Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 511–516 doi:10.7153/mia-2018-21-36

ON THE TRIANGLE EQUALITY

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(*Communicated by M. Praljak*)

Abstract. Given a norm on a vector space a natural question is when for given vectors the triangle equality holds, i.e., when the norm of a sum of vectors is equal to the sum of norms of vectors. It is well known that if the space is strictly convex, then the triangle equality holds if and only if the vectors belong to the same half-line emanating from the origin. In the note we give necessary and sufficient geometric conditions for the triangle equality in an arbitrary normed vector space. The conditions can be checked once the unit sphere is known.

1. Introduction

The problem when for given vectors x,y of a normed vector space X the *triangle equality*

$$\|x+y\| = \|x\| + \|y\| \tag{1}$$

holds has been studied for different types of vector spaces, see [2, 5, 6, 7] and references within. For instance, it is well known that if X is strictly convex, i.e., $\frac{x+y}{2}$ belongs to the open unit ball $B(1)$ for any x,y from the unit sphere $S(1)$, $x \neq y$, then (1) holds for nonzero vectors $x,y \in X$ if and only if $\frac{x}{\|x\|} = \frac{y}{\|y\|}$. In particular, the triangle equality in every inner product space is characterized in this way. In Banach space setting Abramovich et al. noted in [1, Lemma 2.1] that (1) implies that for any fixed $\lambda > 0$ and $\mu > 0$ one has $\|\lambda x + \mu y\| = \lambda\|x\| + \mu\|y\|$ (see also [5, Theorem 1.6]) and Nakamoto and Takashi proved in [6, Theorem 2] that (1) holds for nonzero vectors $x,y \in X$ if and only if there exists an extremal point f in the closed unit ball $\overline{B}^*(1)$ of the dual space X^* such that $f(x) = \|x\|$ and $f(y) = \|y\|$. However we were not able to find in the existing literature simple geometric necessary and sufficient conditions for the triangle equality (1) in an arbitrary normed vector space.

In this note we give necessary and sufficient geometric conditions for the triangle equality (1) and some of its applications. Namely we prove the following

THEOREM 1. *Let* $(X, \|\cdot\|)$ *be a complex normed vector space or a real asymmetric normed vector space and* $x,y \in X \setminus \{0\}$. *Then the following conditions are equivalent:*

(a) *the triangle equality* (1) *holds;*

(b) *the unit sphere* $S(1)$ *contains the segment* $[\frac{x}{\|x\|}, \frac{y}{\|y\|}]$ *(possibly reduced to a point);*

Mathematics subject classification (2010): 46B20.
Keywords and phrases: Triangle equality, asymmetric normed spaces.

(c) $x+y\neq 0$ *and* $\frac{x+y}{\|x+y\|}=\alpha\frac{x}{\|x\|}+(1-\alpha)\frac{y}{\|y\|}$ *with some* $0<\alpha<1$.

2. Normed vector spaces

Let X be a complex vector space. By a *norm* on X we mean a function $\|\cdot\|: X\to[0,\infty)$ satisfying

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|\lambda x\|=|\lambda|\|x\|$ for any $\lambda\in\mathbb{C}$ and $x\in X$;

(iii) $\|x+y\|\leqslant\|x\|+\|y\|$ for any $x,y\in X$.

The condition (ii) is called the *homogeneity* and (iii) the *triangle inequality*.

If X is a real vector space, then one can replace the condition (ii) by a weaker one

(ii') $\|\lambda x\|=\lambda\|x\|$ for any $\lambda>0$ and $x\in X$.

In this case one says that X is an *asymmetric normed vector space*. Such spaces were introduced as a tool for the constructions of models for the complexity analysis of algorithms in Computer Science, [10], and used in the study of some extremal problems, [8]. Basic theory of these spaces can be found in a monograph by Cobzas, [3].

Given a norm $\|\cdot\|$ on X a translation invariant metric ρ on X is defined by

$$\rho(x,y)=\|x-y\|.$$

$\rho(x,y)$ can be interpreted as a distance from x to y, in particular, $\rho(x,0)=\|x\|$ is a distance from x to 0. Note here that, in asymmetric case $\rho(x,y)$ need not be equal to $\rho(y,x)$.

By an *open ball* of center $x\in X$ and radius $r>0$ we mean the set

$$B(x,r)=\{y\in X:\|y-x\|<r\}.$$

Respectively, $S(x,r)=\{y\in X:\|y-x\|=r\}$ is a *sphere* of center x and radius r. By properties $(i)-(iii)$ the unit ball $B(1)=B(0,1)$ is an open, bounded, convex, balanced (i.e., $\lambda B(1)\subset B(1)$ for any $|\lambda|\leqslant 1$), and absorbing (i.e., for each $x\in X$ there is $\lambda_x>0$ such that $\lambda x\in B(1)$ for all $|\lambda|\leqslant\lambda_x$) set. On the other hand if B is an open, bounded, convex, balanced, and absorbing subset of $\mathbb{C}^n$, then $\|x\|=\inf\{\lambda>0: x/\lambda\in B\}$ defines a norm on $\mathbb{C}^n$ for which $B(1)=B$, [4, Theorem 1.3.4]. In the real case the unit ball $B(1)$ is an open, bounded, convex, and absorbing set, and any such a subset B of $\mathbb{R}^n$ is the unit ball for some asymmetric norm.

3. The proof of Theorem 1

Proof. $(b)\Rightarrow(a)$. By homogeneity of the norm we can assume that $\|x\|+\|y\|=1$. (Indeed if $\|x\|+\|y\|=\lambda>0$, then putting $\widetilde{x}=x/\lambda$, $\widetilde{y}=y/\lambda$ we have $\|\widetilde{x}\|+\|\widetilde{y}\|=1$, $x/\|x\|=\widetilde{x}/\|\widetilde{x}\|$ and $y/\|y\|=\widetilde{y}/\|\widetilde{y}\|$.) So $\|x\|=\alpha$ and $\|y\|=1-\alpha$ with some

$0 < \alpha < 1$. Since by assumption for any $0 < \beta < 1$, $\|\beta \cdot x/\|x\| + (1-\beta)\cdot y/\|y\|\| = 1$ we get $\|x\| + \|y\| = 1 = \|\alpha \cdot x/\|x\| + (1-\alpha)\cdot y/\|y\|\| = \|x+y\|$.

$(a) \Rightarrow (b)$. Assume that $x/\|x\| \neq y/\|y\|$ and $S(1)$ does not contain the segment $[x/\|x\|, y/\|y\|]$. By the convexity of the closed unit ball $\overline{B}(1)$ there exists $0 < \alpha < 1$ such that $\|z_\alpha\| < 1$ where $z_\alpha = \alpha \cdot x/\|x\| + (1-\alpha)\cdot y/\|y\|$. Take a point $\zeta \in (x/\|x\|, z_\alpha]$. Then $\zeta = \beta \cdot x/\|x\| + (1-\beta)\cdot z_\alpha$ with some $0 \leqslant \beta < 1$. By the triangle inequality we get $\|\zeta\| \leqslant \beta + (1-\beta)\|z_\alpha\| < 1$. Analogously $\|\zeta\| < 1$ for any point $\zeta \in [z_\alpha, y/\|y\|)$. Hence for any $0 < \beta < 1$ we get

$$\left\|\beta \frac{x}{\|x\|} + (1-\beta)\frac{y}{\|y\|}\right\| < 1.$$

Putting $\beta = \frac{\|x\|}{\|x\|+\|y\|}$ in the above inequality we get

$$\left\|\frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|}\right\| < 1.$$

Hence $\|x+y\| < \|x\| + \|y\|$ and we get a contradiction.

$(a) \Rightarrow (c)$. The implication is clear since

$$\frac{x+y}{\|x+y\|} = \frac{\|x\|}{\|x+y\|}\frac{x}{\|x\|} + \frac{\|y\|}{\|x+y\|}\frac{y}{\|y\|} = \alpha \frac{x}{\|x\|} + (1-\alpha)\frac{y}{\|y\|} \tag{2}$$

with $0 < \alpha = \frac{\|x\|}{\|x+y\|} < 1$, and $\|x+y\| > 0$ if (a) holds.

$(c) \Rightarrow (a)$. If vectors x and y are linearly independent, then by (2) we get $\alpha = \frac{\|x\|}{\|x+y\|}$ and $1-\alpha = \frac{\|y\|}{\|x+y\|}$. So $1 = \frac{\|x\|}{\|x+y\|} + \frac{\|y\|}{\|x+y\|}$ and (a) follows. If x and y are linearly dependent it can be assumed that $y = \lambda x$ with $\lambda \neq 0$ and $\lambda \neq -1$. Then in the complex case we get $\frac{y}{\|y\|} = e^{\imath \arg \lambda}\frac{x}{\|x\|}$ and $\frac{x+y}{\|x+y\|} = e^{\imath \arg(1+\lambda)}\frac{x}{\|x\|}$. Hence (c) implies that

$$e^{\imath \arg(1+\lambda)} = \alpha + (1-\alpha)e^{\imath \arg \lambda}$$

with some $0 < \alpha < 1$ and this is possible only if $\lambda > 0$ in which case (1) clearly holds. In the real asymmetric case we get same conclusion when $\lambda > 0$. If $-1 < \lambda < 0$, then $\frac{y}{\|y\|} = \frac{-x}{\|-x\|}$ and $\frac{x+y}{\|x+y\|} = \frac{x}{\|x\|}$. If $\lambda < -1$, then $\frac{y}{\|y\|} = \frac{-x}{\|-x\|}$ and $\frac{x+y}{\|x+y\|} = \frac{-x}{\|-x\|}$. In both cases (c) implies that $\frac{x}{\|x\|} = \frac{-x}{\|-x\|}$ which is a contradiction since $x \neq 0$. □

4. A generalization

As a direct generalization of Theorem 1 we have

THEOREM 2. *Let $(X, \|\cdot\|)$ be a complex normed vector space or a real asymmetric normed vector space and $x_i \in X \setminus \{0\}$ for $i = 1,\ldots,N$ with $N \geqslant 2$. Then the equality*

$$\left\|\sum_{i=1}^{N} x_i\right\| = \sum_{i=1}^{N} \|x_i\| \tag{3}$$

holds if and only if the unit sphere $S(1)$ contains the convex hull of the set $\{x_i/\|x_i\|, i = 1,\ldots,N\}$.

Proof. For $N=2$ the statement is exactly the equivalence of (a) and (b) in Theorem 1. Assume that the unit sphere $S(1)$ contains the convex hull of the set $\{x_i/\|x_i\|, i=1,\cdots,N\}$ with $N \geqslant 3$. Then it contains the segment $[x_1/\|x_1\|, x_2/\|x_2\|]$. So by Theorem 1, $\|x_1+x_2\| = \|x_1\|+\|x_2\|$ and $\frac{x_1+x_2}{\|x_1+x_2\|} \in [\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}]$. Furthermore $S(1)$ contains the convex hull of the set $\{\frac{x_1+x_2}{\|x_1+x_2\|}, \frac{x_3}{\|x_3\|}, \cdots, \frac{x_N}{\|x_N\|}\}$. So by induction we get (3).

Now assume (3) and observe that it implies that for any $1 \leqslant m \leqslant N$ and $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant N$ we have

$$\Big\| \sum_{j=1}^{m} x_{i_j} \Big\| = \sum_{j=1}^{m} \|x_{i_j}\|. \tag{4}$$

Take a point z from the convex hull of the set $V := \{x_i/\|x_i\|, i=1,\ldots,N\}$. Then one can find $1 \leqslant m \leqslant N$, indices $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant N$ and numbers $\alpha_1 > 0, \ldots, \alpha_m > 0$ such that $\alpha_1 + \cdots + \alpha_m = 1$ and

$$z = \alpha_1 \frac{x_{i_1}}{\|x_{i_1}\|} + \cdots + \alpha_m \frac{x_{i_m}}{\|x_{i_m}\|}.$$

Assume that $\|z\| < 1$ and take ζ of the form

$$\zeta = \beta_1 \frac{x_{i_1}}{\|x_{i_1}\|} + \cdots + \beta_m \frac{x_{i_m}}{\|x_{i_m}\|} \tag{5}$$

with $\beta_1 > 0, \ldots, \beta_m > 0$ and $\beta_1 + \cdots + \beta_m = 1$. Set $\gamma = \min\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_m}{\alpha_m}\right)$. Since $\gamma > 0$, $\beta_j - \gamma\alpha_j \geqslant 0$ for $j = 1, \ldots, m$ and $\zeta = \gamma z + \sum_{j=1}^{m} (\beta_j - \gamma\alpha_j) x_{i_j}/\|x_{i_j}\|$ we get

$$\|\zeta\| \leqslant \gamma\|z\| + \sum_{j=1}^{m} (\beta_j - \gamma\alpha_j) \Big\| \frac{x_{i_j}}{\|x_{i_j}\|} \Big\| < \gamma + \sum_{j=1}^{m} \beta_j - \gamma \sum_{j=1}^{m} \alpha_j = 1.$$

Putting

$$\beta_j = \frac{\|x_{i_j}\|}{\|x_{i_1}\| + \cdots + \|x_{i_m}\|}, \quad j = 1, \ldots, m,$$

in (5) we get

$$\frac{\|\sum_{j=1}^{m} x_{i_j}\|}{\sum_{j=1}^{m} \|x_{i_j}\|} < 1$$

which gives a contradiction with (4). Hence $\|z\| = 1$ for any z from the convex hull of the set V, which means that $S(1)$ contains the convex hull of V. □

5. Corollaries

Recall that a point $z \in \overline{B}(0,r)$ is called an *extremal point* of $\overline{B}(0,r)$ if for any $x, y \in \overline{B}(0,r)$, $x \neq y$, and $\alpha \in [0,1]$ if $\alpha x + (1-\alpha) y = z$, then $\alpha = 0$ or $\alpha = 1$. Clearly extremal points belong to $S(0,r)$. Equivalently $z \in S(0,r)$ is an extremal point of $S(0,r)$ if $z = \alpha x + (1-\alpha) y$ with $x, y \in S(0,r)$ and $0 < \alpha < 1$, then $x = y = z$.

COROLLARY 1. *Let $x,y \in X \setminus \{0\}$ be such that $x+y$ be an extremal point of $S(0,r)$ with some $r>0$. Then $\|x+y\| = \|x\|+\|y\|$ if and only if $y = \lambda x$ with some $\lambda > 0$. Hence if X is strictly convex, then* (1) *holds for $x,y \neq 0$ if and only if $\frac{x}{\|x\|} = \frac{y}{\|y\|}$.*

Proof. Clearly if $y = \lambda x$ with some $\lambda > 0$, then $\|x+y\| = (1+\lambda)\|x\| = \|x\| + \|y\|$. Conversely, assume that $\|x+y\| = \|x\|+\|y\|$. Then by Theorem 1, $\frac{x+y}{\|x+y\|}$ belongs to the segment $[x/\|x\|, y/\|y\|] \subset S(1)$. But since $\frac{x+y}{\|x+y\|}$ is an extremal point of $S(1)$ we get $x/\|x\| = y/\|y\|$. Hence $y = \lambda x$ with $\lambda = \|y\|/\|x\|$.
To justify the last statement note that in a strictly convex space every point of $S(1)$ is extremal, [4, Theorem 2.1.4]. □

A frequently used norm on $\mathbb{C}^n$ or $\mathbb{R}^n$ is the l_p-norm, where $1 \leqslant p \leqslant \infty$. It is defined by

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup_{i=1,\ldots,n} |x_i| & \text{if } p = \infty. \end{cases}$$

The l_p-norm $\|\cdot\|_p$ satisfies conditions $(i)-(iii)$, [9]. If $1 < p < \infty$, then every point of $S(1)$ is extremal. So we get

COROLLARY 2. [9, Theorem 3.5] *Let $1 < p < \infty$ and $x,y \in \mathbb{C}^n$. Then $\|x+y\|_p = \|x\|_p + \|y\|_p$ if and only if $x = 0$ or $y = \lambda x$ with some $\lambda \geqslant 0$.*

If $p = 1$ Corollary 2 no longer holds. Indeed if $x = (1,0) \in \mathbb{R}^2$ and $y = (0,1) \in \mathbb{R}^2$, then $x+y = (1,1)$. So $2 = \|x\|_1 + \|y\|_1 = \|x+y\|_1$ but $y \neq \lambda x$ for any $\lambda \geqslant 0$. However it is easy to observe that the unit sphere $S(1)$ in $\mathbb{R}^n$ with l_1-norm contains the segment $[x/\|x\|, y/\|y\|]$ if and only if x and y belong to the same closed orthant. So we have

COROLLARY 3. *Let $x,y \in \mathbb{R}^n$. Then $\|x+y\|_1 = \|x\|_1 + \|y\|_1$ if and only if for every $i = 1,\ldots,n$, $x_i = 0$ or $y_i = 0$ or $\operatorname{sgn} x_i = \operatorname{sgn} y_i$.*

In $\mathbb{C}^n$ with l_1-norm we have

PROPOSITION 1. *Let $x,y \in \mathbb{C}^n$. Then $\|x+y\|_1 = \|x\|_1 + \|y\|_1$ if and only if for every $i = 1,\ldots,n$, $x_i = 0$ or $y_i = \lambda_i x_i$ with some $\lambda_i \geqslant 0$.*

Proof. By the triangle inequality for $x,y \in \mathbb{C}^n$ we have

$$\|x+y\|_1 = \sum_{i=1}^n |x_i + y_i| \leqslant \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|x\|_1 + \|y\|_1,$$

and the equality holds if and only if for every $i = 1,\ldots,n$, $|x_i + y_i| = |x_i| + |y_i|$, which is equivalent to the condition in Proposition 1. □

On $\mathbb{R}^n$ equipped with the l_∞-norm the unit ball is the cube Q with vertices at $(\varepsilon_1, \cdots, \varepsilon_n)$ where $\varepsilon_i \in \{1,-1\}$ for $i = 1,\cdots,n$. So we get

COROLLARY 4. *Let* $x, y \in \mathbb{R}^n$. *Then* $\|x+y\|_\infty = \|x\|_\infty + \|y\|_\infty$ *if and only if* $x = 0$ *or* $y = 0$ *or* $\frac{x}{\|x\|_\infty}$ *and* $\frac{y}{\|y\|_\infty}$ *belong to the same face of the cube* Q.

Finally let us describe conditions for the triangle equality on $\mathbb{C}^n$ with l_∞-norm.

PROPOSITION 2. *Let* $x, y \in \mathbb{C}^n$ *and let* $k, l \in \{1, \ldots, n\}$ *be such that for* $j = 1, \ldots, n$, $|x_j| \leqslant |x_k|$ *and* $|y_j| \leqslant |y_l|$. *Then* $\|x+y\|_\infty = \|x\|_\infty + \|y\|_\infty$ *if and only if* $x = 0$ *or there exists* $i \in \{1, \ldots, n\}$ *such that* $y_i = \lambda x_i$ *with some* $\lambda \geqslant 0$ *and* $|x_i| = |x_k|$ *and* $|y_i| = |y_l|$.

Proof. Take $x, y \in \mathbb{C}^n \setminus \{0\}$. Then by the assumption $\|x\|_\infty = |x_k|$ and $\|y\|_\infty = |y_l|$. Hence $\|x+y\|_\infty = \|x\|_\infty + \|y\|_\infty$ if and only if there exists $i \in \{1, \ldots, n\}$ such that $|x_i + y_i| = |x_k| + |y_l|$. Since $|x_i + y_i| \leqslant |x_i| + |y_i| \leqslant |x_k| + |y_l|$ the last equality holds if and only if $y_i = \lambda x_i$ with some $\lambda \geqslant 0$ and $|x_i| = |x_k|$ and $|y_i| = |y_l|$. $\square$

Acknowledgement. The author is thankful to the referee for pointing out some inaccuracies in the first version of the paper.

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(Received August 28, 2017)

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