v_0)=\Pi^DK$ for every $v_0\in\mathbb{R}^{2n}$, is the same. □

Acknowledgements. The authors are indebted to the referee for many valuable suggestions and comments.

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(Received September 11, 2016)

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