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SOME INEQUALITIES AND AN APPLICATION OF EXPONENTIAL POLYNOMIALS

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To the memory of my mother

(*Communicated by N. Elezović*)

Abstract. In the paper, with the help of the Faà di Bruno formula, properties of the Bell polynomials of the second kind, and the inversion theorem for the Stirling numbers of the first and second kinds, the author presents an explicit formula and an identity for higher order derivatives of generating functions of exponential polynomials; consequently, the author recovers an explicit formula and finds an identity for exponential polynomials in terms of the Stirling numbers of the fist and second kinds; furthermore and importantly, with the assistance of the complete monotonicity of generating functions of exponential polynomials and other known conclusions, the author constructs some determinantal inequalities and product inequalities and deduces the logarithmic convexity and logarithmic concavity of two sequences related to exponential polynomials; finally, the author gives an application of exponential polynomials by confirming that exponential polynomials satisfy conditions for sequences required in white noise distribution theory.

1. Introduction

In combinatorics, the Bell numbers, usually denoted by B_k for $k \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, count the number of ways a set with k elements can be partitioned into disjoint and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s. The Bell numbers B_k for $k \geqslant 0$ can be generated by

$$e^{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 + t + t^2 + \frac{5}{6}t^3 + \frac{5}{8}t^4 + \frac{13}{30}t^5 + \frac{203}{720}t^6 + \frac{877}{5040}t^7 + \cdots$$

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and the first ten Bell numbers B_k for $0 \leqslant k \leqslant 9$ are

$$\begin{array}{lllll} B_0 = 1, & B_1 = 1, & B_2 = 2, & B_3 = 5, & B_4 = 15, \\ B_5 = 52, & B_6 = 203, & B_7 = 877, & B_8 = 4140, & B_9 = 21147. \end{array}$$

For more information on the Bell numbers B_k, please refer to [1, 9, 11, 22, 25, 30] and plenty of references therein.

The Touchard polynomials $T_k(x)$ for $k \geqslant 0$ can be generated by

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} T_k(x)\frac{t^k}{k!} = 1 + xt + \frac{1}{2}x(x+1)t^2 + \frac{1}{6}x(x^2+3x+1)t^3 \\ + \frac{1}{24}x(x^3+6x^2+7x+1)t^4 + \frac{1}{120}x(x^4+10x^3+25x^2+15x+1)t^5 + \cdots$$

and the first seven Touchard polynomials $T_k(x)$ for $0 \leqslant k \leqslant 6$ are

$$1, \quad x, \quad x(x+1), \quad x(x^2+3x+1), \quad x(x^3+6x^2+7x+1), \\ x(x^4+10x^3+25x^2+15x+1), \quad x(x^5+15x^4+65x^3+90x^2+31x+1).$$

Since $T_k(1) = B_k$, the Touchard polynomials $T_k(x)$ are generalizations of the Bell numbers B_k for $k \geqslant 0$. Occasionally the polynomials $T_n(x)$ are also called [17] the Bell polynomials and denoted by $B_n(x)$. There has been research on interesting applications of the Touchard polynomials $T_n(x)$ in nonlinear Fredholm-Volterra integral equations [17] and soliton theory in [14, 15, 16], including connections with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. Therefore, applications of the Touchard polynomials $T_n(x)$ to integrable nonlinear equations are greatly expected and any amendment on multi-linear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the community. For more information about the Touchard polynomials $T_n(x)$, please refer to [27, 28] and closely related references therein.

On 6 September 2017, Boyadzhiev wrote an e-mail to the author and clarified the history of the Touchard polynomials $T_n(x)$ as follows. The polynomials $T_n(x)$ were used as early as 1843 in the works of Grunert (see [7]) and possibly could have been used before him. Bell [5] called them "exponential polynomials", so did Touchard [33], Rota [32], and Boyadzhiev [7]. Touchard has not contributed much to the theory. Most properties were found by Grunert, Bell, and, for example, in the papers [7, 10]. Using the name "Touchard polynomials" could be misleading.

In this paper, continuing the paper [25], we present an explicit formula and an identity for higher order derivatives with respect to t of generating functions $e^{xe^{\pm t}}$ for exponential polynomials $T_k(x)$ with the help of the Faà di Bruno formula, properties of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,\dots,x_{n-k+1})$, and the inversion theorem for the Stirling numbers $s(n,k)$ and $S(n,k)$, recover an explicit formula and find an identity for exponential polynomials $T_k(x)$ in terms of the Stirling numbers $s(n,k)$ and $S(n,k)$, construct some determinantal inequalities and product inequalities for exponential polynomials $T_k(x)$, deduce the logarithmic convexity and logarithmic concavity related to exponential polynomials $T_k(x)$, and find an application of exponential polynomials $T_k(x)$ to white noise distribution theory by confirming that the polynomial sequence $\{T_k(x), x > 0\}_{k \geqslant 0}$ satisfies conditions for sequences required in white noise distribution theory.

2. Derivatives, an explicit formula, and an inversion formula

In this section, by the Faà di Bruno formula, properties of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$, and the inversion theorem for the Stirling numbers $s(n,k)$ and $S(n,k)$, we present an explicit formula and an identity for higher order derivatives with respect to t of the generating functions $e^{xe^{\pm t}}$. Consequently, we recover an explicit formula and find an identity for exponential polynomials $T_n(x)$ in terms of the Stirling numbers $s(n,k)$ and $S(n,k)$.

THEOREM 2.1. *For $n \geqslant 0$, the nth derivative of the generating functions $e^{xe^{\pm t}}$ with respect to t can be computed by*

$$\frac{\partial^n e^{xe^{\pm t}}}{\partial t^n} = (\pm 1)^n e^{xe^{\pm t}} \sum_{k=0}^{n} S(n,k) \left(xe^{\pm t}\right)^k \tag{2.1}$$

and the generating functions $e^{xe^{\pm t}}$ satisfy the identity

$$\sum_{k=0}^{n} (\pm 1)^k s(n,k) \frac{\partial^k e^{xe^{\pm t}}}{\partial t^k} = e^{xe^{\pm t}} \left(xe^{\pm t}\right)^n, \tag{2.2}$$

where $x \in \mathbb{C}$, $S(n,k)$ for $n \geqslant k \geqslant 0$, which can be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!},$$

represent the Stirling numbers of the second kind, and $s(n,k)$ for $n \geqslant k \geqslant 0$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1,$$

stand for the Stirling numbers of the first kind. Consequently, exponential polynomials $T_n(x)$ for $n \geqslant 0$ can be computed by

$$T_n(x) = \sum_{k=0}^{n} S(n,k) x^k \tag{2.3}$$

and satisfy

$$\sum_{k=0}^{n} s(n,k) T_k(x) = x^n. \tag{2.4}$$

Proof. In combinatorics, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\leqslant i\leqslant n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\geqslant k\geqslant 0$, see [9, p. 134, Theorem A], and satisfy identities

$$\mathrm{B}_{n,k}\left(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\right)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{2.5}$$

and

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k), \tag{2.6}$$

see [9, p. 135], where a and b are any complex numbers. The Faà di Bruno formula for computing higher order derivatives of composite functions can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\circ g(x)=\sum_{k=0}^{n}f^{(k)}(g(x))\mathrm{B}_{n,k}\left(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\right), \tag{2.7}$$

see [9, p. 139, Theorem C]. Applying $f(u)=e^{xu}$ and $u=g(t)=e^t$ to (2.7) and making use of identities (2.5) and (2.6) yield

$$\begin{aligned}\frac{\partial^n e^{xe^t}}{\partial t^n}&=\sum_{k=0}^{n}\frac{\partial^k e^{xu}}{\partial u^k}\mathrm{B}_{n,k}\left(e^t,e^t,\dots,e^t\right)\\ &=\sum_{k=0}^{n}x^ke^{xu}e^{kt}\mathrm{B}_{n,k}(1,1,\dots,1)=e^{xe^t}\sum_{k=0}^{n}x^ke^{kt}S(n,k).\end{aligned}$$

The explicit formula (2.1) for the plus sign case is thus proved.

In [31, p. 171, Theorem 12.1], it is stated that, if b_α and a_k are a collection of constants independent of n, then

$$a_n=\sum_{\alpha=0}^{n}S(n,\alpha)b_\alpha \quad \text{if and only if} \quad b_n=\sum_{k=0}^{n}s(n,k)a_k. \tag{2.8}$$

Combining this inversion theorem for the Stirling numbers with (2.1) arrives at equations

$$\sum_{k=0}^{n}s(n,k)\frac{\partial^k e^{xe^t}}{\partial t^k}=e^{xe^t}\left(xe^t\right)^n$$

which is equivalent to that the generating function e^{xe^t} satisfies the family of nonlinear ordinary differential equations in (2.2) for the plus sign case.

The equation (2.12) in the second proof of [25, Theorem 2.2] reads that

$$\mathrm{B}_{n,n}(\alpha)=\alpha^n \quad \text{and} \quad \mathrm{B}_{n+k+1,k}(\alpha,0,\dots,0)=0, \tag{2.9}$$

where $\alpha \in \mathbb{C}$ and $k,n \in \{0\} \cup \mathbb{N}$. Applying $f(u) = e^{xe^u}$ and $u = g(t) = -t$ in (2.7), taking $\alpha = -1$ in (2.9), and utilizing (2.1) lead to

$$\begin{aligned}\frac{\partial^n e^{xe^{-t}}}{\partial t^n} &= \sum_{k=0}^{n} \frac{\partial^k e^{xe^u}}{\partial u^k} \mathrm{B}_{n,k}(-1,0,\ldots,0) = \frac{\partial^n e^{xe^u}}{\partial u^n} \mathrm{B}_{n,n}(-1) \\ &= (-1)^n e^{xe^u} \sum_{k=0}^{n} S(n,k)\big(xe^u\big)^k = (-1)^n e^{xe^{-t}} \sum_{k=0}^{n} S(n,k)\big(xe^{-t}\big)^k.\end{aligned}$$

The formula (2.1) for the minus sign case follows immediately.

Employing the inversion theorem (2.8) for Stirling numbers to consider the formula (2.1) reveals the identity (2.2).

In light of the theory of series, it is easy to see that

$$T_n(x) = (\pm 1)^n \lim_{t \to 0} \frac{\partial^n e^{x(e^{\pm t}-1)}}{\partial t^n} = (\pm 1)^n e^{-x} \lim_{t \to 0} \frac{\partial^n e^{xe^{\pm t}}}{\partial t^n}.$$

Combining this with (2.1) gives

$$T_n(x) = (\pm 1)^n e^{-x} \lim_{t \to 0} (\pm 1)^n e^{xe^{\pm t}} \sum_{k=0}^{n} S(n,k)\big(xe^{\pm t}\big)^k = \sum_{k=0}^{n} S(n,k) x^k.$$

The formula (2.3) follows.

Substituting $f(t) = e^{xe^{\pm t}}$ into (2.2) and taking $t \to 0$ result in

$$\sum_{k=0}^{n} (\pm 1)^k s(n,k) (\pm 1)^k e^x T_k(x) = e^x x^n$$

which can be simplified as (2.4). The proof of Theorem 2.1 is thus complete.

3. Inequalities for exponential polynomials

In light of complete monotonicity of generating functions $e^{xe^{-t}}$ and with the assistance of properties of completely monotonic functions, we can construct some determinantal inequalities and product inequalities for exponential polynomials $T_n(x)$. From these inequalities and other conclusions in [1, 6], we can derive the logarithmic convexity and logarithmic concavity of the sequences $\{T_n(x)\}_{n \geqslant 0}$ and $\big\{\frac{T_n(x)}{n!}\big\}_{n \geqslant 0}$ respectively. These inequalities are our main results in this paper.

THEOREM 3.1. *Let $m \geqslant 1$ be a positive integer, let $|e_{ij}|_m$ denote a determinant of order m with elements e_{ij}, and let $x > 0$.*

1. If a_i for $1 \leqslant i \leqslant m$ are non-negative integers, then

$$\big|T_{a_i+a_j}(x)\big|_m \geqslant 0 \quad \textit{and} \quad \big|(-1)^{a_i+a_j} T_{a_i+a_j}(x)\big|_m \geqslant 0. \tag{3.1}$$

2. *If $a=(a_1,a_2,\dots,a_n)$ and $b=(b_1,b_2,\dots,b_n)$ are non-increasing n-tuples of non-negative integers such that $\sum_{i=1}^{k} a_i \geqslant \sum_{i=1}^{k} b_i$ for $1\leqslant k\leqslant n-1$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$, then*

$$\prod_{i=1}^{n} T_{a_i}(x) \geqslant \prod_{i=1}^{n} T_{b_i}(x). \tag{3.2}$$

Proof. Recall from [20, Chapter XIII] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t)\geqslant 0$ for $t\in I$ and $k\in\mathbb{N}$. Recall from [20, Chapter XIII] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^k f^{(k)}(x)\geqslant 0$ on I for all $k\geqslant 0$. It is clear that, if $f_1(x)$ is absolutely monotonic and $f_2(x)$ is completely monotonic on their defined intervals, then their composite function $f_1(f_2(x))$ is completely monotonic on its defined interval. Consequently, the function $e^{xe^{-t}}$ for $x>0$ is completely monotonic with respect to $t\in[0,\infty)$.

In [19] and [20, p. 367], it was obtained that if f is completely monotonic on $[0,\infty)$, then

$$\left|f^{(a_i+a_j)}(t)\right|_m \geqslant 0 \quad \text{and} \quad \left|(-1)^{a_i+a_j} f^{(a_i+a_j)}(t)\right|_m \geqslant 0. \tag{3.3}$$

Applying $f(t)$ to the function $e^{xe^{-t}}$ in (3.3) and taking the limit $t\to 0^+$ give

$$\lim_{t\to 0^+}\left|\left(e^{xe^{-t}}\right)^{(a_i+a_j)}\right|_m = \left|(-1)^{a_i+a_j} e^x T_{a_i+a_j}(x)\right|_m \geqslant 0$$

and

$$\lim_{t\to 0^+}\left|(-1)^{a_i+a_j}\left(e^{xe^{-t}}\right)^{(a_i+a_j)}\right|_m = \left|(-1)^{a_i+a_j}(-1)^{a_i+a_j} e^x T_{a_i+a_j}(x)\right|_m \geqslant 0.$$

The determinantal inequalities in (3.1) follow.

In [20, p. 367, Theorem 2], it was stated that if f is a completely monotonic function on $[0,\infty)$, then

$$\prod_{i=1}^{n}\left[(-1)^{a_i} f^{(a_i)}(t)\right] \geqslant \prod_{i=1}^{n}\left[(-1)^{b_i} f^{(b_i)}(t)\right]. \tag{3.4}$$

Applying $f(t)$ to the function $e^{xe^{-t}}$ in (3.4) and taking the limit $t\to 0^+$ give

$$\begin{aligned}\lim_{t\to 0^+}\prod_{i=1}^{n}\left[(-1)^{a_i}\left(e^{xe^{-t}}\right)^{(a_i)}\right] &= \prod_{i=1}^{n}[e^x T_{a_i}(x)]\\ &\geqslant \lim_{t\to 0^+}\prod_{i=1}^{n}\left[(-1)^{b_i}\left(e^{xe^{-t}}\right)^{(b_i)}\right] = \prod_{i=1}^{n}[e^x T_{b_i}(x)].\end{aligned}$$

The product inequality (3.2) follows. The proof of Theorem 3.1 is complete.

COROLLARY 3.1. *For $x>0$, if $\ell\geqslant 0$ and $n\geqslant k\geqslant 0$, then*

$$[T_{n+\ell}(x)]^k [T_\ell(x)]^{n-k} \geqslant [T_{k+\ell}(x)]^n.$$

Proof. This follows from taking

$$a=(\overbrace{n+\ell,\dots,n+\ell}^{k},\overbrace{\ell,\dots,\ell}^{n-k}) \quad \text{and} \quad b=(k+\ell,k+\ell,\dots,k+\ell)$$

in the inequality (3.2). The proof of Corollary 3.1 is complete.

THEOREM 3.2. *For $x>0$, the sequence $\{T_n(x)\}_{n\geqslant 0}$ is logarithmically convex and the sequence $\left\{\frac{T_n(x)}{n!}\right\}_{n\geqslant 0}$ is logarithmically concave. Consequently, for $x>0$ and $m,n\geqslant 0$,*

$$T_m(x)T_n(x)\leqslant T_{m+n}(x)\leqslant \binom{m+n}{m}T_m(x)T_n(x). \tag{3.5}$$

Proof. In [20, p. 369] and [21, p. 429, Remark], it was stated that if $f(t)$ is a completely monotonic function such that $f^{(k)}(t)\neq 0$ for $k\geqslant 0$, then the sequence

$$\ln\left[(-1)^{k-1}f^{(k-1)}(t)\right], \quad k\geqslant 1 \tag{3.6}$$

is convex. Applying this result to the function $e^{xe^{-t}}$ for $x>0$ implies that the sequence

$$\ln\left[(-1)^{k-1}\left(e^{xe^{-t}}\right)^{(k-1)}\right]\to x+\ln T_{k-1}(x), \quad t\to 0^+$$

is convex for $k\geqslant 1$. Hence, the sequence $\{T_n(x)\}_{n\geqslant 0}$ is logarithmically convex.

Alternatively, letting

$$\ell\geqslant 1, \quad n=2, \quad a_1=\ell+2, \quad a_2=\ell, \quad \text{and} \quad b_1=b_2=\ell+1$$

in the inequality (3.2) leads to $T_\ell(x)T_{\ell+2}(x)\geqslant T_{\ell+1}^2(x)$ which means that the sequence $\{T_k(x)\}_{k\in\mathbb{N}}$ is logarithmically convex.

If $\{1,X_1,X_2,\dots\}$ is a logarithmically concave sequence of nonnegative real numbers and the sequences $\{A_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ are defined by

$$\sum_{n=0}^{\infty}A_nu^n=\sum_{n=0}^{\infty}\frac{P_n}{n!}u^n=\exp\left(\sum_{i=1}^{\infty}X_i\frac{u^i}{i}\right),$$

then it was proved in [6, p. 58, Theorem 1] that the sequence $\{A_n\}_{n\geqslant 0}$ is logarithmically concave and the sequence $\{P_n\}_{n\geqslant 0}$ is logarithmically convex. By the definition of exponential polynomials $T_n(x)$, we see that

$$\sum_{n=0}^{\infty}\frac{T_n(x)}{n!}t^n=e^{x(e^t-1)}=\exp\left[x\left(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\frac{t^n}{n}\right)\right].$$

Moreover, once can easily verify that the sequence $\left\{1,\frac{1}{(n-1)!}\right\}_{n\geqslant 1}$ is logarithmically concave. Therefore, when $x>0$, the sequence $\{T_n(x)\}_{n\geqslant 0}$ is logarithmically convex and the sequence $\left\{\frac{T_n(x)}{n!}\right\}_{n\geqslant 0}$ is logarithmically concave.

Theorem 2 in [1] states that

1. if $\{\alpha_n\}_{n\geqslant 0}$ is logarithmically convex with $\alpha_0=1$, then $\alpha_m\alpha_n\leqslant\alpha_{m+n}$ for $m,n\geqslant 0$;

2. if $\left\{\frac{\alpha_n}{n!}\right\}_{n\geqslant 0}$ is logarithmically concave with $\alpha_0=1$, then $\alpha_{m+n}\leqslant\binom{m+n}{m}\alpha_m\alpha_n$ for $m,n\geqslant 0$.

Combining this theorem with the logarithmic convexity and logarithmic concavity of the sequences $\{T_n(x)\}_{n\geqslant 0}$ and $\left\{\frac{T_n(x)}{n!}\right\}_{n\geqslant 0}$ respectively leads to the double inequality (3.5). The proof of Theorem 3.2 is complete.

THEOREM 3.3. *For $x>0$, $k\geqslant 0$, and $n\in\mathbb{N}$, we have*

$$\left[\prod_{\ell=0}^{n}T_{k+2\ell}(x)\right]^{1/(n+1)}\geqslant\left[\prod_{\ell=0}^{n-1}T_{k+2\ell+1}(x)\right]^{1/n}. \tag{3.7}$$

Proof. If $f(t)$ is a completely monotonic function on $(0,\infty)$, then, by the convexity of the sequence (3.6) and Nanson's inequality listed in [18, p. 205, 3.2.27],

$$\left[\prod_{\ell=0}^{n}(-1)^{k+2\ell+1}f^{(k+2\ell+1)}(t)\right]^{1/(n+1)}\geqslant\left[\prod_{\ell=1}^{n}(-1)^{k+2\ell}f^{(k+2\ell)}(t)\right]^{1/n}$$

for $k\geqslant 0$. Replacing $f(t)$ by $e^{xe^{-t}}$ in the above inequality results in

$$\left[\prod_{\ell=0}^{n}(-1)^{k+2\ell+1}\left(e^{xe^{-t}}\right)^{(k+2\ell+1)}\right]^{1/(n+1)}\geqslant\left[\prod_{\ell=1}^{n}(-1)^{k+2\ell}\left(e^{xe^{-t}}\right)^{(k+2\ell)}\right]^{1/n}$$

for $k\geqslant 0$. Letting $t\to 0^+$ in the above inequality leads to (3.7). The proof of Theorem 3.3 is complete.

THEOREM 3.4. *For $x>0$, if $\ell\geqslant 0$, $n\geqslant k\geqslant m$, $2k\geqslant n$, and $2m\geqslant n$, then*

$$T_{k+\ell}(x)T_{n-k+\ell}(x)\geqslant T_{m+\ell}(x)T_{n-m+\ell}(x). \tag{3.8}$$

Proof. In [34, p. 397, Theorem D], it was recovered that if $f(t)$ is a completely monotonic function on $(0,\infty)$ and if $n\geqslant k\geqslant m$, $k\geqslant n-k$, and $m\geqslant n-m$, then

$$(-1)^nf^{(k)}(t)f^{(n-k)}(t)\geqslant(-1)^nf^{(m)}(t)f^{(n-m)}(t).$$

Replacing $f(t)$ by the function $(-1)^\ell\left(e^{xe^{-t}}\right)^{(\ell)}$ in the above inequality leads to

$$(-1)^n\left(e^{xe^{-t}}\right)^{(k+\ell)}\left(e^{xe^{-t}}\right)^{(n-k+\ell)}\geqslant(-1)^n\left(e^{xe^{-t}}\right)^{(m+\ell)}\left(e^{xe^{-t}}\right)^{(n-m+\ell)}.$$

Further taking $t\to 0^+$ finds the inequality (3.8). The proof of Theorem 3.4 is complete.

THEOREM 3.5. *For $x>0$, $\ell \geqslant 0$, and $m,n\in\mathbb{N}$, let*

$$\begin{aligned}
\mathscr{G}_{\ell,m,n}(x) &= T_{\ell+2m+n}(x)[T_\ell(x)]^2 - T_{\ell+m+n}(x)T_{\ell+m}(x)T_\ell(x)\\
&\quad - T_{\ell+n}(x)T_{\ell+2m}(x)T_\ell(x) + T_{\ell+n}(x)[T_{\ell+m}(x)]^2,\\
\mathscr{H}_{\ell,m,n}(x) &= T_{\ell+2m+n}(x)[T_\ell(x)]^2 - 2T_{\ell+m+n}(x)T_{\ell+m}(x)T_\ell(x) + T_{\ell+n}(x)[T_{\ell+m}(x)]^2,\\
\mathscr{I}_{\ell,m,n}(x) &= T_{\ell+2m+n}(x)[T_\ell(x)]^2 - 2T_{\ell+n}(x)T_{\ell+2m}(x)T_\ell(x) + T_{\ell+n}(x)[T_{\ell+m}(x)]^2.
\end{aligned}$$

Then

$$\begin{gathered}
\mathscr{G}_{\ell,m,n}(x) \geqslant 0, \quad \mathscr{H}_{\ell,m,n}(x) \geqslant 0,\\
\mathscr{H}_{\ell,m,n}(x) \lesseqgtr \mathscr{G}_{\ell,m,n}(x) \quad \textit{when } m \lesseqgtr n,\\
\mathscr{I}_{\ell,m,n}(x) \geqslant \mathscr{G}_{\ell,m,n}(x) \geqslant 0 \quad \textit{when } n \geqslant m.
\end{gathered} \tag{3.9}$$

Proof. In [35, Theorem 1 and Remark 2], it was obtained that if $f(t)$ is completely monotonic on $(0,\infty)$ and

$$\begin{aligned}
G_{m,n}(t) &= (-1)^n\big\{f^{(n+2m)}(t)f^2(t) - f^{(n+m)}(t)f^{(m)}(t)f(t)\\
&\quad - f^{(n)}(t)f^{(2m)}(t)f(t) + f^{(n)}(t)\big[f^{(m)}(t)\big]^2\big\},\\
H_{m,n}(t) &= (-1)^n\big\{f^{(n+2m)}(t)f^2(t) - 2f^{(n+m)}(t)f^{(m)}(t)f(t) + f^{(n)}(t)\big[f^{(m)}(t)\big]^2\big\},\\
I_{m,n}(t) &= (-1)^n\big\{f^{(n+2m)}(t)f^2(t) - 2f^{(n)}(t)f^{(2m)}(t)f(t) + f^{(n)}(t)\big[f^{(m)}(t)\big]^2\big\}
\end{aligned}$$

for $n,m\in\mathbb{N}$, then

$$\begin{gathered}
G_{m,n}(t) \geqslant 0, \quad H_{m,n}(t) \geqslant 0;\\
H_{m,n}(t) \lesseqgtr G_{m,n}(t) \quad \text{when } m \lesseqgtr n;\\
I_{m,n}(t) \geqslant G_{m,n}(t) \geqslant 0 \quad \text{when } n \geqslant m.
\end{gathered} \tag{3.10}$$

Replacing $f(t)$ by $(-1)^\ell\big(e^{xe^{-t}}\big)^{(\ell)}$ in $G_{m,n}(t)$, $H_{m,n}(t)$, and $I_{m,n}(t)$ and simplifying produce

$$\begin{aligned}
G_{m,n}(t) &= (-1)^{\ell+n}\Big\{\big(e^{xe^{-t}}\big)^{(\ell+2m+n)}\Big[\big(e^{xe^{-t}}\big)^{(\ell)}\Big]^2 - \big(e^{xe^{-t}}\big)^{(\ell+m+n)}\big(e^{xe^{-t}}\big)^{(\ell+m)}\big(e^{xe^{-t}}\big)^{(\ell)}\\
&\quad - \big(e^{xe^{-t}}\big)^{(\ell+n)}\big(e^{xe^{-t}}\big)^{(\ell+2m)}\big(e^{xe^{-t}}\big)^{(\ell)} + \big(e^{xe^{-t}}\big)^{(\ell+n)}\Big[\big(e^{xe^{-t}}\big)^{(\ell+m)}\Big]^2\Big\},\\
H_{m,n}(t) &= (-1)^{\ell+n}\Big\{\big(e^{xe^{-t}}\big)^{(\ell+2m+n)}\Big[\big(e^{xe^{-t}}\big)^{(\ell)}\Big]^2\\
&\quad - 2\big(e^{xe^{-t}}\big)^{(\ell+m+n)}\big(e^{xe^{-t}}\big)^{(\ell+m)}\big(e^{xe^{-t}}\big)^{(\ell)} + \big(e^{xe^{-t}}\big)^{(\ell+n)}\Big[\big(e^{xe^{-t}}\big)^{(\ell+m)}\Big]^2\Big\},\\
I_{m,n}(t) &= (-1)^{\ell+n}\Big\{\big(e^{xe^{-t}}\big)^{(\ell+2m+n)}\Big[\big(e^{xe^{-t}}\big)^{(\ell)}\Big]^2\\
&\quad - 2\big(e^{xe^{-t}}\big)^{(\ell+n)}\big(e^{xe^{-t}}\big)^{(\ell+2m)}\big(e^{xe^{-t}}\big)^{(\ell)} + \big(e^{xe^{-t}}\big)^{(\ell+n)}\Big[\big(e^{xe^{-t}}\big)^{(\ell+m)}\Big]^2\Big\}.
\end{aligned}$$

Further taking $t\to 0^+$ reveals

$$\lim_{t\to 0^+} G_{m,n}(t) = e^{3x}\mathscr{G}_{\ell,m,n}(x), \quad \lim_{t\to 0^+} H_{m,n}(t) = e^{3x}\mathscr{H}_{\ell,m,n}(x),$$

and $\lim_{t\to0^+} I_{m,n}(t) = e^{3x}\mathscr{I}_{\ell,m,n}(x)$. Substituting these quantities into (3.10) and simplifying bring about inequalities in (3.9). The proof of Theorem 3.5 is complete.

4. An application to white noise distribution theory

In this section, we finally find an application of exponential polynomials $T_k(x)$ by confirming that the polynomial sequence $\{T_k(x), x>0\}_{k\geqslant0}$ satisfies conditions for sequences required in white noise distribution theory.

Let $\{\alpha_n\}_{n\geqslant0}$ be a sequence of positive numbers. In [1, 3, 4, 8, 12] and closely related references therein, for studying the spaces of test and generalized functions and their characterization theorems in white noise distribution theory [13], the following conditions for the sequence $\{\alpha_n\}_{n\geqslant0}$ are required:

$$\alpha_0=1, \quad \inf_{n\geqslant0}(\alpha_n\sigma^n)>0, \quad \lim_{n\to\infty}\left(\frac{\alpha_n}{n!}\right)^{1/n}=0, \quad \lim_{n\to\infty}\left(\frac{1}{n!\alpha_n}\right)^{1/n}=0, \tag{4.1}$$

$$\limsup_{n\to\infty}\left[\frac{n!}{\alpha_n}\inf_{x>0}\frac{G_\alpha(x)}{x^n}\right]^{1/n}<\infty, \quad \limsup_{n\to\infty}\left[n!\alpha_n\inf_{x>0}\frac{G_{1/\alpha}(x)}{x^n}\right]^{1/n}<\infty, \tag{4.2}$$

$$\text{the sequence } \left\{\tfrac{\alpha_n}{n!}\right\}_{n\geqslant0} \text{ is logarithmically concave,} \tag{4.3}$$

$$\text{the sequence } \left\{\tfrac{1}{n!\alpha_n}\right\}_{n\geqslant0} \text{ is logarithmically concave,} \tag{4.4}$$

$$\text{the sequence } \{\alpha_n\}_{n\geqslant0} \text{ is logarithmically convex,} \tag{4.5}$$

$$\text{there exists a constant } c_1 \text{ such that } \alpha_n\leqslant c_1^m\alpha_m \text{ for all } n\leqslant m, \tag{4.6}$$

$$\text{there exists a constant } c_2 \text{ such that } \alpha_{m+n}\leqslant c_2^{m+n}\alpha_m\alpha_n \text{ for all } m,n\geqslant0, \tag{4.7}$$

$$\text{there exists a constant } c_3 \text{ such that } \alpha_m\alpha_n\leqslant c_3^{m+n}\alpha_{m+n} \text{ for all } m,n\geqslant0, \tag{4.8}$$

where $\sigma\geqslant1$ is a constant,

$$G_\alpha(x)=\sum_{n=0}^{\infty}\frac{\alpha_n}{n!}x^n, \quad G_{1/\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!\alpha_n}.$$

For details, please read [4, Appendix A] and closely related references therein.

Theorem 4.3 in [8] proved that the condition (4.3) implies the first one in (4.2). It is easy to check that the first two conditions in (4.1) implies the fourth one in (4.1). In [2], it was showed that the condition (4.4) implies the second one in (4.2), while (4.5) implies (4.4). In [12], it was pointed out that the condition (4.8) implies (4.6). In [4, p. 83], it was concluded that the essential conditions for distribution theory on a CKS-space are the first three in (4.1) and the conditions (4.3), (4.4), (4.7), and (4.8).

It is clear that the sequence $\{T_k(x), x>0\}_{k\geqslant0}$ satisfies the first two conditions in (4.1). Theorem 3.2 in this paper shows that the sequence $\{T_k(x), x>0\}_{k\geqslant0}$ satisfies the conditions (4.3) and (4.5). The left inequality in (3.5) means that taking $c_3=1$ in (4.8) is sound. Since $\binom{m+n}{m}\leqslant 2^{m+n}$ for $m,n\geqslant0$, the right inequality in (3.5) implies that the condition (4.7) applied to the sequence $\{T_k(x), x>0\}_{k\geqslant0}$ is valid for $c_2=2$. Since the generating function $e^{x(e^t-1)}$ of exponential polynomials $T_k(x)$ is an entire function of $t\in\mathbb{C}$, by the root test, the sequence $\{T_k(x), x>0\}_{k\geqslant0}$ satisfies the third condition in (4.1). In conclusion, the polynomial sequence $\{T_k(x), x>0\}_{k\geqslant 0}$ satisfies all the essential conditions for sequences required in distribution theory on a CKS-space.

REMARK 4.1. When taking $x=1$, all results for exponential polynomials $T_n(x)$ in this paper become those for the Bell numbers B_n, especially including those in [25].

REMARK 4.2. By the way, exponential numbers $B_n(x)$ and exponential polynomials $T_n(x)$ have been generalized in the papers [23, 24, 29] and closely related references therein.

REMARK 4.3. This paper is a revised version of the preprint [26] and closely-related preprints therein.

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