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ANOTHER CHARACTERIZATION OF ORTHOGONALITY IN HILBERT C^*–MODULES

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Abstract. We discuss a certain relation related to the Roberts orthogonality in Hilbert C^*-modules which turns out to be equivalent to the orthogonality with respect to the C^*-valued inner product. We also describe Hilbert C^*-modules in which the Birkhoff–James orthogonality coincides with the Roberts orthogonality.

1. Introduction and preliminaries

There are many different concepts of orthogonality in normed linear spaces which, in the case when the norm originates from the inner product, all coincide with the usual orthogonality with respect to that inner product. Here we mention just the Birkhoff–James and the Roberts orthogonality as two most important concepts of orthogonality in normed spaces: we say that x *is Birkhoff–James orthogonal to* y ([5, 8, 9, 10]), and we write $x \perp_B y$, if

$$\|x\| \leqslant \|x+\lambda y\|, \quad \forall \lambda \in \mathbb{C}, \tag{1}$$

and that x *is Roberts orthogonal to* y ([13]), and we write $x \perp_R y$, if

$$\|x+\lambda y\| = \|x-\lambda y\|, \quad \forall \lambda \in \mathbb{C}. \tag{2}$$

The Roberts orthogonality implies the Birkhoff–James orthogonality, while the converse does not hold in general. If $(H,(\cdot|\cdot))$ is a Hilbert space and $\|x\| = \sqrt{(x|x)}$ the induced norm, then $(x|y)=0$ is equivalent to any of the relations (1) and (2).

Hilbert C^*-modules are often regarded as a generalization of Hilbert spaces in a way that instead of a standard inner product we have a C^*-valued inner product. In the same way as any standard inner product induces a norm on a Hilbert space, any C^*-valued inner product $\langle \cdot,\cdot \rangle$ induces a mapping defined by $|x| = \langle x,x\rangle^{\frac{1}{2}}$ which is called a C^*-norm on a Hilbert C^*-module. If x and y are elements of a Hilbert C^*-module X over a C^*-algebra $\mathscr{A}$, then we say that x and y are *orthogonal* to each other if $\langle x,y\rangle = 0$. Just as in the Hilbert space case, some relations similar to (1) and (2) describe the orthogonality with respect to the C^*-valued inner product of two elements of a Hilbert C^*-module. For example, it was proved in [2] (see also [7]) that for elements x,y of a Hilbert $\mathscr{A}$-module X any of the following statements:

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(a) $|x|^2 \leqslant |x+ya|^2$ for all $a \in \mathscr{A}$;

(b) $|x|^2 \leqslant |x+\lambda y|^2$ for all $\lambda \in \mathbb{C}$;

(c) $|x| \leqslant |x+ya|$ for all $a \in \mathscr{A}$;

is equivalent to $\langle x,y\rangle = 0$. (Let us mention here that it is still an open question whether the relation $|x| \leqslant |x+\lambda y|$ for all $\lambda \in \mathbb{C}$ also implies $\langle x,y\rangle = 0$ in a general Hilbert C^*-module; see Section 3 in [2].) Since the orthogonality with respect to the C^*-valued inner product is symmetric, it is an interesting consequence of these equivalences that all of the characterizations mentioned above are symmetric.

Further, it was proved in [14] that $\langle x,y\rangle = 0$ is also equivalent to each of the following two statements:

(d) $|x+ya| = |x-ya|$ for all $a \in \mathscr{A}$;

(e) $|x+\lambda y| = |x-\lambda y|$ for all $\lambda \in \mathbb{C}$.

The following two relations fit naturally into this picture; the first one is discussed in [2] and it describes elements x and y of a Hilbert $\mathscr{A}$-module X satisfying

$$\|x\| \leqslant \|x+ya\|, \quad \forall a \in \mathscr{A}, \tag{3}$$

while the second one is the subject of the present paper and it is about elements x and y such that

$$\|x+ya\| = \|x-ya\|, \quad \forall a \in \mathscr{A}. \tag{4}$$

When (3) holds, we say that x is *strongly Birkhoff–James orthogonal to* y, and write $x \perp_B^s y$. It turns out that, in general, (3) is indeed stronger than the usual Birkhoff–James orthogonality and weaker than the orthogonality with respect to the C^*-valued inner product; therefore, this is a good way to define an orthogonality in a Hilbert C^*-module which takes into account not just the linear but also the modular structure of a Hilbert C^*-module.

It is clear that (4) implies (3). Indeed, if (4) holds, we have

$$2\|x\| = \|x+ya+x-ya\| \leqslant \|x+ya\| + \|x-ya\| = 2\|x+ya\|, \quad \forall a \in \mathscr{A}.$$

In this short note, we shall prove that (4) is actually as strong as any of the relations (a)–(e), i.e., it is equivalent to $\langle x,y\rangle = 0$. It will enable us to describe Hilbert C^*-modules in which the Birkhoff–James orthogonality coincide with the Roberts orthogonality.

We end this introductory section with basic definitions that we shall need in the sequel.

A *C^*-algebra* $\mathscr{A}$ is a Banach $*$-algebra with the norm satisfying the C^*-condition $\|a^*a\| = \|a\|^2$ for all $a \in \mathscr{A}$. We say that $a \in \mathscr{A}$ is *positive*, and write $a \geqslant 0$, if a is a self-adjoint element whose spectrum is positive. When $a,b \in \mathscr{A}$ are self-adjoint elements such that $b-a \geqslant 0$, we write $a \leqslant b$. A *positive linear functional* of $\mathscr{A}$ is a linear map $\varphi : \mathscr{A} \to \mathbb{C}$ such that $\varphi(a) \geqslant 0$ for every positive element $a \in \mathscr{A}$. By $S(\mathscr{A})$ we denote the set of all *states* of $\mathscr{A}$, that is, the set of all positive linear functionals of

$\mathscr{A}$ whose norm is 1. The *numerical range* of an element a of a unital C^*-algebra $\mathscr{A}$ is the set $V(a) = \{\varphi(a) : \varphi \in S(\mathscr{A})\}$.

A (*right*) *Hilbert C^*-module over a C^*-algebra* $\mathscr{A}$ (or a (*right*) *Hilbert $\mathscr{A}$-module*) is a linear space which is a right $\mathscr{A}$-module equipped with an $\mathscr{A}$-valued inner-product $\langle\cdot,\cdot\rangle : X \times X \to \mathscr{A}$ that is sesquilinear, positive definite and respects the module action, i.e.,

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$ for $x,y,z \in X$, $\alpha,\beta \in \mathbb{C}$;

(ii) $\langle x, ya\rangle = \langle x,y\rangle a$ for $x,y \in X$, $a \in \mathscr{A}$;

(iii) $\langle x,y\rangle^* = \langle y,x\rangle$ for $x,y \in X$;

(iv) $\langle x,x\rangle \geqslant 0$ for $x \in X$; if $\langle x,x\rangle = 0$ then $x = 0$;

and such that X is complete with respect to the norm defined by $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$, $x \in X$. In fact, for all $x,y \in X$ it holds $\langle y,x\rangle\langle x,y\rangle \leqslant \|x\|^2\langle y,y\rangle$, wherefrom $\|\langle x,y\rangle\| \leqslant \|x\|\|y\|$.

A Hilbert $\mathscr{A}$-module X is *full* if the norm closure of the span of $\{\langle x,y\rangle : x,y \in X\}$ is equal to $\mathscr{A}$. X is *singly generated* if there is $x \in X$ such that X is equal to the norm closure of $x\mathscr{A} = \{xa : a \in \mathscr{A}\}$.

For every two elements x and y of a Hilbert $\mathscr{A}$-module X, we define a map $\theta_{x,y} : X \to X$ by setting $\theta_{x,y}(z) = x\langle y,z\rangle$. By $\mathbf{K}(X)$ we denote the C^*-algebra spanned by $\{\theta_{x,y} : x,y \in X\}$. Every right Hilbert $\mathscr{A}$-module can be regarded as a left Hilbert $\mathbf{K}(X)$-module, and it holds $\|\theta_{x,x}\| = \|x\|^2$, $x \in X$.

Every Hilbert space is a Hilbert $\mathbb{C}$-module. Also, every C^*-algebra $\mathscr{A}$ can be regarded as a Hilbert C^*-module over itself with the inner product $\langle a,b\rangle := a^*b$, and the corresponding norm is just the norm on $\mathscr{A}$ because of the C^*-condition. For general theory about C^*-algebras see e.g [6, 12]; for Hilbert C^*-modules we refer the reader to [11, 15].

2. Results

Before proving our results let us first take a look at a very special case of (4), i.e. the case when X is a unital C^*-algebra $\mathscr{A}$ regarded as a Hilbert C^*-module over itself.

Let x be the unit e in $\mathscr{A}$, and $y = b$ an arbitrary element of $\mathscr{A}$ such that $\|e + ba\| = \|e - ba\|$ for all $a \in \mathscr{A}$. Then putting $a = \lambda b^*$, $\lambda \in \mathbb{C}$, we get $\|e + \lambda bb^*\| = \|e - \lambda bb^*\|$ for all $\lambda \in \mathbb{C}$. This means that e and bb^* are Roberts orthogonal. By [1, Proposition 2.5], normal elements of a C^*-algebra that are Roberts orthogonal to e are precisely those whose numerical range is a symmetric set with respect to the origin. In particular, positive nonzero elements cannot be Roberts orthogonal to e. This implies that $bb^* = 0$, i.e. $b = 0$.

This is just a special case of Theorem 2. In our proof we shall use the following characterization of the Birkhoff–James orthogonality which was obtained in [3].

THEOREM 1. ([3]) *Let X be a Hilbert $\mathscr{A}$-module and $x,y \in X$. Then $x \perp_B y$ if and only if there is $\varphi \in S(\mathscr{A})$ such that $\varphi(\langle x,x\rangle) = \|x\|^2$ and $\varphi(\langle x,y\rangle) = 0$.*

THEOREM 2. *Let X be a Hilbert $\mathscr{A}$-module and $x,y\in X$. The following conditions are mutually equivalent*:

(i) $\|x+ya\|=\|x-ya\|$ *for all* $a\in\mathscr{A}$;

(ii) $\langle x,y\rangle=0$.

Proof. Suppose that (i) holds. Then, in particular,

$$\|x+\lambda y\langle y,x\rangle\|=\|x-\lambda y\langle y,x\rangle\|,\quad \forall\lambda\in\mathbb{C},$$

which means that $y\langle y,x\rangle$ and x are Roberts orthogonal. Since the Roberts orthogonality is stronger than the Birkhoff–James orthogonality, it follows that $y\langle y,x\rangle$ is Birkhoff–James orthogonal to x. By Theorem 1, there is $\varphi\in S(\mathscr{A})$ such that

$$\varphi\big(\langle y\langle y,x\rangle,y\langle y,x\rangle\rangle\big)=\|y\langle y,x\rangle\|^2\quad\text{and}\quad\varphi\big(\langle x,y\rangle\langle y,x\rangle\big)=0.$$

Note that

$$0\leqslant\big\langle y\langle y,x\rangle,y\langle y,x\rangle\big\rangle=\langle x,y\rangle\langle y,y\rangle\langle y,x\rangle\leqslant\|y\|^2\langle x,y\rangle\langle y,x\rangle,$$

so we have

$$\|y\langle y,x\rangle\|^2=\varphi\big(\langle y\langle y,x\rangle,y\langle y,x\rangle\rangle\big)\leqslant\|y\|^2\varphi\big(\langle x,y\rangle\langle y,x\rangle\big)=0.$$

Then we have $0=\big\langle x,y\langle y,x\rangle\big\rangle=\langle x,y\rangle\langle y,x\rangle$, and therefore $\langle x,y\rangle=0$.

Conversely, suppose that (ii) holds, i.e., $\langle x,y\rangle=0$. Then

$$\|x+ya\|^2=\|\langle x+ya,x+ya\rangle\|=\|\langle x,x\rangle+a^*\langle y,y\rangle a\|,$$

$$\|x-ya\|^2=\|\langle x-ya,x-ya\rangle\|=\|\langle x,x\rangle+a^*\langle y,y\rangle a\|,$$

for every $a\in\mathscr{A}$, and therefore (i) follows. This proves our theorem. □

The following result follows from the proof of Theorem 2.

COROLLARY 1. *Let X be a Hilbert $\mathscr{A}$-module and $x,y\in X$. The following conditions are mutually equivalent*:

(i) $y\langle y,x\rangle\perp_B x$;

(ii) $y\langle y,x\rangle\perp_R x$;

(iii) $\langle x,y\rangle=0$.

REMARK 1. (a) It is obvious that the Roberts orthogonality (2) is a symmetric relation. However, it is not clear at the first sight that its generalization, i.e., the relation (4) is symmetric. Since the relation $\langle x,y\rangle=0$ is symmetric, by the preceding theorem we see that (4) is symmetric as well. In other words, Theorem 2 can be extended with the following equivalent statement:

(iii) $\|y+xa\|=\|y-xa\|$ *for all* $a\in\mathscr{A}$.

(b) It was proved in [2, Theorem 2.5] that $x \perp_B y\langle y,x\rangle$ is equivalent to $x \perp_B^s y$. Since the strong Birkhoff–James orthogonality is, in general, weaker than the orthogonality with respect to the C^*-valued inner product, Corollary 1 shows that $y\langle y,x\rangle \perp_B x$ always implies $x \perp_B y\langle y,x\rangle$, but the converse does not hold in general.

As we know, the Roberts orthogonality is, in general, stronger than the Birkhoff–James orthogonality. However, Corollary 1 says that the Birkhoff–James orthogonality of certain elements is as strong as their C^*-valued (and then also Roberts) orthogonality. We shall use this to describe Hilbert C^*-modules in which the Birkhoff–James orthogonality coincides with the Roberts orthogonality.

THEOREM 3. *Let $X \neq \{0\}$ be a full Hilbert $\mathscr{A}$-module such that for all $x,y \in X$, $x \perp_B y$ if and only if $x \perp_R y$. Then $\mathscr{A}$ or $\mathbf{K}(X)$ is isomorphic to $\mathbb{C}$. Thereby, $\mathbf{K}(X)$ is isomorphic to $\mathbb{C}$ if and only if X is a singly generated Hilbert $\mathscr{A}$-module.*

In particular, if $\mathscr{A}$ is a C^*-algebra in which the Birkhoff–James orthogonality coincides with the Roberts orthogonality, then $\mathscr{A}$ is isomorphic to $\mathbb{C}$.

Proof. In [4, Corollary 4.9], it was proved that if for all $x,y \in X$ the following two conditions

1. $\|x\| \leqslant \|x+ya\|$ for all $a \in \mathscr{A}$;

2. $\langle x,y\rangle = 0$;

are mutually equivalent, then $\mathscr{A}$ or $\mathbf{K}(X)$ is isomorphic to $\mathbb{C}$; thereby, the second option happens exactly when X is a singly generated $\mathscr{A}$-module.

Taking this into account, it is enough to prove that our assumption implies that the above two conditions are equivalent (which, evidently, reduces to showing that (1) implies (2)).

So, suppose $\|x\| \leqslant \|x+ya\|$ for all $a \in \mathscr{A}$. This means that $x \perp_B ya$ for all $a \in \mathscr{A}$. Then, by our assumption, $x \perp_R ya$ for all $a \in \mathscr{A}$, that is, $\|x+ya\| = \|x-ya\|$ for all $a \in \mathscr{A}$. By Theorem 2, it follows that $\langle x,y\rangle = 0$. This proves our statement. □

At the end, let us state a result about orthogonality preserving maps.

Let X,Y be Hilbert $\mathscr{A}$-modules and $T : X \to Y$. We say that T is *$\mathscr{A}$-linear* if $T(xa) = (Tx)a$ for all $x \in X$ and $a \in \mathscr{A}$. We say that T is an *orthogonality preserving mapping* if for $x,y \in X$ such that $\langle x,y\rangle = 0$ it holds $\langle Tx,Ty\rangle = 0$.

Theorem 2 and Corollary 1 enable us to obtain another characterization of orthogonality preserving mappings, similar to that in [7, Corollary 2.2].

COROLLARY 2. *Let X,Y be Hilbert $\mathscr{A}$-modules and $T : X \to Y$. The following statements are mutually equivalent.*

(i) T is an orthogonality preserving mapping.

(ii) If $x,y \in X$ satisfy $\|x+ya\| = \|x-ya\|$ for all $a \in \mathscr{A}$, then $\|Tx+(Ty)a\| = \|Tx-(Ty)a\|$ for all $a \in \mathscr{A}$.

(iii) If $x,y \in X$ *are such that* $x \perp_R y\langle y,x\rangle$, *then* $Tx \perp_R Ty\langle Ty,Tx\rangle$.

(iv) If $x,y \in X$ *are such that* $y\langle y,x\rangle \perp_B x$, *then* $Ty\langle Ty,Tx\rangle \perp_B Tx$.

In particular, if $T : X \to Y$ *is an* $\mathscr{A}$*-linear mapping such that* $\|x\| \leqslant \|y\|$ *implies* $\|Tx\| \leqslant \|Ty\|$, *then* T *is orthogonality preserving.*

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