ON THE BANACH–MAZUR DISTANCE BETWEEN
THE CUBE AND THE CROSSPOLYTOPE

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Abstract. In this note we study the Banach-Mazur distance between the n-dimensional cube and the crosspolytope. Previous work shows that the distance has order $\sqrt{n}$, and here we will prove some explicit bounds improving on former results. Even in dimension 3 the exact distance is not known, and based on computational results it is conjectured to be $\frac{9}{5}$. Here we will also present computer based potentially optimal results in dimension 4 to 8.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and in this paper an n-dimensional vector $x \in \mathbb{R}^n$ is always treated as a column vector. We call $K \subset \mathbb{R}^n$ an n-dimensional convex body, if K is compact, and for any $x, y \in K$ and $\lambda \in [0,1]$, it holds $\lambda x + (1 - \lambda)y \in K$. The set of all n-dimensional convex bodies is denoted by $\mathbb{K}^n$. The set of n-dimensional 0-symmetric convex bodies is denoted by $\mathbb{K}_o^n$. A convex polytope P is defined as the convex hull of finitely many points

$$P = conv\{u_1, \ldots, u_k\},$$

and the set of all n-dimensional convex polytopes is denoted by $\mathbb{P}^n$.

The Hausdorff distance between two convex bodies K and L is defined as:

$$d_H(K,L) = \max\{\sup_{x \in K} \inf_{y \in L} d(x,y), \sup_{y \in L} \inf_{x \in K} d(x,y)\},$$

where $d(x,y)$ is the usual Euclidean distance.

For a real number $p \geqslant 1$, the p-norm of $x \in \mathbb{R}^n$ is defined by

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

The maximum norm is the limit of the p-norm for $p \to \infty$. It is equivalent to

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Denote by

$$C_n = \{x \in \mathbb{R}^n : \|x\|_\infty \leqslant 1\} = [-1,1]^n$$

the n-dimensional unit cube, and denote the vertices of the n-dimensional unit cube by $\{-1,1\}^n$. Denote by

$$C_n^\star = \{x \in \mathbb{R}^n : \|x\|_1 \leqslant 1\}$$

the n-dimensional unit crosspolytope. Denote by

$$B_n = \{x \in \mathbb{R}^n : \|x\|_2 \leqslant 1\}$$

the n-dimensional unit ball. For example, the Hausdorff distance between C_n and $C_n^\star$ is $\frac{n-1}{\sqrt{n}}$, and the Hausdorff distance between C_n and B_n is $\sqrt{n}-1$.

The Banach-Mazur distance between two 0-symmetric convex bodies K and L is defined as:

$$d_{BM}(K,L) = \min\{r > 0 : K \subset gL \subset rK, g \in \mathrm{GL}(n,\mathbb{R})\}$$

where $\mathrm{GL}(n,\mathbb{R})$ is the group of invertible linear operators. It can be deduced that

$$d_{BM}(K_1, K_3) \leqslant d_{BM}(K_1, K_2) d_{BM}(K_2, K_3).$$

There are some results on the Banach-Mazur distance for some special convex bodies. John's theorem on the maximal volume ellipsoid contained in a convex body gives the estimate:

THEOREM 1.1. (John's theorem [5]) *The Banach-Mazur distance between an n-dimensional 0-symmetric convex body K and the n-dimensional ball is at most $\sqrt{n}$.*

As a corollary, for any two 0-symmetric convex bodies K and L,

$$d_{BM}(K,L) \leqslant d_{BM}(K,B_n) d_{BM}(B_n,L) \leqslant n.$$

As a matter of fact, the diameter of $(\mathbb{K}_o^n, d_{BM})$ is still unknown, but E. Gluskin [3] proved that the diameter is bounded from below by cn for some universal constant $c > 0$.

Exploiting the symmetries of the ℓ^p norms, one can easily prove that:

THEOREM 1.2. ([8]) *The Banach-Mazur distance between B_n and C_n is $\sqrt{n}$. The Banach-Mazur distance between B_n and $C_n^\star$ is $\sqrt{n}$.*

There are also some results on the Banach-Mazur distance from any convex body to the cube [1, 2].

We are interested in the Banach-Mazur distance between C_n and $C_n^\star$. There are results in [7, 8] showing that the distance has order $\sqrt{n}$:

THEOREM 1.3. ([7, 8]) *There exist constants $c, C > 0$ such that*

$$c\sqrt{n} \leqslant d_{BM}(C_n, C_n^\star) \leqslant C\sqrt{n}.$$

To be exact, for the upper bound one can get

$$C = \frac{1}{\sqrt[4]{2} - 1} = 5.2852\cdots$$

from Proposition 37.6 in [7]. For the lower bound, the constant c is not explicitly stated in [7].

In this paper we discuss the upper and the lower bounds of this distance. Our main results are:

THEOREM 1.4. *(1) There is a maximum absolute constant α, such that for any $x \in \mathbb{R}^n$,*

$$\frac{1}{2^n} \sum_{v \in \{-1,1\}^n} |\langle x,v \rangle| \geqslant \alpha \|x\|_2.$$

(2) $\alpha > \frac{1}{1.71881\cdots} \approx 0.5818\cdots$.

THEOREM 1.5. *Let α be as above. Then*

$$\alpha\sqrt{n} \leqslant d_{BM}(C_n, C_n^\star) \leqslant (\sqrt{2}+1)\sqrt{n}.$$

We observe that $\alpha \leqslant \frac{1}{\sqrt{2}}$ by evaluating the inequality for $x = (1,1,0,0,\ldots,0)$, and we conjecture that α can be at most $\frac{1}{\sqrt{2}}$. Furthermore, we show that if the Hadamard matrix conjecture holds true, then the upper bound can be reduced to $\sqrt{n}+3$.

2. Some computational results

In order to find the Banach-Mazur distance between the cube and the crosspolytope, one needs to find the minimum $r > 0$ such that there exists $g \in \mathrm{GL}(n,\mathbb{R})$ with

$$\frac{1}{r}C_n \subset gC_n^\star \subset C_n.$$

Assume that g is the linear transformation $g = (x_{ij})_{n \times n}$, then the crosspolytope

$$gC_n^\star = conv\{\pm(x_{i1},\ldots,x_{in})^T : i = 1,\ldots,n\},$$

and $gC_n^\star \subset C_n$ implies that $|x_{ij}| \leqslant 1$ for $i,j = 1,\ldots,n$. The left part $\frac{1}{r}C_n \subset gC_n^\star$ with miminum r implies that the vertices of the cube $\frac{1}{r}C_n$ are contained in the crosspolytope $gC_n^\star$, which is

$$\max_{v \in \{-1,1\}^n} \|g^{-1}v\|_1 = r.$$

Therefore the Banach-Mazur distance is

$$d_{BM}(C_n, C_n^\star) = \min_{\substack{g=(x_{ij})_{n \times n} \in \mathrm{GL}(n,\mathbb{R}) \\ |x_{ij}| \leqslant 1, i,j=1,\ldots,n}} \max_{v \in \{-1,1\}^n} \|g^{-1}v\|_1.$$

An approximate solution can be obtained via a computer program like Wolfram Mathematica v.11.2.0. We can use the code here on Mathematica:

```
dim = 3;
T = Array[Subscript[TT, ##] &, {dim, dim}];
B1 = IdentityMatrix[dim];
B1 = Join[-B1, B1];
Binf = Tuples[{-1, 1}, dim];
NMinimize[
Join[{Max[Table[Norm[Inverse[T].Binf[[j]], 1],
{j, Length[Binf]}]], Det[T] != 0},
Table[Norm[T.B1[[i]], Infinity] <= 1, {i, Length[B1]}]],
Flatten[T]]
```

where we can change 3 to any dimension we need. Since the computer only gives the numerical results, we made some adjustment to make them to be the probably optimal ones. We also need to point out that, for the same code in the same dimension, the numerical result might change slightly when we run it again.

In dimension 3 the numerical result shows that the distance is at most $\frac{9}{5}$ and the crosspolytope is determined by:

$$\begin{pmatrix} 1 & 1 & -1/3 \\ -1/3 & 1 & 1 \\ 1 & -1/3 & 1 \end{pmatrix}.$$

In dimension 4 the numerical result shows that the distance is at most 2.26515 and the crosspolytope is determined by:

$$\begin{pmatrix} -0.164392 & 0.902819 & 1 & -1 \\ -1 & -0.0286877 & -0.999908 & -0.760687 \\ 0.192848 & -1 & 0.16027 & -1 \\ -1 & -0.70927 & 1 & 0.518805 \end{pmatrix}.$$

But, we know that the distance is at most 2 if we choose the crosspolytope to be determined by:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

In dimension 5 the numerical result shows that the distance is at most 2.32871 and the crosspolytope is determined by:

$$\begin{pmatrix} 0.792559 & 1 & 0.0387439 & -1 & -0.704555 \\ 1 & 0.792092 & 0.999411 & 0.855944 & 1 \\ -1 & -0.0773263 & 1 & -1 & 0.888962 \\ 0.925403 & -1 & 1 & -0.115724 & -0.822648 \\ 1 & -0.79255 & -0.999989 & -0.856439 & 1 \end{pmatrix}.$$

It seems to be highly irregular.

In dimension 6 the numerical result shows that the distance is at most 2.45449 and the crosspolytope is determined by:

$$\begin{pmatrix} -1 & 1 & 0.999902 & 0.999988 & -0.331954 & 0.436841 \\ 0.991908 & 0.339038 & -1 & 1 & -0.454488 & 1 \\ 0.971694 & 1 & -0.319982 & 0.454287 & 1 & -1 \\ -1 & 1 & -0.999995 & -0.998472 & 0.976994 & 0.999489 \\ 0.998897 & 1 & 0.435783 & -1 & -1 & 0.266908 \\ -1 & 0.429375 & -0.999995 & 0.335729 & -1 & 1 \end{pmatrix}.$$

It is always appropriate to switch some rows or some columns, as well as to change the sign of some row or some column. Then, we replace the numbers that are close to ± 1, ± 0.33, and ± 0.45, by ± 1, $\pm x$, and $\pm y$, respectively. Finally, we calculate the minimum value with respect to the variables x, y, and get a probably optimal result: the distance is at most 2.4488 and the crosspolytope is determined by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & x & 1 & y & -1 & 1 \\ -1 & 1 & x & 1 & y & -1 \\ -1 & -1 & 1 & x & 1 & y \\ -1 & y & -1 & 1 & x & 1 \\ -1 & 1 & y & -1 & 1 & x \end{pmatrix}$$

where $x = 0.324842$, $y = -0.434446$.

In dimension 7 the numerical result shows that the distance is at most 2.6 and the crosspolytope is determined by:

$$\begin{pmatrix} 1 & 1 & 0.736632 & 1 & -1 & 3.22206 \times 10^{-7} & -0.903055 \\ -0.763516 & 0.763516 & -1 & 0.763516 & 0.763516 & -0.763516 & -1 \\ -1 & 2.60819 \times 10^{-7} & 0.736632 & -1 & -1 & -1 & -0.903054 \\ 1 & 1 & -0.736632 & -1 & 8.08984 \times 10^{-8} & -1 & 0.903055 \\ 1 & -1 & 0.736632 & -3.02552 \times 10^{-7} & 1 & -1 & -0.903055 \\ -3.25631 \times 10^{-7} & -1 & -0.736632 & 1 & -1 & -1 & 0.903054 \\ 0.833018 & -0.833018 & -1 & -0.833019 & -0.833019 & 0.833018 & -1 \end{pmatrix}.$$

We replace the numbers that are close to 0, ± 0.73, ± 0.76, ± 0.83, and ± 0.90, by 0, $\pm x$, $\pm y$, $\pm z$, and $\pm w$, respectively. Up to a change of rows and columns, we find that by simply changing these variables to 1 or -1 we can get a beautiful matrix, whereas the distance does not change. Therefore, we get a probably optimal result: the distance is at most 2.6 and the crosspolytope is determined by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

In dimension 8 we are not sure how long it takes to wait for the numerical result. However, we find an example with a Hadamard matrix, showing that the distance is at most 2.5, smaller than in dimension 7, and the crosspolytope is determined by:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & 1 & -1
\end{pmatrix}.
$$

3. Upper bound

Recall that the Banach-Mazur distance between the cube and the crosspolytope is

$$
d_{BM}(C_n, C_n^\star) = \min_g \max_{v \in \{-1,1\}^n} \|g^{-1}v\|_1
$$

where $g = (x_{ij})_{n \times n}$ with $|x_{ij}| \leqslant 1$. By giving a special g one can get an upper bound of the distance.

3.1. Hadamard matrix

A Hadamard matrix is a square matrix whose entries are either $+1$ or -1, whose rows are mutually orthogonal and has maximal determinant among matrices with entries of absolute value less than or equal to 1.

Sylvester [6] provided one way to construct Hadamard matrices. Let

$$
H_1 = (1)
$$

$$
H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
$$

and

$$
H_{2^k} = \begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix}
$$

for $k \geqslant 2$, then H_{2^k} are all Hadamard matrices.

The Hadamard conjecture proposes that a Hadamard matrix of order $4k$ exists for every positive integer k. Sylvester's construction yields Hadamard matrices of order 2^k. A generalization of Sylvester's construction proves that if H_n and H_m are Hadamard matrices of orders n and m respectively, then there exists a Hadamard matrix of order nm [6]. So far the Hadamard conjecture is still open.

3.2. Proof of the upper bound in Theorem 1.5

In dimension $n = 2^k$, there exists a Hadamard matrix H_n. Choose the matrix $g_n = H_n$, then $g_n^{-1} = \frac{1}{n} g_n^T$ where g_n^T is still a Hadamard matrix with row vectors $r_1, \ldots, r_n$. So

$$\max_{v \in \{-1,1\}^n} \|g_n^{-1} v\|_1$$

$$= \frac{1}{n} \max_{v \in \{-1,1\}^n} (|\langle r_1, v \rangle| + \cdots + |\langle r_n, v \rangle|)$$

$$\leqslant \frac{1}{n} \max_{v \in \{-1,1\}^n} \sqrt{n(\langle r_1, v \rangle^2 + \cdots + \langle r_n, v \rangle^2)}$$

$$= \max_{v \in \{-1,1\}^n} \frac{1}{n} \sqrt{n \cdot n \cdot \|v\|_2^2}$$

$$= \sqrt{n}.$$

By induction, assume that in dimension $t \leqslant 2^k$ the upper bound is not bigger than $(\sqrt{2}+1)\sqrt{t}$ with crosspolytope determined by g_t. Then in dimension $n = 2^k + t$ where $t \leqslant 2^k$, let

$$g_{2^k+t} = \begin{pmatrix} g_{2^k} & 0 \\ 0 & g_t \end{pmatrix}.$$

The distance is therefore

$$\max_{v \in \{-1,1\}^n} \|g_{2^k+t}^{-1} v\|_1$$

$$= \max_{v \in \{-1,1\}^{2^k}} \|g_{2^k}^{-1} v\|_1 + \max_{v \in \{-1,1\}^t} \|g_t^{-1} v\|_1$$

$$\leqslant \sqrt{2^k} + (\sqrt{2}+1)\sqrt{t}$$

$$\leqslant (\sqrt{2}+1)\sqrt{2^k+t}$$

$$= (\sqrt{2}+1)\sqrt{n}.$$

The proof for the upper bound is finished.

The Hadamard conjecture predicts the existence of a Hadamard matrix in dimension $n = 4k$. When the Hadamard matrix exists in dimension $n = 4k$, denoted by H_n, the distance between C_n and the crosspolytope determined by H_n will be $\sqrt{n}$.

When $n = 4k + j$, $j < 4$, let the crosspolytope be determined by

$$g_{4k+j} = \begin{pmatrix} I_j & 0 \\ 0 & H_{4k} \end{pmatrix}.$$

Then the distance is

$$\max_{v \in \{-1,1\}^n} \|g_{4k+j}^{-1} v\|_1$$

$$= \max_{v \in \{-1,1\}^{4k}} \|H_{4k}^{-1} v\|_1 + \max_{v \in \{-1,1\}^j} \|I_j^{-1} v\|_1$$

$$\leqslant \sqrt{4k} + j < \sqrt{n} + 3.$$

Therefore the upper bound will be $\sqrt{n}+3$ for all n.

4. The proof of the lower bound in Theorem 1.5

The Banach-Mazur distance of the cube and the crosspolytope is the minimum value of

$$\max_{v\in\{-1,1\}^n}\|g^{-1}v\|_1$$

with respect to g. Without loss of generality, consider only $\det(g)>0$. Write $g^{-1}=\det(g^{-1})^{1/n}N$, where $N\in\mathrm{SL}(n,\mathbb{R})$, the group of special linear operators. Let the row vectors of N be N_j, i.e. $N=(N_j)_{n\times1}$, then we have

$$\|Nv\|_1=|\langle N_1,v\rangle|+\cdots+|\langle N_n,v\rangle|.$$

Also, since $\det(N)=1$, by the definition of determinant we have:

$$\prod_{j=1}^n\|N_j\|_2\geqslant1$$

and by the arithmetic-geometric mean inequality

$$\sum_{j=1}^n\|N_j\|_2\geqslant n(\prod_{j=1}^n\|N_j\|_2)^{1/n}\geqslant n.$$

Based on this result, we can infer that:

$$\max_{v\in\{-1,1\}^n}\|g^{-1}v\|_1$$
$$=\det(g^{-1})^{1/n}\max_{v\in\{-1,1\}^n}\|Nv\|_1$$
$$=\det(g^{-1})^{1/n}\max_{v\in\{-1,1\}^n}\sum_{j=1}^n|\langle N_j,v\rangle|$$
$$\geqslant\det(g^{-1})^{1/n}\frac{1}{2^n}\sum_{v\in\{-1,1\}^n}\sum_{j=1}^n|\langle N_j,v\rangle|$$
$$=\det(g^{-1})^{1/n}\frac{1}{2^n}\sum_{j=1}^n\sum_{v\in\{-1,1\}^n}|\langle N_j,v\rangle|$$
$$\geqslant\alpha\det(g^{-1})^{1/n}\sum_{j=1}^n\|N_j\|_2$$
$$\geqslant\alpha\det(g^{-1})^{1/n}n(\prod_{j=1}^n\|N_j\|_2)^{1/n}$$
$$\geqslant\alpha\det(g^{-1})^{1/n}n$$
$$\geqslant\alpha\sqrt{n}.$$

The last inequality comes from

$$\det(g) \leqslant n^{n/2}$$

since $|x_{ij}| \leqslant 1$.

5. The proof of Theorem 1.4

We are looking for the maximal absolute constant α such that

$$\frac{1}{2^n} \sum_{v \in \{-1,1\}^n} |\langle x, v \rangle| \geqslant \alpha \|x\|_2$$

holds for all $x \in \mathbb{R}^n$ and all dimension n.

A convex polytope may be defined as a bounded intersection of a finite number of half-spaces. That is, for any convex polytope P, there exist vectors u_j $(1 \leqslant j \leqslant k)$ such that

$$P = \{x \in \mathbb{R}^n : \langle x, u_j \rangle \leqslant 1; 1 \leqslant j \leqslant k\}.$$

For the same reason, for any symmetric convex polytope C, there exist vectors u_j $(1 \leqslant j \leqslant k)$ such that

$$C = \{x \in \mathbb{R}^n : |\langle x, u_j \rangle| \leqslant 1; 1 \leqslant j \leqslant k\}.$$

Consider the set

$$K = \{x \in \mathbb{R}^n : \sum_{j=1}^{k} |\langle x, u_j \rangle| \leqslant 1\}$$

where u_j $(1 \leqslant j \leqslant k)$ are non-zero vectors such that K is bounded. As the intersection of 2^k halfspaces K is a convex polytope.

For general dimension n, the problem is equivalent to find a point x with the maximal value of $\|x\|_2$ in the polytope

$$\left\{x \in \mathbb{R}^n : F_n(x) = \frac{1}{2^n} \sum_{v \in \{-1,1\}^n} |\langle x, v \rangle| \leqslant 1\right\}.$$

The maximal value is attained at a special vertex of this polytope. Moreover, if x is a vertex of this convex polytope, then it is the intersection of at least n facets.

First we prove that, for any vertex x, there are $n-1$ linearly independent vectors $v \in \{-1,1\}^n$ such that $\langle x, v \rangle = 0$.

For any $x, y \in \mathbb{R}^n$ and $\varepsilon > 0$ small enough,

$$F_n(x + \varepsilon y) + F_n(x - \varepsilon y) - 2F_n(x) = 2 \sum_{\substack{v \in \{-1,1\}^n \\ \langle x,v \rangle = 0}} |\langle \varepsilon y, v \rangle|.$$

Notice that if $\langle y, v \rangle = 0$ for all v such that $\langle x, v \rangle = 0$, and if $F_n(x) = F_n(x + \varepsilon y) = 1$, then we have $F_n(x - \varepsilon y) = 1$, which means that x is not a vertex of the polytope.

If there are at most $n-2$ linearly independent $v \in \{-1,1\}^n$ such that $\langle x,v \rangle = 0$, then there exists y not linear to x, such that $\langle y,v \rangle = 0$ whenever $\langle x,v \rangle = 0$, meaning that

$$F_n(x+\varepsilon y) + F_n(x-\varepsilon y) - 2 = 0$$

for $\varepsilon > 0$ small enough. Now we choose ε to be arbitrarily small, and let $y = y' + y''$, where $x + \varepsilon y'$ lies on some facet of the polytope containing x, whereas y'' is proportional to x. Since $\langle y'', v \rangle = 0$ whenever $\langle x,v \rangle = 0$, we also have $\langle y', v \rangle = 0$ and hence

$$F_n(x-\varepsilon y') = 2 - F_n(x+\varepsilon y') = 1.$$

Thus $x - \varepsilon y'$ lies also on some facet of the polytope. Therefore x is not a vertex of the polytope.

With this observation, we can find out the vertices of the convex polytope with $F_n(x) \leqslant 1$.

In dimension 2, $F_2(x) = \frac{|x_1+x_2|+|x_1-x_2|}{2} \leqslant 1$ is the cube C_2, and the maximal value of $\|x\|_2$ is $\sqrt{2}$.

In dimension 3, without loss of generality, we assume that $x = (x_1, x_2, x_3)$, where $x_1 \geqslant x_2 \geqslant x_3 \geqslant 0$. When $x_2 + x_3 \geqslant x_1$, we have

$$F_3(x) = \frac{1}{4}(|x_1+x_2+x_3| + |x_1+x_2-x_3| + |x_2+x_3-x_1| + |x_3+x_1-x_2|)$$
$$= \frac{x_1+x_2+x_3}{2} \leqslant 1.$$

When $x_1 \geqslant x_2 + x_3$, we have

$$F_3(x) = \frac{1}{4}(|x_1+x_2+x_3| + |x_1+x_2-x_3| + |x_2+x_3-x_1| + |x_3+x_1-x_2|)$$
$$= x_1 \leqslant 1.$$

So the convex polytope is

$$conv\{(\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1), (0, \pm 1, \pm 1)\}$$

and the maximum value of $\|x\|_2$ is $\sqrt{2}$.

In dimension 4, consider the vertex $x = (x_1, x_2, x_3, x_4)$. We know that there are three linearly independent $v \in \{-1,1\}^4$ such that $\langle x,v \rangle = 0$, denoted by v_1, v_2, v_3, then:

(1) if v_i and v_j have 1 or 3 coordinate(s) in common, for example $v_1 = (1,1,1,1)$ and $v_2 = (1,1,1,-1)$, then $x_4 = 0$, and (x_1,x_2,x_3) is a vertex of the polytope $F_3(x) \leqslant 1$.

(2) if all pairs of v_i and v_j have 2 coordinates in common, then x has the form $(\pm t, \pm t, \pm t, \pm t)$, with (by calculation) $t = \frac{2}{3}$.

Therefore the convex polytope contains only one more set of vertices:

$$\left(\pm \frac{2}{3}, \pm \frac{2}{3}, \pm \frac{2}{3}, \pm \frac{2}{3}\right).$$

The maximal value of $\|x\|_2$ is still $\sqrt{2}$.

According to the results in dimension 2, 3, 4, we conjecture that:

CONJECTURE 5.1.

$$\frac{1}{2^n} \sum_{v \in \{-1,1\}^n} |\langle x, v \rangle| \geqslant \frac{1}{\sqrt{2}} \|x\|_2,$$

i.e., $\alpha = 1/\sqrt{2}$.

Finally we prove Theorem 1.4. Let α_n be such that, for any $z \in \mathbb{R}^n$,

$$\frac{\|z\|_2}{F_n(z)} \leqslant \alpha_n.$$

Let $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ be such that

$$\frac{\|x\|_2}{F_{n+1}(x)} = \alpha_{n+1}.$$

Without loss of generality, we assume that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n+1} \geqslant 0$. Let $y_{(1)} = (x_1, \ldots, x_n + x_{n+1})$, $y_{(2)} = (x_1, \ldots, x_n - x_{n+1})$. By definition we have

$$F_{n+1}(x) = \frac{F_n(y_{(1)}) + F_n(y_{(2)})}{2}$$

and

$$\|x\|_2^2 = \frac{\|y_{(1)}\|_2^2 + \|y_{(2)}\|_2^2}{2}.$$

Therefore

$$\frac{\|x\|_2}{F_{n+1}(x)} = \frac{\sqrt{2}\sqrt{\|y_{(1)}\|_2^2 + \|y_{(2)}\|_2^2}}{F_n(y_{(1)}) + F_n(y_{(2)})}$$

$$\leqslant \alpha_n \frac{\sqrt{2}\sqrt{\|y_{(1)}\|_2^2 + \|y_{(2)}\|_2^2}}{\|y_{(1)}\|_2 + \|y_{(2)}\|_2}.$$

Since $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n+1} \geqslant 0$, we have

$$\frac{\|y_{(1)}\|_2^2}{\|y_{(2)}\|_2^2} = 1 + \frac{4x_n x_{n+1}}{x_1^2 + \cdots + x_{n-1}^2 + (x_n - x_{n+1})^2}$$

$$\leqslant 1 + \frac{4x_n^2}{(n-1)x_n^2}$$

$$= 1 + \frac{4}{n-1}.$$

Therefore by monotonicity we have

$$\alpha_{n+1} = \frac{\|x\|_2}{F_{n+1}(x)} \leqslant \alpha_n \frac{\sqrt{2}\sqrt{2 + \frac{4}{n-1}}}{1 + \sqrt{1 + \frac{4}{n-1}}}$$

$$\leqslant \alpha_n \left(1 + \frac{1}{2(n-1)^2} \right).$$

Since we already know that $\alpha_4 = \sqrt{2}$, by recurrence we have:

$$\alpha_n \leqslant \sqrt{2} \prod_{j=4}^{n-1} \left(1 + \frac{1}{2(j-1)^2} \right)$$

$$< \sqrt{2} \prod_{j=4}^{\infty} \left(1 + \frac{1}{2(j-1)^2} \right)$$

$$\approx 1.71881.$$

This concludes the proof of Theorem 1.4.

REMARK 1. We can also use

$$\alpha_{n+1} \leqslant \alpha_n \frac{\sqrt{2}\sqrt{2 + \frac{4}{n-1}}}{1 + \sqrt{1 + \frac{4}{n-1}}}$$

to get a better bound:

$$\alpha_n \leqslant \sqrt{2} \prod_{j=4}^{n-1} \frac{\sqrt{2}\sqrt{2 + \frac{4}{j-1}}}{1 + \sqrt{1 + \frac{4}{j-1}}}$$

$$< \sqrt{2} \prod_{j=4}^{\infty} \frac{\sqrt{2}\sqrt{2 + \frac{4}{j-1}}}{1 + \sqrt{1 + \frac{4}{j-1}}}.$$

From the inequalities above, we already know that this infinite product converges, but so far we cannot get the exact value of this infinite product. A reference value is:

$$\sqrt{2} \prod_{j=4}^{10000} \frac{\sqrt{2}\sqrt{2 + \frac{4}{j-1}}}{1 + \sqrt{1 + \frac{4}{j-1}}}$$

$$\approx \sqrt{2} * 1.120166\ldots \approx 1.584154\ldots$$

REMARK 2. In order to get a better value, if we can prove that $\alpha_k = \sqrt{2}$ for some $k > 4$, then

$$\alpha_n < \sqrt{2} \prod_{j=k}^{\infty} \frac{\sqrt{2}\sqrt{2 + \frac{4}{j-1}}}{1 + \sqrt{1 + \frac{4}{j-1}}}.$$

So far we still believe that $\alpha_n = \sqrt{2}$ for all n.

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