FRACTIONAL DE LA VALLÉE POUSSIN INEQUALITIES

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Abstract. In this work we derive some inequalities for fractional boundary value problems, that generalize the well-known de la Vallée Poussin inequality. With our results we also were able to improve the intervals where some Mittag–Leffler functions don't possess real zeros.

1. Introduction

When considering a second order linear boundary value problem with Dirichlet boundary conditions, the following result is known as the de la Vallée Poussin inequality (see e.g. [10]):

THEOREM 1. *Suppose that $x \in C^2[a,b]$ is a nontrivial solution of the BVP*

$$x'' + g(t)x' + f(t)x = 0, \quad t \in (a,b)$$
$$x(a) = 0 = x(b), \tag{1}$$

where $f,g \in C[a,b]$. Then, the following inequality holds:

$$1 < M_1(b-a) + M_2 \frac{(b-a)^2}{2}, \tag{2}$$

where $M_1 = \max_{t \in [a,b]} |g(t)|$ and $M_2 = \max_{t \in [a,b]} |f(t)|$.

Cohn [4], Harris [9], Hartman and Wintner [10], and most recently the author [8] obtained generalizations of theorem 1 in these referenced works, respectively. A survey about the de La Vallée Poussin work on boundary value problems maybe found in [13]. The research in order to find de la Vallée Poussin or Lyapunov type inequalities is an endless subject (see e.g. [14]), but until 2013, it was done exclusively for classical ordinary differential equations. However, in that year the author presented for the first time in the literature [5] an inequality for a fractional differential equation depending on a fractional derivative. His result generalized the classical Lyapunov inequality (see [5, theorem 2.1]). Since then, many other researchers dedicated their time to find

Lyapunov-type inequalities for boundary value problems in which fractional deriva-
tives are present (see [1, 2, 3, 12] and the references therein). It is, nevertheless, worth
mentioning that there are some open problems within the subject [7].

In this work we consider the fractional differential equation (see Section 2 for a
brief introduction to fractional calculus)

$$(D_a^\alpha x) + g(t)(D_a^\beta x) + f(t)x = 0, \quad 1 < \alpha \leqslant 2,\ 0 < \beta \leqslant 1, \tag{3}$$

together with the boundary conditions (1), and make an attempt to derive inequalities
of de la Vallée Poussin type for such a problem. To the best of our knowledge it is
the first time such results appear in the literature for an equation of the type given in
(3). We divide our main results into two sections: in the first section we consider the
differential equation $x'' + g(t)(D_a^\beta x) + f(t)x = 0$, while in the second one, we consider
the differential equation $(D_a^\alpha x) + g(t)(D_a^\beta x) + f(t)x = 0$. The main reason to do it so is
that, when considering the first equation we were able to obtain results that generalize
the ones by Hartman and Wintner [10] (and consequently of the de la Vallée Poussin),
while when considering the second equation we were *only* able to generalize the results
of de la Vallée Poussin. Nevertheless, it is worth mentioning it that we obtain as a
particular case from (3)–(1), i.e. considering $g = 0$ on $[a,b]$, the Lyapunov fractional
inequality [5, theorem 2.1]. Finally, we revisit some results (and provide some new
ones) related with the zeros of certain Mittag–Leffler functions.

It is the first time that these type of inequalities appear in the literature for differen-
tial equations with a middle term (cf. (3)) and as such we believe that this work might
be a cornerstone for future research within this interesting subject.

2. Fractional calculus

We introduce here to the reader the basics about fractional integrals and deriva-
tives, namely, what will be used throughout this work. A thorough introduction to the
subject may be found in [11].

DEFINITION 1. Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann–
Liouville fractional integral of order α is defined by $(I_a^0 f)(x) = f(x)$ and

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t \in [a,b],$$

provided the integral exists.

DEFINITION 2. The Riemann–Liouville fractional derivative of order $n - 1 <
\alpha \leqslant n$, $n \in \mathbb{N}$ of a function f is defined by $(D_a^\alpha f)(t) = (D^n I_a^{n-\alpha} f)(t)$, provided the
right hand side of the equality exists.

The following result may be found in [11, Property 2.2].

PROPOSITION 1. *Suppose that $f \in C[a,b]$ and let $q \geqslant p > 0$. Then,*

$$(D_a^p I_a^q f)(t) = (I_a^{q-p} f)(t), \quad t \in [a,b].$$

A version of the mean value theorem is contained in the following

THEOREM 2. [15, theorem 3.1] *Let $0 < \beta \leqslant 1$. Suppose that $f \in C[a,t]$ is such that $(D_a^\beta f) \in C[a,t]$. Let $f(a) = 0$. Then, there exists $\tau \in (a,t)$ such that*

$$f(t) = \frac{(t-a)^\beta}{\Gamma(\beta+1)} (D_a^\beta f)(\tau). \tag{4}$$

3. Main results

3.1. The equation $x'' + g(t)(D_a^\beta x) + f(t)x = 0$

In this section we shall consider the following boundary value problem:

$$x'' + g(t)(D_a^\beta x) + f(t)x = 0, \quad t \in (a,b), \ \beta \in (0,1], \tag{5}$$

$$x(a) = 0 = x(b), \tag{6}$$

where $f, g \in C[a,b]$. It follows the main result of this section:

THEOREM 3. *Suppose that $x \in C^2[a,b]$ is a solution of (5)–(6) such that $x(t) \neq 0$ for $t \in (a,b)$. Then, the following inequality holds:*

$$b - a < \max \left\{ \int_a^b \frac{(s-a)^{2-\beta}}{\Gamma(2-\beta)} |g(s)| ds, \int_a^b \frac{(s-a)^{1-\beta}}{\Gamma(2-\beta)} (b-s)|g(s)| ds \right\}$$

$$+ \int_a^b (s-a)(b-s)|f(s)| ds. \tag{7}$$

Proof. We start by writing the BVP (5)–(6) in an equivalent integral form. Indeed, we know that $x \in C^2[a,b]$ is a solution of (5) if and only if it is a solution of

$$x(t) = c_1 + c_2(t-a) - \int_a^t (t-s)[g(s)(D_a^\beta x)(s) + f(s)x(s)] ds,$$

with $c_1, c_2 \in \mathbb{R}$.

Now, since $x(a) = 0$, then $c_1 = 0$. Also, since $x(b) = 0$, then

$$c_2 = \frac{1}{b-a} \int_a^b (b-s)[g(s)(D_a^\beta x)(s) + f(s)x(s)] ds.$$

Therefore,

$$x(t) = \int_a^t \left[\frac{t-a}{b-a}(b-s) - (t-s) \right] [g(s)(D_a^\beta x)(s) + f(s)x(s)] ds$$

$$+ \int_t^b \frac{t-a}{b-a}(b-s)[g(s)(D_a^\beta x)(s) + f(s)x(s)]ds,$$

which after some simplifications finally yields

$$(b-a)x(t) = \int_a^t (b-t)(s-a)[g(s)(D_a^\beta x)(s) + f(s)x(s)]ds$$
$$+ \int_t^b (t-a)(b-s)[g(s)(D_a^\beta x)(s) + f(s)x(s)]ds.$$

Differentiating both sides of the previous equality gives

$$(b-a)x'(t) = - \int_a^t (s-a)[g(s)(D_a^\beta x)(s) + f(s)x(s)]ds$$
$$+ \int_t^b (b-s)[g(s)(D_a^\beta x)(s) + f(s)x(s)]ds. \tag{8}$$

Let $v = \max_{t \in [a,b]} |x'(t)| > 0$. Then, by the mean value theorem and the fact that $x(a) = 0 = x(b)$, we know that

$$|x(t)| \leqslant v(t-a),$$

and

$$|x(t)| \leqslant v(b-t),$$

for $t \in [a,b]$. Therefore,

$$|x(t)| \leqslant v\phi(t), \tag{9}$$

where $\phi(t) = \min(t-a, b-t)$, and it is clear that the $\leqslant$ in (9) is a $<$ for some $t \in (a,b)$. Moreover, in view of $x(a) = 0$, we have that[1]

$$|(D_a^\beta x)(t)| = \left| \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} x'(s)ds \right| \leqslant \frac{v}{\Gamma(2-\beta)}(t-a)^{1-\beta}, \quad 0 < \beta < 1,$$

where again the inequality is strict for some $t \in (a,b)$. Therefore,

$$(b-a)|x'(t)| < v \int_a^t (s-a) \left[|g(s)| \frac{(s-a)^{1-\beta}}{\Gamma(2-\beta)} + |f(s)|\phi(s) \right] ds$$
$$+ v \int_t^b (b-s) \left[|g(s)| \frac{(s-a)^{1-\beta}}{\Gamma(2-\beta)} + |f(s)|\phi(s) \right] ds.$$

Note that the definition of ϕ shows that $(s-a)\phi(s)$ and $(b-s)\phi(s)$ are majorized by $(s-a)(b-s)$ on $[a,b]$, hence

$$(b-a)|x'(t)| < v \left(\int_a^t \frac{(s-a)^{2-\beta}}{\Gamma(2-\beta)}|g(s)|ds + \int_t^b \frac{(s-a)^{1-\beta}}{\Gamma(2-\beta)}(b-s)|g(s)|ds \right)$$

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[1]Note that if $\beta = 1$ we immediately see that $|x'(t)| \leqslant v$ for all $t \in [a,b]$.

$$+ v \int_a^b (s-a)(b-s)|f(s)|ds. \tag{10}$$

Now, we define $S(t) = \int_a^t \frac{(s-a)^{2-\beta}}{\Gamma(2-\beta)}|g(s)|ds + \int_t^b \frac{(s-a)^{1-\beta}}{\Gamma(2-\beta)}(b-s)|g(s)|ds$ for $t \in [a,b]$. Then,

$$S'(t) = \frac{(t-a)^{2-\beta}}{\Gamma(2-\beta)}|g(t)| - \frac{(t-a)^{1-\beta}}{\Gamma(2-\beta)}(b-t)|g(t)| = (2t-(a+b))\frac{(t-a)^{1-\beta}}{\Gamma(2-\beta)}|g(t)|,$$

which means that $\max_{t \in [a,b]} S(t)$ is obtained either at $t = a$ or at $t = b$. It follows from (10) that

$$b - a < \max \left\{ \int_a^b \frac{(s-a)^{2-\beta}}{\Gamma(2-\beta)}|g(s)|ds, \int_a^b \frac{(s-a)^{1-\beta}}{\Gamma(2-\beta)}(b-s)|g(s)|ds \right\}$$
$$+ \int_a^b (s-a)(b-s)|f(s)|ds,$$

which concludes the proof. $\quad\square$

If we let $\beta = 1$ in the previous theorem, then we immediately get Hartman and Wintner's result [10]:

COROLLARY 1. *Suppose that $x \in C^2[a,b]$ is a solution of*

$$x'' + g(t)x' + f(t)x = 0, \quad t \in (a,b),$$
$$x(a) = 0 = x(b),$$

such that $x(t) \neq 0$ for $t \in (a,b)$. Then, the following inequality holds:

$$b - a < \max \left\{ \int_a^b (s-a)|g(s)|ds, \int_a^b (b-s)|g(s)|ds \right\} + \int_a^b (s-a)(b-s)|f(s)|ds.$$

REMARK 1. We note that if we assume in theorem 3 x to be only nontrivial, then we may derive the inequality (7) but with non-strict sign.

We will end this section showing that, for certain values of the parameter β, we can improve a result obtained in [6]. For the sake of completeness we recall it now:

THEOREM 4. *Let $1 < \alpha \leqslant 2$. Then, the Mittag–Leffler function*

$$E_{\alpha,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 2)}, \quad x \in \mathbb{C},$$

has no real zeros for

$$x \in \left[-\Gamma(\alpha)\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}}, 0 \right).$$

In order to complete our goal, we first need the following

Lemma 1. *Define the function*

$$f(x) = \frac{x^x}{(x-1)^{x-1}}, \quad x \in (1,2].$$

There exists a unique $x^\star \in (1,2)$ such that

$$f(x) < x+1, \ \forall x \in (1,x^\star), \ \text{and} \ f(x) > x+1, \ \forall x \in (x^\star,2].$$

Proof. The function $g(x) = x+1$ is a straight line with $g(1) = 2$ and $g(2) = 3$. Now we show that f is an increasing and concave function, with $\lim_{x\to 1} f(x) = 1$ and $f(2) = 4$, which in turn proves the result.

First, note that $x^x = e^{x\ln(x)}$. Therefore, $\lim_{x\to 0} x^x = 1$, hence $\lim_{x\to 1} f(x) = 1$. Now, standard calculations show that

$$f'(x) = \frac{x^x}{(x-1)^{x-1}}(\ln(x) - \ln(x-1)).$$

Since $x/(x-1) > 1$, then $f' > 0$ and that shows that f is increasing. Differentiating again and performing some simplifications, we obtain

$$f''(x) = \frac{x^{x-1}}{(x-1)^{x-1}}\left(x(\ln(x) - \ln(x-1))^2 - \frac{1}{x-1}\right).$$

Defining the auxiliary function

$$h(x) = x(\ln(x) - \ln(x-1))^2 - \frac{1}{x-1},$$

and differentiating it, we see that

$$h'(x) = \frac{((1-x)\ln(x-1) - 1 + (x-1)\ln(x))^2}{(x-1)^2} > 0, \quad x \in (1,2].$$

Since $h(2) < 0$ we conclude that $h(x) < 0$ on $(1,2]$, i.e. $f'' < 0$ or, in other words, f is concave on $(1,2]$. The proof is done. $\square$

Remark 2. A numerical approximation of $x^\star$ of the previous lemma is given[2] by 1.447.

The following result improves theorem 4 in the sense that, for certain values of the parameter α, the given Mittag–Leffler function cannot have zeros on a larger interval of real numbers.

Theorem 5. *Let $1 < \alpha < \overline{\alpha}$, where $\overline{\alpha} \in (1,2)$ is defined implicitly by $\frac{\overline{\alpha}^{\overline{\alpha}}}{(\overline{\alpha}-1)^{\overline{\alpha}-1}} = \overline{\alpha}+1$. Then, the Mittag–Leffler function $E_{\alpha,2}(x)$ has no real zeros for*

$$x \in (-\Gamma(\alpha)(1+\alpha),0) \supset \left[-\Gamma(\alpha)\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}},0\right).$$

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[2]This value was calculated using Maple Software

Proof. By lemma 1, the number $\overline{\alpha}$ is well defined.

Consider $a = 0$ and $b = 1$. Let $f = 0$ in (5) and suppose that x is a nontrivial solution of the following BVP

$$x''(t) + \lambda \, (D_0^\beta x)(t) = 0, \quad t \in (0,1), \ \beta \in (0,1), \ \lambda \in \mathbb{R},$$
$$x(0) = 0 = x(1).$$

By [11, corollary 5.3] we may conclude that λ must satisfy $E_{2-\beta,2}(-\lambda) = 0$. It is clear that, if such λ exist, it must be positive. By theorem 3 and remark 1, we get that

$$1 \leqslant \lambda \max \left\{ \int_0^1 \frac{s^{2-\beta}}{\Gamma(2-\beta)} ds, \int_0^1 \frac{s^{1-\beta}}{\Gamma(2-\beta)} (1-s) ds \right\} = \frac{\lambda}{\Gamma(2-\beta)} \frac{1}{3-\beta}.$$

Therefore, putting $\alpha = 2 - \beta$ we conclude that if $x \in (-\Gamma(\alpha)(1+\alpha), 0)$, then $E_{\alpha,2}(x)$ cannot have zeros. Since $\alpha < \overline{\alpha}$ we know, by lemma (1), that

$$\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} < \alpha + 1,$$

which concludes the proof. $\square$

3.2. The equation $(D_a^\alpha x) + g(t)(D_a^\beta x) + f(t)x = 0$

In this section we shall consider the following boundary value problem:

$$(D_a^\alpha x) + g(t)(D_a^\beta x) + f(t)x = 0, \quad t \in (a,b), \ \beta \in (0,1], \ \alpha \in (1,2], \tag{11}$$
$$x(a) = 0 = x(b), \tag{12}$$

where $f, g \in C[a,b]$ and $\alpha - \beta - 1 \geqslant 0$. This BVP brings many differences in its study when compared to the one described in Section 3.1. For example, now, we don't even expect to have continuously differentiable solutions on $[a,b]$. But more importantly, the analysis becomes much more complex and we could not obtain a *sharp* result, in the sense that, when $\alpha = 2$ and $\beta = 1$, our result would reduce to the one by Hartman and Wintner (cf. corollary 1). Nevertheless, our results generalize the well known de la Vallée Poussin inequality as well as the Fractional Lyapunov inequality.

We prove a series of lemmas before stating (and proving) our main result.

LEMMA 2. *Let* $x \in E_\beta := \{f \in C^1(a,b] \cap C[a,b] : (D_a^\beta f) \in C[a,b]\}$ *be a solution of* (11)–(12). *Put* $G(t) = g(t)(D_a^\beta x)(t) + f(t)x(t)$. *Then,*

$$(D_a^\beta x)(t) = \frac{1}{\Gamma(\alpha-\beta)} \left\{ \int_a^t \left[\frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1} \right] G(s) ds \right.$$
$$\left. + \int_t^b \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} G(s) ds \right\} \tag{13}$$

Proof. It is standard that $x \in E_\beta$ is a solution of (11)–(12) if and only if it satisfies the integral equation

$$x(t) = c(t-a)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} G(s) ds.$$

The boundary condition at $t = b$ in (12) determines the constant c and we get,

$$x(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1} \Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} G(s) ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} G(s) ds.$$

Finally, applying the Riemann–Liouville fractional derivative operator to both sides of the previous equality and having in mind that $(D_a^\beta (s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}$ and proposition 1, we get (13). $\square$

LEMMA 3. *Suppose that $\alpha - \beta - 1 \geqslant 0$. Define the function*

$$f(t,s) = \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1}, \quad a \leqslant s \leqslant t \leqslant b.$$

Then,

$$|f(t,s)| \leqslant \max \left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha - \beta - 1 > 0, (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \right\}.$$

Proof. We start by noticing that, if $\alpha - \beta - 1 = 0$, then

$$|f(t,s)| = \left| \frac{(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - 1 \right| = 1 - \frac{(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}.$$

Suppose now that $\alpha - \beta - 1 > 0$. Differentiating f with respect to t and make some rearrangements gives

$$
\begin{aligned}
f_t(t,s) &= \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (\alpha-\beta-1)(t-s)^{\alpha-\beta-2} \\
&= \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} \\
&\quad - (\alpha-\beta-1)\frac{(t-a)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \left(b - \left(a + \frac{(s-a)(b-a)}{t-a} \right) \right)^{\alpha-\beta-2} \\
&= \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \left[\frac{(b-s)^{\alpha-1}}{(b-a)^{\beta+1}} - \left(b - \left(a + \frac{(s-a)(b-a)}{t-a} \right) \right)^{\alpha-\beta-2} \right],
\end{aligned}
$$

$a \leqslant s < t \leqslant b$. Now, it is easy to see that

$$a + \frac{(s-a)(b-a)}{t-a} \geqslant s \iff s \geqslant a,$$

hence

$$f_t(t,s) \leqslant \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \left[\frac{(b-s)^{\alpha-1}}{(b-a)^{\beta+1}} - (b-s)^{\alpha-\beta-2} \right].$$

Observe now that

$$\frac{(b-s)^{\alpha-1}}{(b-a)^{\beta+1}} - (b-s)^{\alpha-\beta-2} \leqslant 0 \iff s \geqslant a,$$

which implies that $f_t(t,s) \leqslant 0$, i.e. f is a decreasing function. Therefore,

$$|f(t,s)| \leqslant \max\{f(s,s), |f(b,s)|\},$$

from which the result follows. $\quad \square$

LEMMA 4. *Let $\alpha - \beta - 1 \geqslant 0$. Suppose that $G : [a,b] \to \mathbb{R}_0^+$. Define $F : [a,b] \to \mathbb{R}_0^+$ by*

$$F(t)$$
$$= \int_a^t \max\left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \right\}$$
$$G(s)ds + \int_t^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} G(s)ds.$$

Then,

$$F(t)$$
$$\leqslant \max\left\{ \int_a^b \max\left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \right\} \right.$$
$$\left. G(s)ds, \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} G(s)ds \right\}.$$

Proof. We start by differentiating F on (a,b) to obtain

$$F'(t) = \left[\max\left\{ \frac{(t-a)^{\alpha-\beta-1}(b-t)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, (b-t)^{\alpha-\beta-1} - \frac{(b-t)^{\alpha-1}}{(b-a)^\beta} \right\} \right.$$
$$\left. - \frac{(t-a)^{\alpha-\beta-1}(b-t)^{\alpha-1}}{(b-a)^{\alpha-1}} \right] G(t).$$

We claim that

$$p(t) = \frac{(t-a)^{\alpha-\beta-1}(b-t)^{\alpha-1}}{(b-a)^{\alpha-1}}, \quad \alpha-\beta-1 > 0,$$

and

$$r(t) = (b-t)^{\alpha-\beta-1} - \frac{(b-t)^{\alpha-1}}{(b-a)^{\beta}},$$

coincide in exactly one point on (a,b): indeed, it is easy to check that

$$p(t) = r(t) \iff \hat{p}(t) = \frac{(t-a)^{\alpha-\beta-1}(b-t)^{\beta}}{(b-a)^{\alpha-\beta-1}} = (b-a)^{\beta} - (b-t)^{\beta} = \hat{r}(t).$$

Differentiating twice the previous functions, it is not difficult to conclude that $\hat{p}(t)$ is concave while $\hat{r}(t)$ is convex. Noticing that $\hat{p}(a) = \hat{p}(b) = 0$ and $\hat{r}(a) = 0$, $\hat{r}(b) = (b-a)^{\beta} > 0$ we conclude that p and r coincide in at most one point on (a,b). However, it is not hard to see that $\hat{p}(\frac{a+b}{2}) > \hat{r}(\frac{a+b}{2})$ and, since $\hat{p}(b) < \hat{r}(b)$, then continuity implies that there is a point $t^{\star} \in (\frac{a+b}{2},b)$ such that $p(t^{\star}) = r(t^{\star})$, which concludes the proof of our claim.

Therefore, if

$$\max\left\{ \frac{(t-a)^{\alpha-\beta-1}(b-t)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha - \beta - 1 > 0, (b-t)^{\alpha-\beta-1} - \frac{(b-t)^{\alpha-1}}{(b-a)^{\beta}} \right\} = p(t),$$

then $F'(t) = 0$ for all $t \in (a,t^{\star})$, which implies that $F(t) = F(a)$ on that interval. On the other hand, if

$$\max\left\{ \frac{(t-a)^{\alpha-\beta-1}(b-t)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha - \beta - 1 > 0, (b-t)^{\alpha-\beta-1} - \frac{(b-t)^{\alpha-1}}{(b-a)^{\beta}} \right\} = r(t),$$

then we define the function X by

$$X(t) = r(t) - p(t) = (b-t)^{\alpha-1}\left[(b-t)^{-\beta} - \frac{(t-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}} - (b-a)^{-\beta} \right].$$

Let $K(t) = (b-t)^{-\beta} - \frac{(t-a)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}} - (b-a)^{-\beta}$. Then,

$$K'(t) = \beta(b-t)^{-\beta-1} - \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}}{(b-a)^{\alpha-1}},$$

and

$$K''(t) = \beta(\beta+1)(b-t)^{-\beta-2} - \frac{(\alpha-\beta-1)(\alpha-\beta-2)(t-a)^{\alpha-\beta-3}}{(b-a)^{\alpha-1}}.$$

We see that $K'' > 0$ on (a,b), which means that K' is increasing. Now, if $\alpha - \beta - 1 = 0$, then $K' > 0$, hence K is increasing. Since $K(a) = -\frac{1}{(b-a)^{\alpha-1}}$ and $\lim_{t \to b} K(t) = \infty$, then X has a unique zero $t_{\star} \in (a,b)$ and $X(t) < 0$ on $(a,t_{\star})$, $X(t) > 0$ on $(t_{\star},b)$. Finally, suppose that $\alpha - \beta - 1 > 0$. Since $\lim_{t \to a} K'(t) = -\infty$ and $\lim_{t \to b} K'(t) = \infty$ we conclude that K' has a unique zero $\hat{t} \in (a,b)$. Moreover, we have that $X(t) < 0$ on $(a,\hat{t})$, $X(t) > 0$ on $(\hat{t},b)$. Therefore, $F(t) \leqslant \max\{F(a),F(b)\}$ and the proof is done. $\square$

It follows the main result of this section.

THEOREM 6. *Fix $\alpha - \beta - 1 \geqslant 0$, with $1 < \alpha \leqslant 2$ and $0 < \beta \leqslant 1$. Suppose that $x \in E_\beta$ is a nontrivial solution of the BVP* (11)–(12). *Then, the following inequality holds*

$$\Gamma(\alpha - \beta)$$
$$\leqslant \max\left\{ \int_a^b \max\left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \right\} \right.$$
$$|g(s)|ds, \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}|g(s)|ds \Bigg\}$$
$$+ \max\left\{ \int_a^b \max\left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, \right. \right.$$
$$(b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \Bigg\} \cdot |f(s)|\frac{(s-a)^\beta}{\Gamma(\beta+1)}ds, \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}|f(s)|$$
$$\frac{(s-a)^\beta}{\Gamma(\beta+1)}ds \Bigg\}.$$

Proof. We have by (13) that

$$|(D_a^\beta x)(t)|\Gamma(\alpha-\beta) \leqslant \int_a^t \left| \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1} \right| |G(s)|ds$$
$$+ \int_t^b \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}|G(s)|ds,$$

where $G(t) = g(t)(D_a^\beta x)(t) + f(t)x(t)$. Now, let $\mu = \max_{t \in [a,b]} |(D_a^\beta x)(t)| > 0$. Using theorem 2, we get

$$|G(t)| \leqslant |g(t)|\mu + |f(t)|\frac{(t-a)^\beta}{\Gamma(\beta+1)}\mu.$$

Inserting this inequality in the previous one, we achieve

$$\Gamma(\alpha-\beta) \leqslant \int_a^t \left| \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1} \right| \left[|g(s)| + |f(s)|\frac{(s-a)^\beta}{\Gamma(\beta+1)} \right] ds$$
$$+ \int_t^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} \left[|g(s)| + |f(s)|\frac{(s-a)^\beta}{\Gamma(\beta+1)} \right] ds$$
$$= \int_a^t \left| \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1} \right| |g(s)|ds$$
$$+ \int_t^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}|g(s)|ds$$

$$+ \int_a^t \left| \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1} \right| |f(s)| \frac{(s-a)^\beta}{\Gamma(\beta+1)} ds$$

$$+ \int_t^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} |f(s)| \frac{(s-a)^\beta}{\Gamma(\beta+1)} ds.$$

An application of lemma 3 and afterwards of lemma 4 finally yields

$$\Gamma(\alpha-\beta)$$

$$\leqslant \max \left\{ \int_a^b \max \left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \right\} \right.$$

$$|g(s)|ds, \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} |g(s)|ds \right\}$$

$$+ \max \left\{ \int_a^b \max \left\{ \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1 > 0, (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^\beta} \right\} \right.$$

$$\left. \cdot |f(s)| \frac{(s-a)^\beta}{\Gamma(\beta+1)} ds, \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} |f(s)| \frac{(s-a)^\beta}{\Gamma(\beta+1)} ds \right\}.$$

The proof is done. $\square$

The following result shows that theorem 6 is a generalization of the de la Vallée Poussin inequality.

COROLLARY 2. *Theorem* 1 *is a consequence of theorem* 6.

Proof. Put $\alpha = 2$ and $\beta = 1$ in theorem 6. Then,

$$1 \leqslant \max \left\{ \int_a^b \frac{s-a}{b-a} |g(s)|ds, \int_a^b \frac{b-s}{b-a} |g(s)|ds \right\}$$

$$+ \max \left\{ \int_a^b \frac{(s-a)^2}{b-a} |f(s)|ds, \int_a^b \frac{(b-s)(s-a)}{b-a} |f(s)|ds \right\}$$

$$< (b-a)M_1 + M_2 \max \left\{ \frac{(b-a)^2}{3}, \frac{(b-a)^2}{2} \right\} = M_1(b-a) + M_2 \frac{(b-a)^2}{2},$$

which concludes the proof. $\square$

Another consequence of theorem 6 is the fractional Lyapunov inequality, that was firstly established by the author in [5].

COROLLARY 3. *If the following fractional boundary value problem*

$$(D_a^\alpha x) + f(t)x = 0, \quad t \in (a,b), \ 1 < \alpha \leqslant 2,$$
$$x(a) = 0 = x(b),$$

where $q \in C[a,b]$ has a nontrivial solution, then

$$\int_a^b |f(s)|ds > \Gamma(\alpha) \left(\frac{4}{b-a}\right)^{\alpha-1}.$$

Proof. In theorem 6 we let $g = 0$ on $[a,b]$. Then, we may take $\beta = 0$ and we have that

$$\Gamma(\alpha) \leqslant \int_a^b \frac{(s-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} |f(s)|ds$$

Now, note that f cannot be zero on the entire interval $[a,b]$, otherwise, x would be the trivial solution. Therefore, by using [5, lemma 2.2], we get

$$\int_a^b \frac{(s-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} |f(s)|ds < \left(\frac{b-a}{4}\right)^{\alpha-1} \int_a^b |f(s)|ds,$$

from which the result follows. □

We end this work establishing a result analogous to theorem 5.

THEOREM 7. *Let $1 < \alpha \leqslant 2$ and $0 < \beta \leqslant 1$ be such that $\alpha - \beta - 1 \geqslant 0$. Then, the Mittag–Leffler function*

$$E_{\alpha-\beta,\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k(\alpha-\beta)+\alpha)},$$

has no real zeros for $x \in (-v, 0)$, where

$$v = \frac{\Gamma(\alpha-\beta)}{\max\left\{\int_0^1 \Delta(s)ds, B(\alpha-\beta,\alpha)\right\}},$$

with $\Delta(s) = \max\left\{s^{\alpha-\beta-1}(1-s)^{\alpha-1} : \alpha-\beta-1 > 0, (1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right\}$ and $B(x,y)$ being the Beta function.

Proof. Consider $a = 0$ and $b = 1$. Let $f = 0$ in (11) and suppose that x is a nontrivial solution of the following BVP

$$D_0^\alpha x(t) + \lambda (D_0^\beta x)(t) = 0, \quad t \in (0,1), \ \lambda \in \mathbb{R},$$
$$x(0) = 0 = x(1).$$

By [11, corollary 5.3] we know that λ must satisfy $E_{\alpha-\beta,\alpha}(-\lambda) = 0$. It is clear that, if such λ exist, it must be positive. Using theorem 6, we obtain

$$\Gamma(\alpha-\beta)$$
$$\leqslant \lambda \max\left\{\int_0^1 \max\left\{s^{\alpha-\beta-1}(1-s)^{\alpha-1} : \alpha-\beta-1 > 0, (1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right\}ds\right.$$

$$, \int_0^1 s^{\alpha-\beta-1}(1-s)^{\alpha-1}ds \Bigg\}.$$

Noting that $\int_0^1 s^{\alpha-\beta-1}(1-s)^{\alpha-1}ds = B(\alpha-\beta,\alpha)$, where $B(x,y)$ is the Beta function, we finally achieve the result we wanted to prove. $\square$

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