Mathematical
Inequalities
& Applications
Volume 23, Number 1 (2020), 71–76 doi:10.7153/mia-2020-23-06

A HARMONIC MEAN INEQUALITY FOR THE POLYGAMMA FUNCTION

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(*Communicated by N. Elezović*)

Abstract. In this work, we discuss some new inequalities and a concavity property of the polygamma function $\psi^{(n)}(x) = \frac{d^n}{dx^n}\psi(x)$, $x > 0$, where $\psi(x)$ represents the digamma function (i.e. logarithmic derivative of the gamma function $\Gamma(x)$). Using these inequalities, minimum value of harmonic mean of $(-1)^n\psi^{(n)}(x)$ and $(-1)^n\psi^{(n)}(1/x)$ is obtained in terms of the Riemann zeta function and the Bernoulli numbers. Further new characterizations of π and the Apéry's constant are also presented as a consequence.

Gamma function started its journey in 1729 as an interpolatory value of $n!$ for non-integers with positive n. Gamma function, digamma function (psi function) and polygamma function play vital role in mathematical physics, quantum physics, theoretical physics, approximation theory and in many branches of applied science and engineering. Gamma functions are also important in the definition of hypergeometric functions and are also useful in defining the Riemann zeta function [1]. The hypergeometric functions satisfy respective differential equations and they are useful in mathematical modeling. The Riemann zeta function is a main tool in analytic number theory and studied extensively in cryptography. Besides these extensions, Gamma functions and polygamma functions are involved in several applications on their own. In view of their importance in the Theory of Special Functions, Inequalities and in Mathematical Physics, they have been intensively investigated by various researchers and mathematicians [6, 11, 17, 12, 10, 3, 4, 5, 13, 9, 7, 15, 8, 16, 2].

In 1974, W. Gautschi [11] proved a very interesting mean value inequality for the Euler's gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0.$$

He [11] proved that 1 is the minimum value of harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ for any positive real x, i.e.

$$\frac{2}{1/(\Gamma(x)) + 1/(\Gamma(1/x))} \geqslant 1 \quad \text{for all } x > 0. \tag{1}$$

The sign of equality holds for $x = 1$.

Mathematics subject classification (2010): 33B15, 39B62, 26D07.
Keywords and phrases: Polygamma function, harmonic mean, inequalities, monotonicity properties.
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Paper MIA-23-06

With the help of well known relations between harmonic, geometric and arithmetic means the following is immediate for $x>0$:

$$\Gamma(x)\Gamma(1/x) \geqslant 1 \quad \text{and} \quad \Gamma(x)+\Gamma(1/x) \geqslant 2.$$

The famous and interesting inequality (1) of Gautschi was generalized in [12] for several variables.

In 1997 H. Alzer [2], while studying the monotonicity properties of functions related to digamma function $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, refined the inequality (1) by

$$\frac{2}{1/(\Gamma(x))^2+1/(\Gamma(1/x))^2} \geqslant 1 \quad \text{for all } x>0. \tag{2}$$

Since the arithmetic mean is less than or equal to quadratic mean, it can be observed that inequality (2) is stronger than inequality (1). For more inequalities involving $\Gamma(x)$ and $\Gamma(1/x)$ we refer to [6] and references therein.

Recently, H. Alzer and G. Jameson [9] proved that the minimum of harmonic mean of $\psi(x)$ and $\psi(1/x)$ is $-\gamma$, where γ denotes the Euler-Mascheroni constant. That is

$$\frac{2}{1/\psi(x)+1/\psi(1/x)} \geqslant -\gamma \quad \text{for all } x>0 \tag{3}$$

and the sign of equality holds for $x=1$. Several generalizations of (1) can be found in the literature. For example, see [2, 3, 6, 4, 5, 7, 9, 13, 15].

In this article we are interested to establish the inequality closely related to (1) and (3) for the polygamma function given by

$$\begin{aligned}\psi^{(m)}(x) &= \frac{d^m}{dx^m}\psi(x)\\ &= (-1)^{m+1}m!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{m+1}} = (-1)^{m+1}\int_0^{\infty}\frac{t^m e^{-xt}}{1-e^{-t}}dt, \quad x>0,\, m\in\mathbb{N}.\end{aligned}$$

It can be noted that $|\psi^{(m)}(x)|=(-1)^{m+1}\psi^{(m)}(x)$ and $\psi^{(m)}(1)=(-1)^{m+1}m!\zeta(m+1)$, where $\zeta(x)$ is the Riemann zeta function [1]. Monotonicity properties and bounds of $|\psi^{(m)}(x)|$ are obtained in [17]. Bounds for the ratio of polygamma functions can be found in [10]. Various other interesting properties of $\psi^{(m)}(x)$ can be found in [18].

Our main objective in this article is to prove the following result.

THEOREM 1. *For all $x>0$ and $m\in\mathbb{N}$ we have*

$$\frac{(-1)^m 2\psi^{(m)}(x)\psi^{(m)}(1/x)}{\psi^{(m)}(x)+\psi^{(m)}(1/x)} \geqslant (-1)^m\psi^{(m)}(1) = -m!\zeta(m+1) \tag{4}$$

and the sign of equality holds if and only if $x=1$.

In particular,

$$\frac{2\psi^{(2m)}(x)\psi^{(2m)}(1/x)}{\psi^{(2m)}(x)+\psi^{(2m)}(1/x)} \geqslant -\frac{B_{2m}(2\pi)^{2m}}{2(2m)!},$$

where B_{2m} is the $2m$th Bernoulli number [1].

REMARK 1. For odd positive integers, such simple expression is known for $\zeta(n)$. For this reason, only one particular case is considered in the previous theorem.

The following lemmas will be useful in proving the main result.

LEMMA 1. (Lemma 3, [17]) *For $m \in \mathbb{N}$ and $x > 0$, the following double inequality holds:*

$$\frac{(m-1)!}{x^m} + \frac{m!}{2x^{m+1}} < (-1)^{m+1}\psi^{(m)}(x) < \frac{(m-1)!}{x^m} + \frac{m!}{x^{m+1}}.$$

LEMMA 2. (Theorem 1, [17]) *Let $m \in \mathbb{N}, c \in \mathbb{R}$ and $d \geqslant 0$. Then*

(i) *The function $x^c|\psi^{(m)}(x)|$ is strictly increasing (or strictly decreasing, respectively) on $(0,\infty)$ if and only if $c \geqslant m+1$ (or $c \leqslant m$, respectively).*

(ii) *For $d \geqslant 1/2$, the function $x^c|\psi^{(m)}(x+d)|$ is strictly increasing on $(0,\infty)$ if and only if $c \geqslant m$.*

LEMMA 3. *The function*

$$f(x) = (-1)^m\left(\psi^{(m)}(x) + \psi^{(m)}(1/x)\right) \tag{5}$$

is strictly concave on $(0,\infty)$ for all $m \in \mathbb{N}$.

Proof. Differentiating (5) with respect to x we have

$$x^4 f''(x) = (-1)^m\left(2x\psi^{(m+1)}(1/x) + \psi^{(m+2)}(1/x) + x^4\psi^{(m+2)}(x)\right). \tag{6}$$

Now combining Lemma 1 and (6) we have

$$\begin{aligned}
x^4 f''(x) &< 2m!x^{m+2} + 2(m+1)!x^{m+3} - (m+1)!x^{m+2} - (m+2)!\frac{x^{m+3}}{2} \\
&\quad - \frac{(m+1)!}{x^{m-2}} - \frac{(m+2)!}{2x^{m-1}} \\
&= -(m-1)m!x^{m+2} - (m-2)(m+1)!\frac{x^{m+3}}{2} - \frac{(m+1)!}{x^{m-2}} - \frac{(m+2)!}{2x^{m-1}} \\
&< 0 \quad \text{for } m \geqslant 2 \text{ and } x > 0,
\end{aligned}$$

which shows that f is strictly concave on $(0,\infty)$ for $m \geqslant 2$.

Now it remains to prove the result for $m = 1$. For this we use the following recurrence relation [17, eqn. 3.3]

$$\psi^{(m)}(x+1) = \psi^{(m)}(x) + \frac{(-1)^m m!}{x^{m+1}} \qquad \text{for } m \in \mathbb{N}. \tag{7}$$

Taking $m = 1$ in the above relation and combining (6), (7) and Lemma 1 we have

$$\begin{aligned}
x^4 f''(x) &= -\Big(2x\Big(\psi^{(2)}(1+1/x) - (m+1)!x^{m+2}\Big) + \psi^{(3)}(1+1/x) \\
&\qquad +(m+2)!x^{m+3} + x^4\psi^{(3)}(x)\Big) \\
&< 2x\left(\frac{x}{1+x} + \frac{2x^3}{(1+x)^3}\right) - 2x^4 - \frac{2x^3}{(1+x)^3} - \frac{3x^4}{(1+x)^4} - 2x^3 - 3x^2 \\
&= -\frac{x^2(1+10x+23x^2+26x^3+23x^4+10x^5+2x^6)}{(1+x)^4} \\
&< 0 \quad \text{for } x > 0,
\end{aligned}$$

which proves the statement for all $m \in \mathbb{N}$ and $x > 0$.

LEMMA 4. *For all $x > 0$ $(x \neq 1)$ we have*

$$(-1)^m\left(\psi^{(m)}(x) + \psi^{(m)}(1/x)\right) < (-1)^m 2\psi^{(m)}(1).$$

The upper bound is sharp.

Proof. For $f(x)$ given in (5), Lemma 3 provides $f''(x) < 0$ on $(0,\infty)$. Hence we have

$$\begin{aligned}
f'(x) &> f'(1) = 0 \quad \text{for } x \in (0,1) \\
f'(x) &< f'(1) = 0 \quad \text{for } x > 1.
\end{aligned}$$

which implies that f is strictly increasing on $(0,1]$ and strictly decreasing on $[1,\infty)$. Therefore $f(x) < f(1) = 2(-1)^m\psi^{(m)}(1)$ for $x > 0$ and $x \neq 1$.

LEMMA 5. *For all $x > 0$, $x \neq 1$ and $m \in \mathbb{N}$ we have*

$$\psi^{(m)}(x)\psi^{(m)}(1/x) > \left(\psi^{(m)}(1)\right)^2.$$

Proof. We write $1/x = y$ and consider the case $x < 1$ (or equivalently $y > 1$). Now, for $c = 0$, Lemma 2 gives $|\Psi^{(m)}(x)| = (-1)^{m+1}\Psi^{(m)}(x)$ is strictly decreasing. This means

$$(-1)^{m+1}\psi^{(m)}(x) > (-1)^{m+1}\psi^{(m)}(1) \quad \text{for } x < 1.$$

On the other hand, we have

$$(-1)^{m+1}\psi^{(m)}(y) > (-1)^{m+1}\psi^{(m)}(1) \quad \text{for } y > 1.$$

Combining these two inequalities, we get the required result.
Note that $\psi^{(m)}(1) = (-1)^{m+1}m!\zeta(m+1)$ gives the inequality in terms of the Riemann zeta function [1, 14, 18], which can be written, for $x > 0, x \neq 1$, as

$$\psi^{(m)}(x)\psi^{(m)}(1/x) > (m!\zeta(m+1))^2, \quad m \in \mathbb{N}.$$

Now we proceed with the proof of the main theorem.

Proof of Theorem 1. Using Lemma 4 and Lemma 5 we have

$$\begin{aligned}\frac{(-1)^m 2\psi^{(m)}(x)\psi^{(m)}(1/x)}{\psi^{(m)}(x)+\psi^{(m)}(1/x)} &> \frac{2(-1)^m\left(\psi^{(m)}(1)\right)^2}{\psi^{(m)}(x)+\psi^{(m)}(1/x)}\\ &> \frac{2(-1)^m\left(\psi^{(m)}(1)\right)^2}{2\psi^{(m)}(1)} = (-1)^m\psi^{(m)}(1),\end{aligned}$$

which completes the proof.

An immediate consequence of Theorem 1 is the following.

COROLLARY 1. *For all* $x > 0$ *and* $m \in \mathbb{N}$ *we have*

$$\frac{2\left|\psi^{(m)}(x)\psi^{(m)}(1/x)\right|}{\left|\psi^{(m)}(x)\right|+\left|\psi^{(m)}(1/x)\right|} \leqslant \left|\psi^{(m)}(1)\right| = m!\zeta(m+1). \tag{8}$$

and the sign of equality holds if and only if $x = 1$.

REMARK 2. It can be noted that

(i) $m = 0$ in (4) gives $-\gamma$ (the Euler-Mascheroni constant) as the minimum of harmonic mean of digamma functions $\psi(x)$ and $\psi(1/x)$, already discussed in (3), which is proved by H. Alzer and G. Jameson in [9].

(ii) $m = 1$ in (4) gives $\frac{\pi^2}{6}$ as the maximum of harmonic mean of trigamma functions $\psi'(x)$ and $\psi'(1/x)$:

$$\frac{2}{1/\psi'(x)+1/\psi'(1/x)} \leqslant \frac{\pi^2}{6} \approx 1.64493\ldots.$$

(iii) $m = 2$ in (4) gives $-\zeta(3)$ (the Apéry's constant) as the half of the minimum of harmonic mean of tetra gamma functions $\psi''(x)$ and $\psi''(1/x)$:

$$\frac{1}{1/\psi''(x)+1/\psi''(1/x)} \geqslant -\zeta(3) \approx -1.20205\ldots.$$

Acknowledgements. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, DC, 1964.

[2] H. ALZER, *A harmonic mean inequality for the gamma function*, J. Comput. Appl. Math. **87** (1997), no. 2, 195–198.

[3] H. ALZER, *Inequalities for the gamma function*, Proc. Amer. Math. Soc.**128** (2000), no. 1, 141–147.

[4] H. ALZER, *On a gamma function inequality of Gautschi*, Proc. Edinb. Math. Soc. (2) **45** (2002), no. 3, 589–600.

[5] H. ALZER, *On Gautschi's harmonic mean inequality for the gamma function*, J. Comput. Appl. Math. **157** (2003), no. 1, 243–249.

[6] H. ALZER, *Inequalities involving* $\Gamma(x)$ *and* $\Gamma(1/x)$, J. Comput. Appl. Math. **192** (2006), no. 2, 460–480.

[7] H. ALZER, *Gamma function inequalities*, Numer. Algorithms **49** (2008), no. 1-4, 53–84.

[8] H. ALZER, *Inequalities for the beta function*, Anal. Math. **40** (2014), no. 1, 1–11.

[9] H. ALZER AND G. JAMESON, *A harmonic mean inequality for the digamma function and related results*, Rend. Semin. Mat. Univ. Padova **137** (2017), 203–209.

[10] N. BATIR, *On some properties of digamma and polygamma functions*, J. Math. Anal. Appl. **328** (2007), no. 1, 452–465.

[11] W. GAUTSCHI, *A harmonic mean inequality for the gamma function*, SIAM J. Math. Anal. **5** (1974), 278–281.

[12] W. GAUTSCHI, *Some mean value inequalities for the gamma function*, SIAM J. Math. Anal.**5** (1974), 282–292.

[13] C. GIORDANO AND A. LAFORGIA, *Inequalities and monotonicity properties for the gamma function*, J. Comput. Appl. Math. **133** (2001), no. 1-2, 387–396.

[14] H. IWANIEC, *Lectures on the Riemann zeta function*, University Lecture Series, 62, Amer. Math. Soc., Providence, RI, 2014.

[15] G. J. O. JAMESON AND T. P. JAMESON, *An inequality for the gamma function conjectured by D. Kershaw*, J. Math. Inequal. **6** (2012), no. 2, 175–181.

[16] F. QI AND J.-S. SUN, *A mononotonicity result of a function involving the gamma function*, Anal. Math. **32** (2006), no. 4, 279–282.

[17] F. QI, S. GUO AND B.-N. GUO, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160.

[18] E. T. WHITTAKER AND G. N. WATSON, *A course of modern analysis*, reprint of the fourth (1927) edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1996.

(Received June 1, 2018)

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