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A NOTE ON THE HARDY—LITTLEWOOD INEQUALITIES FOR MULTILINEAR FORMS

ANTONIO GOMES NUNES

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Abstract. The notion of entropy of the Hardy–Littlewood inequalities for multilinear forms was introduced and explored by Pellegrino and Teixeira in [14]. In this note, among other results, we introduce a related notion and obtain some new estimates.

1. Introduction

The Hardy–Littlewood inequalities [10] for m–linear forms (see, for instance, [3, 4, 7, 8, 9, 12, 15]) are natural extensions of the Bohnenblust–Hille inequality [6] when the sequence space c_0 is replaced by the sequence space ℓ_p. These inequalities assert that for any integer $m \geqslant 2$ there exist constants $C_{m,p}^{\mathbb{K}} \geqslant 1$ such that

$$\left(\sum_{j_1,\cdots,j_m=1}^{\infty}\left|T(e_{j_1},\cdots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant C_{m,p}^{\mathbb{K}}\|T\|, \tag{1}$$

for all continuous m–linear forms $T:\ell_p\times\cdots\times\ell_p\to\mathbb{K}$ (here, and henceforth, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) and $p\geqslant 2m$.

A similar is statement is: for any integer $m \geqslant 2$ there exist constants $C_{m,p}^{\mathbb{K}} \geqslant 1$ such that

$$\left(\sum_{j_1,\cdots,j_m=1}^{k}\left|T(e_{j_1},\cdots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant C_{m,p}^{\mathbb{K}}\|T\|, \tag{2}$$

for all m–linear forms $T:\ell_p^k\times\cdots\times\ell_p^k\to\mathbb{K}$, all positive integers k, and $p\geqslant 2m$.

The investigation of the optimal constants of the Hardy–Littlewood and related inequalities is closely related to the fashionable investigation of the optimal Bohnenblust–Hille inequality constants (see, for instance [11, 12, 13, 14] and the references therein).

The notion of entropy in the context of the Hardy–Littlewood inequalities was introduced by Pellegrino and Teixeira [14] and it essentially estimates the number of monomials needed to achieve the optimal constant $C_{m,p}^{\mathbb{K}}$. There are strong evidences that in general the entropy is finite (see [14]). In this note we introduce a similar notion,

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that we name dimensional-entropy; it estimates the minimum of the k in (2) needed to achieve the optimal constant $C_{m,p}^{\mathbb{K}}$. Our first main result improves an estimate of [14]. More precisely, in [14, Lemma 5.1] it is proved that

$$\left(\sum_{j_1,\cdots,j_m=1}^{k}\left|T(e_{j_1},\cdots,e_{j_m})\right|^{\frac{2p}{p-2m+2}}\right)^{\frac{p-2m+2}{2p}} \leqslant \|T\| \tag{3}$$

for all m-linear forms $T:\ell_p^k\times\cdots\times\ell_p^k\to\mathbb{K}$, all positive integers k, and $p>2m$. Using a recent result of Albuquerque and Rezende [1] we improve (3) by showing a sharper inequality in the anisotropic setting. In fact we show that

$$\left(\sum_{j_1=1}^{k}\left(\cdots\cdot\left(\sum_{j_m=1}^{k}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^2\right)^{\frac{1}{2}}\cdots\cdot\right)^{\frac{2p}{p-2m+2}}\right)^{\frac{p-2m+2}{2p}} \leqslant \|T\|,$$

where the precise definition of the intermediary exponents will be clear along this note. We apply this result to explore the notion of entropy introduced in [14], with a different viewpoint. More precisely for each fixed k we define $C_{m,p}^{\mathbb{K}}(k)$ as the sharp constant for the Hardy–Littlewood inequalities when we are restricted to $T:\ell_p^k\times\cdots\times\ell_p^k\to\mathbb{K}$, and

$$ent_{HL}(\mathbb{K}):=\inf\left\{k: C_{m,p}^{\mathbb{K}}=C_{m,p}^{\mathbb{K}}(k)\right\}.$$

We prove that

$$ent_{HL}(\mathbb{K})\geqslant\left(C_{m,p}^{\mathbb{K}}\right)^{\frac{m^2-3m+p}{2p}}.$$

2. Results

Throughout this paper, X,Y shall stand for Banach spaces over the scalar field $\mathbb{K}$ of real or complex numbers. The topological dual of X and its closed unit ball are denoted by X^* and B_{X^*}, respectively. For $r,p\geqslant 1$, a linear operator $T:X\to Y$ is said $(r;p)$-summing if there exists a constant $C>0$ such that

$$\left(\sum_{j=1}^{\infty}\left\|T\left(x_j\right)\right\|^r\right)^{\frac{1}{r}}\leqslant C\left\|\left(x_j\right)_{j=1}^{\infty}\right\|_{w,p},$$

where

$$\left\|\left(x_j\right)_{j=1}^{\infty}\right\|_{w,p}:=\sup_{\varphi\in B_{X^*}}\left(\sum_{j=1}^{\infty}\left|\varphi\left(x_j\right)\right|^p\right)^{\frac{1}{p}}<\infty.$$

A natural anisotropic approach to summing operators is the following: for all $\mathbf{r}=(r_1,\ldots,r_m),\mathbf{p}=(p_1,\ldots,p_m)\in[1,+\infty)^m$, a multilinear operator $T:X_1\times\cdots\times X_m\to Y$

is said to be multiple $(\mathbf{r},\mathbf{p})$-summing if there exists a constant $C>0$ such that for all sequences $x^k := \left(x_j^k\right)_{j\in\mathbb{N}}, k=1,\ldots,m,$ we have

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\cdots\left(\sum_{j_m=1}^{\infty}\left|T\left(x_{\mathbf{j}}\right)\right|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}} \leqslant C\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,p_k},$$

where $T\left(x_{\mathbf{j}}\right) := T\left(x_{j_1}^1,\ldots,x_{j_m}^m\right)$. The class of all multiple $(\mathbf{r},\mathbf{p})$-summing operators is a Banach space with the norm defined by the infimum of all previous constants $C>0$. The space of all such operators is denoted by $\Pi_{(\mathbf{r};\mathbf{p})}^m(X_1,\ldots,X_m,Y)$. When $r_1=\cdots=r_m=r,$ we simply write $(r;\mathbf{p})$. For $\mathbf{p}\in[\mathbf{1},+\infty)^m$ and each $k\in\{1,..,m\}$, we define

$$\left|\frac{1}{\mathbf{p}}\right|_{\geqslant k} := \frac{1}{p_k}+\cdots+\frac{1}{p_m}.$$

Recently Albuquerque and Rezende [1, Theorem 3] have proved the following result, that generalizes recent results of Bayart [5] and Pellegrino–Santos–Serrano–Teixeira [13]:

THEOREM 1. (Albuquerque and Rezende) *Let m be a positive integer, $r\geqslant 1$, and $\mathbf{s},\mathbf{p},\mathbf{q}\in[\mathbf{1},+\infty)^m$ be such that $q_k\geqslant p_k$, for $k=1,\ldots,m$ and*

$$\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|+\left|\frac{1}{\mathbf{q}}\right|>0.$$

Then

$$\Pi_{(r;\mathbf{p})}^m(X_1,\ldots,X_m,Y)\subset\Pi_{(\mathbf{s};\mathbf{q})}^m(X_1,\ldots,X_m,Y),$$

for any Banach spaces $X_1,\ldots,X_m$, with

$$\frac{1}{s_k}-\left|\frac{1}{\mathbf{q}}\right|_{\geqslant k}=\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|_{\geqslant k}$$

for each $k\in\{1,\ldots,m\}$, and the inclusion operator has norm 1.

Using that all m-linear forms from $X_1\times\cdots\times X_m$ to $\mathbb{K}$ are multiple $(2;1,\ldots,1,p^*)$-summing for $p>2m$ and, from Theorem 1 with

$$\begin{aligned} r&=2\\ \mathbf{p}&=(1,\ldots,1,p^*)\\ \mathbf{q}&=(p^*,\ldots,p^*), \end{aligned}$$

we have

$$\Pi_{(2;1,\ldots,1,\,p^*)}^m(E_1,\ldots,E_m;\mathbb{K})\subset\Pi_{(\mathbf{s};\mathbf{q})}^m(E_1,\ldots,E_m;\mathbb{K})$$

with

$$\begin{cases} \frac{1}{s_1} - \left(\frac{m}{p^*}\right) = \frac{1}{2} - \left(\frac{1}{1} + \cdots + \frac{1}{1} + \frac{1}{p^*}\right) \\ \vdots \\ \frac{1}{s_m} - \frac{1}{p^*} = \frac{1}{2} - \left(\frac{1}{p^*}\right). \end{cases}$$

So, we have the following improvement of [14, Lemma 5.1]:

THEOREM 2. *For all $m \geqslant 2$ we have*

$$\left(\sum_{j_1=1}^{k}\left(\cdots\cdot\left(\sum_{j_m=1}^{k}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s_m}\right)^{\frac{1}{s_m}}\cdots\cdot\right)^{\frac{1}{s_2}s_1}\right)^{\frac{1}{s_1}} \leqslant \|T\|$$

for all m-linear forms $T : \ell_p^k \times \cdots \times \ell_p^k \to \mathbb{K}$ and all positive integers k and $p > 2m$, with

$$s_j = \frac{2p}{p-2m+2j}$$

for all $j = 1,\ldots,m$.

The above result is sharp at least when $p = \infty$ because in this case $s_j = 2$ for every j and the estimate

$$\left(\sum_{j_1,\ldots,j_m}^{k}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^2\right)^{\frac{1}{2}} \leqslant \|T\|,$$

for all m-linear forms $T : \ell_\infty^k \times \cdots \times \ell_\infty^k \to \mathbb{K}$, and all positive integers k, is optimal.

Now we state and prove our second main result:

THEOREM 3. *For all $m \geqslant 2$ we have*

$$ent_{HL}(\mathbb{K}) \geqslant \left(C_{m,p}^{\mathbb{K}}\right)^{\frac{m^2-3m+p}{2p}}.$$

Proof. Following the notation of Theorem 2, by the Hölder inequality for mixed norms (see [2]) and Theorem 2, we have

$$
\begin{aligned}
&\left(\sum_{j_1,\cdots,j_m=1}^{k}\left|T(e_{j_1},\cdots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \qquad (4)\\
&\leqslant \left(\sum_{j_1=1}^{k}\left(\ldots\ldots\left(\sum_{j_m=1}^{k}\left|T(e_{j_1},\ldots,e_{j_m})\right|^{s_m}\right)^{\frac{1}{s_m}}\ldots\ldots\right)^{s_1}\right)^{\frac{1}{s_1}}\\
&\quad\times\left(\sum_{j_1=1}^{k}\left(\ldots\ldots\left(\sum_{j_m=1}^{k}|1|^{t_m}\right)^{\frac{1}{t_m}}\ldots\ldots\right)^{t_1}\right)^{\frac{1}{t_1}}\\
&\leqslant k^{\frac{1}{t_1}+\cdots+\frac{1}{t_m}}\|T\|
\end{aligned}
$$

with

$$
\begin{cases}
\frac{1}{\frac{2mp}{mp+p-2m}} = \frac{1}{s_m}+\frac{1}{t_m},\\
\qquad\vdots\\
\frac{1}{\frac{2mp}{mp+p-2m}} = \frac{1}{s_1}+\frac{1}{t_1}.
\end{cases}
$$

Since

$$
\begin{aligned}
\frac{1}{s_1}+\cdots+\frac{1}{s_m} &= \left(\frac{m}{2}\right)-\left(\frac{m(m-1)}{2}+m\left(1-\frac{1}{p}\right)\right)+\left(1-\frac{1}{p}\right)\left(\frac{m(m+1)}{2}\right)\\
&= \frac{m(p-m+1)}{2p},
\end{aligned}
$$

we have

$$
\frac{1}{t_1}+\cdots+\frac{1}{t_m} = \frac{mp+p-2m}{2p}-\frac{m(p-m+1)}{2p} = \frac{1}{2p}\left(m^2-3m+p\right)
$$

and, by (4), we conclude that

$$
\left(\sum_{j_1,\cdots,j_m=1}^{k}\left|T(e_{j_1},\cdots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leqslant \|T\|\,k^{\frac{1}{2p}\left(m^2-3m+p\right)}.
$$

Thus

$$
k^{\frac{1}{2p}\left(m^2-3m+p\right)} \geqslant C_{m,p}^{\mathbb{K}}
$$

and

$$
k \geqslant \left(C_{m,p}^{\mathbb{K}}\right)^{\frac{m^2-3m+p}{2p}}. \quad \square
$$

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Antonio Gomes Nunes
Department of Mathematics
CCEN – UFERSA
Mossoró, RN, Brazil
e-mail: nunesag@gmail.com;
nunesag@ufersa.edu.br