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COMMUTATORS OF CERTAIN FRACTIONAL TYPE OPERATORS WITH HÖRMANDER CONDITIONS, ONE–WEIGHTED AND TWO–WEIGHTED INEQUALITIES

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Abstract. In this paper we study commutators of a certain class of fractional type integral operators. These operators are given by kernels of the form

$$K(x,y) = k_1(x-A_1y)k_2(x-A_2y)\dots k_m(x-A_my),$$

where A_i are invertible matrices and each k_i satisfies a fractional size condition and generalized fractional Hörmander condition. We obtain weighted Coifman estimates and weighted $L^p(w^p)$ - $L^q(w^q)$ estimates. We also give a two-weighted strong type estimate for pairs of weights of the form (u,Su) where u is an arbitrary non-negative function and S is a maximal operator depending on the smoothness of the kernel K. For the singular case we also give a two-weighted endpoint estimate.

1. Introduction

In [27], Ricci and Sjögren obtained the $L^p(\mathbb{R},dx)$ boundedness, $p>1$, for a family of maximal operators on the three dimensional Heisenberg group. Some of these operators arise in the study of the boundary behavior of Poisson integrals on the symmetric space $SL\mathbb{R}^3/SO(3)$. To get the main result, they studied the boundedness on $L^2(\mathbb{R})$ of the operator

$$T_\alpha f(x) = \int_{\mathbb{R}} |x-y|^{-\alpha}|x+y|^{\alpha-1}f(y)dy, \tag{1.1}$$

for $0<\alpha<1$. Later, in [14], Godoy and Urciuolo studied a generalization of (1.1) for $\mathbb{R}^n$.

During the last years, several authors have studied operators that generalize (1.1). Let $0\leqslant\alpha<n$ and $m\in\mathbb{N}$. For $1\leqslant i\leqslant m$, let A_i be matrices such that satisfy

$$(H) \qquad A_i \text{ is invertible and } A_i-A_j \text{ is invertible for } i\neq j, 1\leqslant i,j\leqslant m.$$

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For any locally integrable bounded function $f, f \in L^{\infty}_{\text{loc}}(\mathbb{R}^n)$, define

$$T_{\alpha,m}f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \tag{1.2}$$

where

$$K(x,y) = k_1(x-A_1y)k_2(x-A_2y)...k_m(x-A_my). \tag{1.3}$$

The operators associated to functions k_i, satisfying fractional size and regularity conditions were studied in different settings such as: weighted Lebesgue and Hardy spaces with constant and variable exponent, endpoint estimates and boundedness in BMO and weighted BMO. See for example [13, 15, 16, 29, 31, 33].

These operators generalize classical operators as I_α, the fractional integral operator, and the rough fractional and singular operators. In several cases these type of operators are not bounded in H^p, but instead are bounded from H^p into L^q, $0 < p < 1$ and some q (see [30]). In the case of $\alpha = 0$, $T_{0,m}$ behaves like a singular integral operator. If $0 < \alpha < n$, $m = 1$, $A_1 = I$ and $k_1(x-A_1y) = \frac{1}{|x-y|^{n-\alpha}}$ then $T_{\alpha,1} = I_\alpha$.

In [28], Urciuolo and the second author considered each k_i as a rough fractional kernel. In those papers each k_i satisfied a L^{α_i,r_i}-Hörmander's regularity condition, $k_i \in H_{\alpha_i,r_i}$, that is, if there exists constants $c_{r_i} > 1$ and $C_{r_i} > 0$ such that for all x and $R > c_{r_i}|x|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha_i} \|(k_i(\cdot - x) - k_i(\cdot))\chi_{B(x,2^{m+1}R)\setminus B(x,2^mR)}\|_{r_i,B(x,2^mR)} < C_{r_i}.$$

More recently, in [18], we analyzed operators of the form (1.2) with conditions of regularity generalizing the $L^{\alpha,r}$-Hörmander condition and a fractional size condition. For the definitions of these conditions recall that a function $\Psi : [0,\infty) \to [0,\infty)$ is said to be a Young function if Ψ is continuous, convex, no decreasing and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$.

For each Young function Ψ we can induce an averaged of the Luxemburg norm for a function f, in the ball B, as follows

$$\|f\|_{\Psi,B} := \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Psi\left(\frac{|f|}{\lambda}\right) \leqslant 1\right\},$$

where $|B|$ is the Lebesgue measure of B. This function Ψ has an associated complementary Young function $\overline{\Psi}$ satisfying the generalized Hölder's inequality

$$\frac{1}{|B|}\int_B |fg| \leqslant 2\|f\|_{\Psi,B}\|g\|_{\overline{\Psi},B}.$$

The fractional maximal operator $M_{\alpha,\Psi}$ is defined in the following way. Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $0 \leqslant \alpha < n$, we define

$$M_{\alpha,\Psi}f(x) := \sup_{B\ni x} |B|^{\alpha/n}\|f\|_{\Psi,B}.$$

Now, we present the fractional size condition and the generalize fractional Hörmander condition. For more details see [4] or [18].

Let Ψ be a Young function and let $0 \leqslant \alpha < n$. Let us introduce some notation: $|x| \sim s$ means $s < |x| \leqslant 2s$ and we write

$$\|f\|_{\Psi,|x|\sim s} = \|f\chi_{|x|\sim s}\|_{\Psi,B(0,2s)}.$$

The function K_α is said to satisfy the fractional size condition, if there exists a constant $C > 0$ such that

$$\|K_\alpha\|_{\Psi,|x|\sim s} \leqslant Cs^{\alpha-n}.$$

In this case we denote $K_\alpha \in S_{\alpha,\Psi}$. When $\Psi(t) = t$ we write $S_{\alpha,\Psi} = S_\alpha$. Observe that if $K_\alpha \in S_\alpha$, then there exists a constant $c > 0$ such that

$$\int_{|x|\sim s} |K_\alpha(x)|dx \leqslant cs^\alpha.$$

The function K_α satisfies the $L^{\alpha,\Psi,k}$-Hörmander condition ($K \in H_{\alpha,\Psi,k}$), if there exist constants $c_\Psi > 1$ and $C_\Psi > 0$ such that for all x and $R > c_\Psi|x|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} m^k \|K_\alpha(\cdot - x) - K_\alpha(\cdot)\|_{\Psi,|y|\sim 2^m R} \leqslant C_\Psi.$$

We say that $K_\alpha \in H_{\alpha,\infty,k}$ if K_α satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim 2^m R}$ in place of $\|\cdot\|_{\Psi,|x|\sim 2^m R}$. When $k = 0$, we write $H_{\alpha,\Psi} = H_{\alpha,\Psi,0}$.

When $\Psi(t) = t^r$, $1 \leqslant r < \infty$, we simply write $H_{\alpha,r,k}$ instead of $H_{\alpha,\Psi,k}$.

In this paper, we study the k-order commutators of the operators of the form (1.2) where $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k}$.
Recall that given a locally integrable function b and an operator T_α defined as (1.2), we define the k-order commutator, $k \in \mathbb{N} \cup \{0\}$, by

$$T^k_{\alpha,b}(f)(x) = \left[b, T^{k-1}_{\alpha,b}\right] f(x) = \int (b(x) - b(y))^k K(x,y) f(y) dy$$

where we assume that $T^0_{\alpha,b} = T_\alpha$.

We also consider the following condition for the weights, there exists $c > 0$ such that

$$w(A_i x) \leqslant c w(x), \tag{1.4}$$

$a.e.\ x \in \mathbb{R}^n$ and for all $1 \leqslant i \leqslant m$.

The following is an example of a weight w that satisfies condition (1.4). Observe that also power weights satisfy this condition.

EXAMPLE. *Let* $w(x) = \begin{cases} \log\left(\frac{1}{|x|}\right) & \text{if } |x| \leqslant \frac{1}{e} \\ 1 & \text{if } |x| > \frac{1}{e} \end{cases}$. *Then* $w \in A_1$ *and satisfies* (1.4).

In 1972, R. Coifman established in [6] that a singular integral operator T with regular kernel (that is, $K \in H_{0,\infty}$) is controlled by the Hardy-Littlewood maximal function M and for every $0 < p < \infty$ and every Muckenhoupt weight $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leqslant C \int_{\mathbb{R}^n} Mf(x)^p w(x)\, dx. \tag{1.5}$$

This inequality (1.5) is called the Coifman type estimate. There have been many attempts of controlling a given singular integral operator by an appropriate maximal function (see [9], [8] and the references therein). In [19] (see also [32] and [34]) singular integral operators with less regular kernels are considered. Implicit in their proofs it is shown that the operators in question are controlled, in the sense of (1.5), by a maximal operator $M_{0,r}f(x) = M(|f|^r)(x)^{1/r}$ for some $1 \leqslant r < \infty$. The value of the exponent r is determined by the smoothness of the kernel, namely, the kernel $K \in H_{0,r}$. Let us point out that in [22] it has been proved that this control is sharp in the sense that one cannot write a pointwise smaller operator $M_{0,s}$ with $s < r$. This yields, in particular, that (1.5) do not hold in general with $M_{0,r}$ for any $1 \leqslant r < \infty$ for singular integral operators satisfying only the classical Hörmander condition H_1. Several authors studied this same problem looking for an appropriate maximal operator to control in weighted L^p norms singular, fractional operators and their commutators considering that the kernel belongs to the general class $H_{\alpha,\Psi,k}$ (see for example [4], [21] and [18]).

The main result in this paper is the following Coifman type estimate:

THEOREM 1.1. *Let $b \in BMO$, $0 \leqslant \alpha < n$, $k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $1 \leqslant i \leqslant m$. Let Ψ_i be Young functions and $0 \leqslant \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $T_{\alpha,m}$ be the integral operator defined by (1.2) and $T^k_{\alpha,m,b}$ be the k-order commutator of $T_{\alpha,m}$. Suppose that the matrices A_i satisfy the hypothesis (H) and $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k}$. If $\alpha = 0$, let $T_{0,m}$ be of strong type (p_0, p_0) for some $1 < p_0 < \infty$.Let $\varphi_k(t) = t\log(e+t)^k$ and let ϕ be a Young function such that $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t) \lesssim t$ for $t \geqslant t_0$, for some $t_0 > 0$. Let $0 < p < \infty$. Then there exists $C > 0$ such that, for $f \in L^\infty_c(\mathbb{R}^n)$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} \left|T^k_{\alpha,m,b}f(x)\right|^p w(x)dx \leqslant C\|b\|^{kp}_{BMO} \sum_{i=1}^m \int_{\mathbb{R}^n} |M_{\alpha,\phi}f(x)|^p w(A_i x)dx. \tag{1.6}$$

whenever the left-hand side is finite. The constant C depends on the operator $T^k_{\alpha,m,b}$, n, p and w.If additionally w satisfies (1.4), then

$$\int_{\mathbb{R}^n} \left|T^k_{\alpha,m,b}f(x)\right|^p w(x)dx \leqslant C\|b\|^{kp}_{BMO} \int_{\mathbb{R}^n} |M_{\alpha,\phi}f(x)|^p w(x)dx.$$

To prove this estimate, we need a pointwise estimate that relates the sharp delta maximal of the commutator with a sum of generalized fractional maximal function of f.

As a consequence of the Coifman estimate we get strong weighted estimates for the operator $T^k_{\alpha,m,b}$,

$$\left\|T^k_{\alpha,m,b}f\right\|_{L^q(w^q)} \leqslant c\|b\|^k_{BMO}\|f\|_{L^p(w^p)}$$

for suitable weights. See Theorem 3.1.

We also obtain a Fefferman-Stein type estimates.

$$\left\|T^k_{\alpha,m,b}f\right\|_{L^p(u)} \leqslant c\|f\|_{L^p(Su)},$$

where $1 < p < n/\alpha$, u is any weight and S is a suitable maximal operator. See Theorem 3.2

For $T^k_{0,m,b}$ we also give an endpoint estimate for pairs of weights (u, Su), that is,

$$u\{x \in \mathbb{R}^n : |T^k_{0,m,b}(x)| > \lambda\} \leqslant c \int_{\mathbb{R}^n} \varphi_k\left(\frac{|f(x)|}{\lambda}\right) Su(x)dx. \tag{1.7}$$

See Theorem 3.3.

The plan of the paper is the following. The next section contains some preliminaries, definitions and previous results that are needed to state the main theorems of the paper that are presented in Section 3. Section 4 is devoted to the proof of the Coifman estimate, namely, Theorem 1.1, and to the proof of a fundamental technical pointwise result. In Section 5 we prove strong one weight norm inequalities and in Section 6 the two weight norm inequalities.

2. Preliminaries and previous results

In this section we present some notions about Young functions, Luxemburg norms and weights that will be fundamental throughout the rest of the paper. Also we gather some previously known results.

2.1. Young functions and Luxemburg norms

Now, we present some extra definitions and properties for Young functions. Also we give some examples. For more details of these topics see [23] or [26].

Each Young function Ψ has an associated complementary Young function $\overline{\Psi}$ satisfying the generalized Hölder's inequality

$$\frac{1}{|B|}\int_B |fg| \leqslant 2\|f\|_{\Psi,B}\|g\|_{\overline{\Psi},B}.$$

If $\Psi_1, \dots, \Psi_m, \phi$ are Young functions satisfying $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\phi^{-1}(t) \leqslant ct$, for all $t \geqslant t_0$, some $t_0 > 0$ then

$$\|f_1\cdots f_m g\|_{L^1,B} \leqslant c\|f_1\|_{\Psi_1,B}\cdots\|f_m\|_{\Psi_m,B}\|g\|_{\phi,B}, \tag{2.1}$$

the function ϕ is called the complementary of the functions $\Psi_1, \dots, \Psi_m$.

Here are some examples of maximal operators related to certain Young functions.

- $\Psi(t) = t$, then $\|f\|_{\Psi,Q} = f_Q := \frac{1}{|Q|}\int_Q |f|$ and $M_{\alpha,\Psi} = M_\alpha$, the fractional maximal operator.

- $\Psi(t) = t^r$ with $1 < r < \infty$. In that case $\|f\|_{\Psi,Q} = \|f\|_{r,Q} := \left(\frac{1}{|Q|}\int_Q |f|^r\right)^{1/r}$ and $M_{\alpha,\Psi} := M_{\alpha,r}$, where $M_{0,r}f = M(f^r)^{1/r}$.

- $\Psi(t) = \exp(t) - 1$, then, $M_{\alpha,\Psi} = M_{\alpha,\exp(L)}$.

- If $\beta > 0$ and $1 \leqslant r < \infty$, $\Psi(t) = t^r \log(e+t)^\beta$ is a Young function then $M_{\alpha,\Psi} = M_{\alpha,L^r(\log L)^\beta}$.

- If $\alpha = 0$ and $k \in \mathbb{N}$, $\Psi(t) = t\log(e+t)^k$ it can be proved that $M_\Psi \approx M^{k+1}$, where M^{k+1} is Hardy-Littlewood maximal operator, M, iterated $k+1$ times.

REMARK 2.1. Observe that if $\Psi(t) = t^r$ then a simple computation shows that

$$M_{\alpha,r}f = (M_{\alpha r,1}|f|^r)^{1/r} = (M_{\alpha r}|f|^r)^{1/r}.$$

If $B = B(x_0, r)$, is the ball of center x_0 and radius r, for A a matrix, we set $AB = \{Ay, y \in B\}$.

PROPOSITION 2.2. *Let $\mathscr{D}$ be a Young function and A be a invertible matrix. Let $w_A(x) = w(Ax)$, then*

$$M_{\alpha,\mathscr{D}}(w_A)(A^{-1}x) \leqslant c_{A,n} M_{\alpha,\mathscr{D}}(w)(x)$$

for almost every $x \in \mathbb{R}^n$.

Proof. Fix $x \in \mathbb{R}^n$ and let us consider the ball $B = B\left(A^{-1}x, r\right)$.

$$\frac{1}{|B|}\int_B \mathscr{D}\left(\frac{w(Ay)}{\lambda}\right) dy = \frac{1}{|AB|}\int_{AB} \mathscr{D}\left(\frac{w(z)}{\lambda}\right) dz.$$

Then, $x \in AB$ and

$$\|w_A\|_{\mathscr{D},B} = \|w\|_{\mathscr{D},AB}.$$

Let $\|A\| = \sup_{x:|x|=1} |Ax|$. There exist balls $B_1 = B\left(x, \frac{r}{\|A^{-1}\|}\right)$ and $B_2 = B(x, \|A\|r)$ such that $B_1 \subset AB \subset B_2$, then

$$\|w\|_{\mathscr{D},AB} \leqslant \left\|A^{-1}\right\|^n \|A\|^n \|w\|_{\mathscr{D},B_2}.$$

Hence,

$$M^c_{\alpha,\mathscr{D}}(w_A)(A^{-1}x) \leqslant \left\|A^{-1}\right\|^n \|A\|^n M^c_{\alpha,\mathscr{D}} w(x),$$

where

$$M^c_{\alpha,\mathscr{D}} f(y) := \sup_{r>0} |B(y,r)|^{\alpha/n} \|f\|_{\mathscr{D},B(y,r)}.$$

2.2. Weights

A weight is a non negative locally integrable function in $\mathbb{R}^n$. Let $0 \leqslant \alpha < n$, $1 \leqslant p,q \leqslant \infty$, we say that a weight $w \in A_{p,q}$ if, and only if

$$[w]_{A_{p,q}} = \sup_B \|w\|_{q,B} \|w^{-1}\|_{p',B} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

If $1 \leqslant p < \infty$, A_p denotes the classical Muckenhoupt classes of weights and $A_\infty = \cup_{p \geqslant 1} A_p$. It can be prove (see [17]) that $w \in A_\infty$ if, and only if

$$[w]_\infty = \sup_B \frac{1}{w(B)} \int_B M(w\chi_B)\, dx < \infty.$$

Observe that $w \in A_{p,p}$ if and only if $w^p \in A_p$ and $w \in A_{p,\infty}$ if, and only if $w^{-p'} \in A_1$. Also $w \in A_{\infty,\infty}$ if, and only if $w^{-1} \in A_1$.

The fractional B_p condition, B_p^α, was introduced by Cruz-Uribe and Moen in [10]: Let $1 < p < n/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. A Young function $\phi \in B_p^\alpha$ if

$$\int_1^\infty \frac{\phi(t)^{q/p}}{t^q} \frac{dt}{t} < \infty.$$

They proved that if $\phi \in B_p^\alpha$ then $M_{\alpha,\phi} : L^p(dx) \to L^q(dx)$ and

$$\|M_{\alpha,\phi}\|_{L^p \to L^q} \leqslant c \left(\int_1^\infty \frac{\phi(t)^{q/p}}{t^q} \frac{dt}{t} \right)^{1/q}.$$

We will consider the following bump conditions: let $1 < q < \infty$ and Ψ be a Young function, then a weight $w \in A_{q,\Psi}$ if

$$[w]_{A_{q,\Psi}} = \sup_B \|w\|_{q,B} \left\|w^{-1}\right\|_{\Psi,B} < \infty$$

where the supremum is over all balls $B \subset \mathbb{R}^n$.

Let f be locally integrable function in $\mathbb{R}^n$. The sharp maximal function is defined by

$$M^\# f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B \left| f(y) - \frac{1}{|B|} \int_B f(z) dz \right| dy.$$

A locally integrable function f has bounded mean oscillation ($f \in BMO$) if $M^\# f \in L^\infty$ and the norm $\|f\|_{BMO} = \|M^\# f\|_\infty$.

Observe that the BMO norm is equivalent to

$$\|f\|_{BMO} = \|M^\# f\|_\infty \simeq \sup_B \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(x) - a| dx.$$

Also we set for $\delta > 0$, $M_\delta^\# f(x) := \left(M^\# |f|^\delta(x)\right)^{1/\delta}$.

2.3. Previous results

Here we enounce some known results for the operator $T_{\alpha,m}$. See [18].

THEOREM 2.3. [18] *Let $0 \leqslant \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.2). For $1 \leqslant i \leqslant m$, let Ψ_i be Young functions, $0 \leqslant \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Also suppose $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and let the matrices A_i satisfy the hypothesis (H). If $\alpha = 0$, suppose $T_{0,m}$ is of strong type (p_0, p_0) for some $1 < p_0 < \infty$.If ϕ is the complementary of the functions $\Psi_1, \ldots, \Psi_m$, then there exists $C > 0$ such that, for $0 < \delta \leqslant 1$ and $f \in L_c^\infty(\mathbb{R}^n)$*

$$M_\delta^\sharp(|T_{\alpha,m}f|)(x) \leqslant C \sum_{i=1}^{m} M_{\alpha,\phi} f(A_i^{-1}x). \tag{2.2}$$

THEOREM 2.4. [18] *Let $0 \leqslant \alpha < n$ and $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.2). For $1 \leqslant i \leqslant m$, let Ψ_i be Young functions, $0 \leqslant \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Also suppose $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and that matrices A_i satisfy the hypothesis (H). If $\alpha = 0$, suppose $T_{0,m}$ is of strong type (p_0, p_0) for some $1 < p_0 < \infty$.Let $0 < \infty$. If ϕ is the complementary of the functions $\Psi_1, \ldots, \Psi_m$, then for $f \in L_c^\infty(\mathbb{R}^n)$ and $w \in A_\infty$, there exists $C > 0$, C depending on the operator $T^k_{\alpha,m,b}$, n, p and w, such that,*

$$\int_{\mathbb{R}^n} |T_{\alpha,m}f(x)|^p w(x)dx \leqslant C \sum_{i=1}^{m} \int_{\mathbb{R}^n} |M_{\alpha,\phi} f(x)|^p w(A_i x)dx,$$

whenever the left-hand side is finite.

3. Main results

In this section we present the main results of the paper.

3.1. One weight norm inequalities

In this subsection, we state the boundedness of the operator, $T^k_{\alpha,m,b}$ in two different ways, using the Coifman inequality and using a Cauchy integral formula.

THEOREM 3.1. *Let $b \in BMO$, $0 \leqslant \alpha < n$, $k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $1 \leqslant i \leqslant m$. Let Ψ_i be Young functions and $0 \leqslant \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $T_{\alpha,m}$ be the integral operator defined by (1.2) and $T^k_{\alpha,m,b}$ be the k-order commutator of $T_{\alpha,m}$. Suppose that the matrices A_i satisfy the hypothesis (H) and $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k}$. If $\alpha = 0$, let $T_{0,m}$ be of strong type (p_0, p_0) for some $1 < p_0 < \infty$.Let $\varphi_k(t) = t\log(e + t)^k$ and let ϕ be a Young function such that $\Psi_1^{-1}(t) \cdots \Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t) \lesssim t$ for $t \geqslant t_0$, for some $t_0 > 0$. Let $1 < p < n/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that one of the following hypothesis holds:*

(a) *Suppose there exists* $1 < r < p$ *such that* $\kappa_r < \infty$. *Let* η *be a Young function such that* $\eta^{-1}(t)t^{\frac{\alpha}{n}} \lesssim \phi^{-1}(t)$ *for every* $t > 0$. *If* $\phi^{1+\frac{sn}{n-\alpha}} \in B_{\frac{sn}{n-\alpha}}$ *for every* $s > r(n-\alpha)/(n-\alpha r)$ *and* $w^r \in A_{\frac{p}{r},\frac{q}{r}}$.

(b) *Suppose there exist* B *and* C *be Young functions such that* $B^{-1}(t)C^{-1}(t) \leqslant \tilde{c}\phi^{-1}(t)$, $t > t_0 > 0$, $C \in B_p^{\alpha}$ *and* $w \in A_{q,B}$.

(c) *Suppose that the operator* $T_{\alpha,m}$ *is bounded from* $L^p(w^p)$ *into* $L^q(w^q)$ *for all* $w \in A_{p,q}$.

If w *satisfies the condition* (1.4) *then there exists* $c > 0$ *such that, for every* $f \in L^p(w^p)$,

$$\left\| T^k_{\alpha,m,b} f \right\|_{L^q(w^q)} \leqslant c\|b\|^k_{BMO}\|f\|_{L^p(w^p)}.$$

The constant, c *depends on the operator* $T^k_{\alpha,m,b}$, n, p *and* w.

3.2. Two weight norm inequalities

Next, we obtain two weight inequality for operators such that their adjoints satisfy a Coifman inequality. Here, the weights are no longer in A_∞.

THEOREM 3.2. *Let* ϕ *be a Young function,* $0 \leqslant \alpha < n$ *and* $1 < p < \infty$. *Suppose there exist Young functions* $\mathscr{E}, \mathscr{F}$ *such that* $\mathscr{E} \in B_{p'}$ *and* $\mathscr{E}^{-1}(t)\mathscr{F}^{-1}(t) \leqslant \phi^{-1}(t)$. *Let* T *be a linear operator such that its adjoint* T^* *satisfies*

$$\int_{\mathbb{R}^n} |T^* f(x)|^q w(x)dx \leqslant c \int_{\mathbb{R}^n} \sum_{i=1}^m \left(M_{\alpha,\phi} f(A_i x)\right)^q w(x)dx, \tag{3.1}$$

for all $0 < q < \infty$ *and* $w \in A_\infty$.

Set $\mathscr{D}(t) = \mathscr{F}(t^{1/p})$. *If* $\mathscr{D}$ *is a Young function then for any weight* u,

$$\int_{\mathbb{R}^n} |Tf(x)|^p u(x)dx \leqslant c \int_{\mathbb{R}^n} |f(x)|^p \sum_{i=1}^m M_{\alpha p,\mathscr{D}} u(A_i x)dx. \tag{3.2}$$

For example, if $T = T_{\alpha,m}$ is defined by (1.2), then its adjoint T^* is

$$T^* g(x) = \int \tilde{k}_1(x - A_1^{-1}y) \cdots \tilde{k}_m(x - A_m^{-1}y) g(y)dy,$$

where $\tilde{k}_i(x) = k_i(-A_i x)$. If T^* satisfies hypothesis of Theorem 1.1 then

$$\int_{\mathbb{R}^n} |T^* f(x)|^q w(x)dx \leqslant c \int_{\mathbb{R}^n} \sum_{i=1}^m \left(M_{\alpha,\phi} f(A_i x)\right)^q w(x)dx.$$

for all $0 < q < \infty$ and $w \in A_\infty$. So, we can apply the Theorem 3.1.

In the following table, Table 1, we can see some examples.

Table 1: Examples

$M_{\alpha,\phi}$	Range of p's	$M_{\alpha p,\mathscr{D}}$
$M_{\alpha,L\log L^k}$	$1<p<\infty$	$M_{\alpha,L\log L^{(k+1)p-1+\varepsilon}}$
$M_{\alpha,L^{r'}\log L^{r'(k+1)}}$	$1<p<r$	$M_{\alpha,L^{\left(\frac{r}{p}\right)'}\log L^{\left(\frac{r}{p}\right)'((k+1)+p-1)+\varepsilon}}$

Now we give an endpoint estimate for $T^k_{0,m,b}$ that derives from Theorems 1.1 and 3.2.

THEOREM 3.3. *Let $b\in BMO$, $k\in\mathbb{N}\cup\{0\}$, $m\in\mathbb{N}$ and $1\leqslant i\leqslant m$. Let Ψ_i be Young functions and $0\leqslant\alpha_i<n$ such that $\alpha_1+\cdots+\alpha_m=n$. Let $T_{0,m}$ be the integral operator defined by (1.2) and $T^k_{0,m,b}$ be the k-order commutator of $T_{0,m}$. Suppose that the matrices A_i satisfy the hypothesis (H) and $k_i\in S_{n-\alpha_i,\Psi_i}\cap H_{n-\alpha_i,\Psi_i,k}$, and $T_{0,m}$ be of strong type (p_0,p_0) for some $1<p_0<\infty$. Let $\varphi_k(t)=t\log(e+t)^k$ and let ϕ be a Young function such that $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t)\lesssim t$ for $t\geqslant t_0$, for some $t_0>0$.*

(a) If there exists $r>1$ such that $t^r\leqslant c\phi(t)$ for $t\geqslant t_0>0$, and $\phi\in B_p$ then

$$u\{x\in\mathbb{R}^n:|T^k_{0,m,b}(x)|>\lambda\}\leqslant c\int_{\mathbb{R}^n}\varphi_k\left(\frac{|f(x)|}{\lambda}\right)\sum_{i=1}^m M_\phi u(A_i(x))dx. \tag{3.3}$$

holds for every weight u.

(b) Suppose there exist Young functions $\mathscr{E},\mathscr{F}$ such that $\mathscr{E}\in B_{p'}$ and $\mathscr{E}^{-1}(t)\mathscr{F}^{-1}(t)\leqslant\phi^{-1}(t)$. Set $\mathscr{D}(t)=\mathscr{F}(t^{1/p})$, then

$$u\{x\in\mathbb{R}^n:|T^k_{0,m,b}(x)|>\lambda\}\leqslant c\int_{\mathbb{R}^n}\varphi_k\left(\frac{|f(x)|}{\lambda}\right)\sum_{i=1}^m M_{\mathscr{D}} u(A_i(x))dx. \tag{3.4}$$

holds for for all weight u.

REMARK 3.4. Observe that the pairs of weights given in (a) are better than the one in (b). (See Remark 3.3 in [20])

4. Proof of the Coifman inequality

Recall some classical results concern to functions in BMO. For the proof see for example the John-Niremberg theorem in [12].

LEMMA 4.1. *Let $b\in BMO$.*

1. *For any measurable subsets $B_1 \subset B_2 \subset \mathbb{R}^n$ such that $|B_1|, |B_2| > 0$, we have*

$$|b_{B_1} - b_{B_2}| \leqslant \frac{|B_2|}{|B_1|}\|b\|_{BMO}.$$

Let $1 \leqslant i \leqslant m$. In particular, if A_i are invertible matices, $\tilde{B}$ is a measurable set and $\tilde{B}_i = A_i^{-1}\tilde{B}$, then

$$|b_{\tilde{B}} - b_{(\cup_{i=1}^m \tilde{B}_i)\cup\tilde{B}}| \leqslant (1 + \sum_{i=1}^m |det(A_i^{-1})|)\|b\|_{BMO}.$$

2. *Let $B = B(c_B, R)$ be a ball, centered at c_B with radius R, and $B^j = B(c_B, 2^j R)$. Then,*

$$|b_B - b_{B^j}| \leqslant cj\|b\|_{BMO}.$$

The following lemma is part of the proof of Theorem 3.1 in [18],

LEMMA 4.2. *Let $0 \leqslant \alpha < n$, $m \in \mathbb{N}$ and $1 \leqslant i \leqslant m$. Let Ψ_i be Young functions and $0 \leqslant \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Suppose that the matrices A_i satisfy hypothesis (H) and $k_i \in S_{n-\alpha_i,\Psi_i}$ for $1 \leqslant i \leqslant m$. Let $\varphi_k(t) = t\log(e+t)^k$ and let ϕ be a Young function such that $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t) \lesssim t$ for $t \geqslant t_0$, some $t_0 > 0$. Let $K(x,y) = k_1(x - A_1 y)k_2(x - A_2 y)\ldots k_m(x - A_m y)$. Let $B = B(c_B, R)$ be a ball centered at c_B with radius R. We write $\tilde{B} = B(c_B, 2R)$ and for $1 \leqslant i \leqslant 2$, set $\tilde{B}_i = A_i^{-1}\tilde{B}$. If $z \in \tilde{B}_j$, for some $1 \leqslant j \leqslant m$, then there exists a positive constant C such that*

$$\int_B |K(y,z)|dy \leqslant CR^\alpha.$$

To obtain an appropriate maximal operator, which controls in weighted L^p norms the operator $T^k_{\alpha,m,b}$, we need the following theorem:

THEOREM 4.3. *Let $b \in BMO$, $0 \leqslant \alpha < n$, $k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $1 \leqslant i \leqslant m$. Let Ψ_i be Young functions and $0 \leqslant \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $T_{\alpha,m}$ be the integral operator defined by (1.2) and $T^k_{\alpha,m,b}$ be the k-order commutator of $T_{\alpha,m}$. Suppose that the matrices A_i satisfy the hypothesis (H) and $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k}$. If $\alpha = 0$, let $T_{0,m}$ be of strong type (p_0, p_0) for some $1 < p_0 < \infty$.Let $\varphi_k(t) = t\log(e+t)^k$ and let ϕ be a Young function such that $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t) \lesssim t$ for $t \geqslant t_0$, some $t_0 > 0$. Then, there exists $0 < C = C(n, \alpha, A_1, \ldots, A_m)$ such that, for $0 < \delta < \varepsilon \leqslant 1$ and $f \in L^\infty_c(\mathbb{R}^n)$*

$$M^\sharp_\delta(|T^k_{\alpha,m,b}f|)(x) \leqslant C\sum_{l=0}^{k-1}\|b\|^{k-l}_{BMO}M_\varepsilon(T^l_{\alpha,m,b})(x) + C\|b\|^k_{BMO}\sum_{i=1}^m M_{\alpha,\phi}f(A_i^{-1}x). \quad (4.1)$$

This Theorem is a generalization of several known results. The improvement of having $M_\delta^\#$ was explored in [1], [25]. The following table illustrates some examples of this result.

Table 2: Examples

	$\Psi_i \quad 1 \leqslant i \leqslant m$	ϕ	$M_{\alpha,\phi}$
(i)	∞	$t\log(e+t)^k$	$M_{\alpha,L\log L^k}$
(ii)	t^{r_i}, $1<r_i<\infty$	$t^s\log(e+t)^{sk}$, $\sum_{i=1}^m \frac{1}{r_i}+\frac{1}{s}=1$	$M_{\alpha,L^s\log L^{sk}}$
(iii)	$\Psi_1=t^r$, $1<r<\infty$ $\Psi_2(t)=\exp(t)-1$	$t^{r'}\log(e+t)^{(k+1)r'}$	$M_{\alpha,L^{r'}\log L^{r'(k+1)}}$

The example (i) with $m=1$ is the classical case proved in [4], (ii) with $k=0$ is the example of fractional rough kernel proved in [28]. The last example (iii) is the commutator of the explicit operator given in [18].

In the proof of Theorem 4.3, we follow the original ideas of papers [1] and [25] and for technical details of set partitions we follow Theorem 2.2 in [28].

Proof of Theorem 4.3. We just consider the case $m=2$ and $k=1$, i.e. $T^1_{\alpha,2,b}=[b,T_{\alpha,2}]$, and we will write $[b,T_\alpha]$. The general case is proved in an analogous way.

Let f be a bounded function with compact support, $b\in BMO$ and $0<\delta<\varepsilon\leqslant 1$. Let $x\in\mathbb{R}^n$ and let $B=B(c_B,R)$ be a ball that contains x, centered at c_B with radius R. We write $\tilde{B}=B(c_B,2R)$ and for $1\leqslant i\leqslant 2$, set $\tilde{B}_i=A_i^{-1}\tilde{B}$, $|\tilde{B}_i|=|\det(A_i^{-1})||\tilde{B}|$. Let $f_1=f\chi_{\cup_{i=1}^2\tilde{B}_i}$ and $f_2=f-f_1$.
Suppose that $a:=T_\alpha((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f_2)(c_B)<\infty$.

We write

$$[b,T_\alpha f](x)=(b(x)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})T_\alpha f(x)-T_\alpha((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f)(x).$$

Now, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B|[b,T_\alpha f](y)-a|^\delta dy\right)^{1/\delta}\\
\leqslant&\left(\frac{1}{|B|}\int_B|(b(y)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})T_\alpha f(y)|^\delta dy\right)^{1/\delta}\\
&+\left(\frac{1}{|B|}\int_B|T_\alpha((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f_1)(y)|^\delta dy\right)^{1/\delta}\\
&+\left(\frac{1}{|B|}\int_B|T_\alpha((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f_2)(y)-T_\alpha((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f_2)(c_B)|^\delta dy\right)^{1/\delta}\\
=&I+II+III.
\end{aligned}\tag{4.2}$$

To estimate I, let $q=\varepsilon/\delta>1$, by Hölder's inequality and Lemma 4.1,

$$\begin{aligned}
I &\leqslant \left(\frac{1}{|B|}\int_B |(b(y)-b_{\tilde{B}})T_\alpha f(y)|^\delta dy\right)^{1/\delta} + |b_{\tilde{B}}-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}|\left(\frac{1}{|B|}\int_B |T_\alpha f(y)|^\delta dy\right)^{1/\delta}\\
&\leqslant \left(\frac{1}{|B|}\int_B |(b(y)-b_{\tilde{B}})|^{q'\delta} dy\right)^{1/q'\delta}\left(\frac{1}{|B|}\int_B |T_\alpha f(y)|^{q\delta} dy\right)^{1/q\delta} + C\|b\|_{BMO}M_\delta(T_\alpha f)(x)\\
&\leqslant C\|b\|_{BMO}M_\varepsilon(T_\alpha f)(x) + C\|b\|_{BMO}M_\delta(T_\alpha f)(x)\\
&\leqslant C\|b\|_{BMO}M_\varepsilon(T_\alpha f)(x).
\end{aligned}$$

For II, by Jensen inequality

$$\begin{aligned}
II &\leqslant \frac{1}{|B|}\int_B\int_{\tilde{B}_1\cup\tilde{B}_2} |K(y,z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f_1(z)|dzdy\\
&\leqslant \sum_{i=1}^{2}\frac{1}{|B|}\int_{\tilde{B}_i} |b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f_1(z)|\int_B |K(y,z)|dydz. \qquad (4.3)
\end{aligned}$$

Then, using Lemma 4.2 we obtain

$$\begin{aligned}
II &\leqslant CR^\alpha\sum_{i=1}^{2}\frac{1}{|B|}\int_{\tilde{B}_i} |b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dz\\
&\leqslant CR^\alpha\sum_{i=1}^{2}\frac{1}{|\tilde{B}_i|}\int_{\tilde{B}_i} (|b(z)-b_{\tilde{B}_i}|+|b_{\tilde{B}_i}-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}|)|f(z)|dz\\
&\leqslant C\sum_{i=1}^{2}\left[R^\alpha\|b-b_{\tilde{B}_i}\|_{\exp L,\tilde{B}_i}\|f\|_{\phi,\tilde{B}_i} + \|b\|_{BMO}M_\alpha f(A_i^{-1}x)\right]\\
&\leqslant C\|b\|_{BMO}\sum_{i=1}^{2}M_{\alpha,\phi}f(A_i^{-1}x).
\end{aligned}$$

For III, by Jensen inequality we get

$$\begin{aligned}
III &\leqslant \frac{1}{|B|}\int_B |T_{\alpha,2}((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f_2)(y) - T_{\alpha,2}((b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})f_2)(c_B)|dy\\
&\leqslant \frac{1}{|B|}\int_B\int_{(\tilde{B}_1\cup\tilde{B}_2)^c} |K(y,z)-K(c_B,z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dzdy\\
&\leqslant \frac{1}{|B|}\int_B\sum_{l=1}^{2}\int_{Z^l} |K(y,z)-K(c_B,z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dzdy,
\end{aligned}$$

where

$$Z^l=(\tilde{B}_1\cup\tilde{B}_2)^c\cap\{z:|c_B-A_lz|\leqslant|c_B-A_rz|, r\neq l, r=1,2\}.$$

Let us estimate $|K(y,z)-K(c_B,z)|$ for $y\in B$ and $z\in Z^l$,

$$|K(y,z)-K(c_B,z)|\leqslant|k_1(y-A_1z)-k_1(c_B-A_1z)||k_2(y-A_2z)|$$

$$+|k_1(c_B-A_1z)||k_2(y-A_2z)-k_2(c_B-A_2z)|. \tag{4.4}$$

For simplicity we estimate the first summand of (4.4), the other one follows in an analogous way. For $j\in\mathbb{N}$, let

$$D_j^l=\{z\in Z^l:|c_B-A_lz|\sim 2^{j+1}R\}.$$

Observe that $D_j^l\subset\{z:|c_B-A_lz|\sim 2^{j+1}R\}\subset A_l^{-1}B(c_B,2^{j+2}R)=:\tilde{B}_{l,j}$. Using generalized Hölder's inequality

$$\begin{aligned}
&\int_{Z^l}|k_1(y-A_1z)-k_1(c_B-A_1z)||k_2(y-A_2z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dz\\
&\leqslant\sum_{j=1}^{\infty}\int_{D_j^l}|k_1(y-A_1z)-k_1(c_B-A_1z)||k_2(y-A_2z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dz\\
&\leqslant\sum_{j=1}^{\infty}\frac{|\tilde{B}_{l,j}|}{|\tilde{B}_{l,j}|}\int_{\tilde{B}_{l,j}}\Big[\chi_{D_j^l}|k_1(y-A_1z)-k_1(c_B-A_1z)||k_2(y-A_2z)|\\
&\qquad\Big(|b(z)-b_{\tilde{B}_{l,j}}|+|b_{\tilde{B}_{l,j}}-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}|\Big)|f(z)|\Big]dz\\
&\leqslant\sum_{j=1}^{\infty}|\tilde{B}_{l,j}|\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{D_j^l}\|_{\Psi_1,|c_B-A_lz|\sim 2^{j+1}R}\\
&\qquad\|k_2(y-A_2\cdot)\chi_{Z^l}\|_{\Psi_2,|c_B-A_lz|\sim 2^{j+1}R}\Big(\|b-b_{\tilde{B}_l^j}\|_{\exp L,\tilde{B}_{l,j}}+cj\|b\|_{BMO}\Big)\|f\|_{\phi,\tilde{B}_{l,j}}\\
&\leqslant c\|b\|_{BMO}\sum_{j=1}^{\infty}|\tilde{B}_{l,j}|j\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{D_j^l}\|_{\Psi_1,|c_B-A_lz|\sim 2^{j+1}R}\\
&\qquad\|k_2(y-A_2\cdot)\chi_{D_j^l}\|_{\Psi_2,|c_B-A_lz|\sim 2^{j+1}R}\|f\|_{\phi,\tilde{B}_{l,j}}.
\end{aligned}$$

Observe that $|c_B-A_lz|/2\leqslant|y-A_lz|<2|c_B-A_lz|$ and if $|c_B-A_lz|\sim 2^{j+1}R$ then $2^jR\leqslant|y-A_lz|\leqslant 2^{j+2}R$. Thus, we have

$$\|k_l(y-A_l\cdot)\chi_{D_j^l}\|_{\Psi_l,|c_B-A_lz|\sim 2^{j+1}R}\leqslant\|k_l(\cdot)\|_{\Psi_l,|x|\sim 2^jR}+\|k_l(\cdot)\|_{\Psi_l,|x|\sim 2^{j+1}R}\leqslant c(2^jR)^{-\alpha_l},$$

where the last inequality holds since $k_l\in S_{n-\alpha_l,\Psi_l}$. Also, by hypothesis

$$\|k_l(c_B-A_l\cdot)\chi_{D_j^l}\|_{\Psi_l,|c_B-A_lz|\sim 2^{j+1}R}\leqslant c(2^{j+1}R)^{-\alpha_l}.$$

For $r\neq l$, let us prove that

$$\|k_r(y-A_r\cdot)\chi_{D_j^l}\|_{\Psi_r,|c_B-A_lz|\sim 2^{j+1}R}\leqslant c(2^jR)^{-\alpha_r}. \tag{4.5}$$

If $z\in D_j^l$ then $|c_B-A_rz|\geqslant|c_B-A_lz|\geqslant 2^{j+1}R$, that is $D_j^l\subset A_r^{-1}B(c_B,2^{j+1}R)^c$. Then $D_j^l\subset A_l^{-1}B(c_B,2^{j+2}R)\subset A_r^{-1}B(c_B,2^{t+j+2}R)$ for some $t>1$ such that $2^t\geqslant\|A_rA_l^{-1}\|$. We decompose $D_j^l=\bigcup_{k=j}^{j+t}(D_j^l)_{k,r}$ where

$$(D_j^l)_{k,r}=\{z\in D_j^l:|c_B-A_rz|\sim 2^{k+1}R\}.$$

Observe that $(D_j^l)_{k,r} \subset \{z : |c_B - A_r z| \sim 2^{j+1}R\}$. Then, as $k_r \in S_{n-\alpha_r,\Psi_r}$,

$$\begin{aligned}
&\frac{1}{|A_l^{-1}B(c_B,2^{j+2}R)|}\int_{A_l^{-1}B(c_B,2^{j+2}R)} \Psi_r\left(\frac{k_r(y-A_rz)\chi_{\cup_{k=j}^{j+t}(D_j^l)_{k,r}}(z)}{\lambda}\right)dz\\
=&\frac{1}{|A_l^{-1}B(c_B,2^{j+2}R)|}\int_{A_l^{-1}B(c_B,2^{j+2}R)\cap(\cup_{k=j}^{j+t}(D_j^l)_{k,r})} \Psi_r\left(\frac{k_r(y-A_rz)}{\lambda}\right)dz\\
=&\sum_{k=j}^{j+t}\frac{1}{|A_l^{-1}B(c_B,2^{j+2}R)|}\int_{A_l^{-1}B(c_B,2^{j+2}R)\cap(D_j^l)_{k,r}} \Psi_r\left(\frac{k_r(y-A_rz)}{\lambda}\right)dz\\
=&\sum_{k=j}^{j+t}\frac{1}{|A_l^{-1}B(c_B,2^{j+2}R)|}\int_{(D_j^l)_{k,r}} \Psi_r\left(\frac{k_r(y-A_rz)\chi_{(D_j^l)_{k,r}}(z)}{\lambda}\right)dz\\
\leqslant&\sum_{k=j}^{j+t}\frac{|A_r^{-1}B(c_B,2^{k+2}R)|}{|A_l^{-1}B(c_B,2^{j+2}R)|}\frac{1}{|A_r^{-1}B(c_B,2^{k+2}R)|}\int_{A_r^{-1}B(c_B,2^{k+2}R)} \Psi_r\left(\frac{k_r(y-A_rz)\chi_{(D_j^l)_{k,r}}(z)}{\lambda}\right)dz\\
\leqslant&\sum_{k=j}^{j+t}\frac{|\det(A_r^{-1})|}{|\det(A_l^{-1})|}2^{(k-j)n}\frac{1}{|A_r^{-1}B(c_B,2^{k+2}R)|}\int_{A_r^{-1}B(c_B,2^{k+2}R)} \Psi_r\left(\frac{k_r(y-A_rz)\chi_{(D_j^l)_{k,r}}(z)}{\lambda}\right)dz
\end{aligned}$$

Observe that $R > |c_B - A_r z| \geqslant |c_B - A_l z|$, for every $R > 0$, implies that $A_l^{-1}B(c_B,R) \subset A_r^{-1}B(c_B,R)$. Then $|A_l^{-1}B(c_B,R)| \leqslant |A_r^{-1}B(c_B,R)|$ and $|\det(A_r^{-1})| \geqslant |\det(A_l^{-1})|$. If we consider $\lambda = \frac{|\det(A_r^{-1})|}{|\det(A_l^{-1})|}\mu$, using that Ψ_r is convex we have

$$\begin{aligned}
&\sum_{k=j}^{j+t}\frac{|\det(A_r^{-1})|}{|\det(A_l^{-1})|}2^{(k-j)n}\frac{1}{|A_r^{-1}B(c_B,2^{k+2}R)|}\\
&\qquad\times\int_{A_r^{-1}B(c_B,2^{k+2}R)} \Psi_r\left(\frac{|\det(A_l^{-1})|}{|\det(A_r^{-1})|}\frac{k_r(y-A_rz)\chi_{(D_j^l)_{k,r}}(z)}{\mu}\right)dz\\
\leqslant&\sum_{k=j}^{j+t}2^{(k-j)n}\frac{1}{|A_r^{-1}B(c_B,2^{k+2}R)|}\int_{A_r^{-1}B(c_B,2^{k+2}R)} \Psi_r\left(\frac{k_r(y-A_rz)\chi_{(D_j^l)_{k,r}}(z)}{\mu}\right)dz\\
\leqslant&2^{tn}\sum_{k=j}^{j+t}\frac{1}{|A_r^{-1}B(c_B,2^{k+2}R)|}\int_{A_r^{-1}B(c_B,2^{k+2}R)} \Psi_r\left(\frac{k_r(y-A_rz)\chi_{(D_j^l)_{k,r}}(z)}{\mu}\right)dz \leqslant 1.
\end{aligned}$$

Finally, taking $\mu = (t+1)2^{tn}\sum_t^{j+t}\|k_r(y-A_r\cdot)\|_{\Psi_r,|c_B-A_rz|\sim 2^{k+1}R} \geqslant (t+1)2^{tn}\|k_r(y-A_r\cdot)\|_{\Psi_r,|c_B-A_rz|\sim 2^{k+1}R}$, we obtain

$$\begin{aligned}
\|k_r(y-A_r\cdot)\chi_{D_j^l}\|_{\Psi_r,|c_B-A_lz|\sim 2^{j+1}R} &= \|k_r(y-A_r\cdot)\chi_{\cup_{k=j}^{j+t}(D_j^l)_{k,r}}\|_{\Psi_r,|c_B-A_lz|\sim 2^{j+1}R}\\
&\leqslant \frac{|\det(A_r^{-1})|}{|\det(A_l^{-1})|}(t+1)2^{tn}\sum_{k=j}^{t+j}\|k_r(y-A_r\cdot)\|_{\Psi_r,|c_B-A_rz|\sim 2^{k+1}R}
\end{aligned}$$

$$\lesssim \sum_{k=j}^{j+t} \|k_r(\cdot)\|_{\Psi_r,|x|\sim 2^k R} + \|k_r(\cdot)\|_{\Psi_r,|x|\sim 2^{k+1}R}$$
$$\lesssim \sum_{k\geqslant j} (2^k R)^{-\alpha_r} = c(2^j R)^{-\alpha_r},$$

where the last inequality holds since $k_r \in S_{n-\alpha_r,\Psi_r}$.

Now for $l=1$,

$$\int_{Z^1} |k_1(y-A_1z)-k_1(c_B-A_1z)||k_2(y-A_2z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f_2(z)|dz$$
$$\leqslant c\|b\|_{BMO}\sum_{j=1}^{\infty}(2^jR)^{n-\alpha_2}j\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{D_j^1}\|_{\Psi_1,|c_B-A_1z|\sim 2^{j+1}R}\|f\|_{\phi,\tilde{B}_1^j}$$
$$\leqslant c\|b\|_{BMO}M_{\alpha,\phi}f(A_1^{-1}x)\sum_{j=1}^{\infty}(2^jR)^{n-\alpha_2-\alpha}j\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{D_j^1}\|_{\Psi_1,|c_B-A_1z|\sim 2^{j+1}R}$$
$$\leqslant c\|b\|_{BMO}M_{\alpha,\phi}f(A_1^{-1}x),$$

where the last inequality follows since $k_1 \in H_{n-\alpha_1,\Psi_1,1}$.

For $l=2$, proceeding as (4.5), we obtain

$$\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{D_j^2}\right\|_{\Psi_1,|c_B-A_2z|\sim 2^{j+1}R}$$
$$\leqslant c\sum_{k\geqslant j}\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{(D_j^2)_{k,1}}\right\|_{\Psi_1,|c_B-A_1z|\sim 2^{k+1}R}.$$

Then, we obtain

$$\sum_{j=1}^{\infty}(2^jR)^{\alpha_1}j\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{D_j^1}\right\|_{\Psi_1,|c_B-A_1z|\sim 2^{j+1}R}$$
$$\leqslant \sum_{j=1}^{\infty}(2^jR)^{\alpha_1}j\sum_{k\geqslant j}\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{(D_j^2)_{k,1}}\right\|_{\Psi_1,|c_B-A_1z|\sim 2^{k+1}R}$$
$$\leqslant \sum_{j=1}^{\infty}\sum_{k\geqslant j}2^{\alpha_1(j-k)}(2^kR)^{\alpha_1}k\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{(D_j^2)_{k,1}}\right\|_{\Psi_1,|c_B-A_1z|\sim 2^{k+1}R}$$
$$\leqslant \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}(2^{-\alpha_1})^{k-j}\right)(2^kR)^{\alpha_1}k\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{(D_j^2)_{k,1}}\right\|_{\Psi_1,|c_B-A_1z|\sim 2^{k+1}R}$$
$$\leqslant c\sum_{k=1}^{\infty}(2^kR)^{\alpha_1}k\left\|(k_1(y-A_1\cdot)-k_1(c_B-A_1\cdot))\chi_{(D_j^2)_{k,1}}\right\|_{\Psi_1,|c_B-A_1z|\sim 2^{k+1}R}\leqslant c,$$

where the last inequality follows since $k_1 \in H_{n-\alpha_1,\Psi_1,1}$.

So, as in the case $l=1$, we obtain

$$\int_{Z^2}|k_1(y-A_1z)-k_1(c_B-A_1z)||k_2(y-A_2z)||b(z)-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dz$$

$$\leqslant c\|b\|_{BMO}M_{\alpha,\varphi}f(A_l^{-1}x).$$

Then

$$III \leqslant c\|b\|_{BMO}\sum_{l=1}^{2}M_{\alpha,\varphi}f(A_l^{-1}x).$$

For the case $\alpha = 0$, the argument above can be adapted as follows. Estimates for terms I and III are analogous to the ones in the case $0 < \alpha < n$. For II, observe that T_0 is of weak-type $(1,1)$ with respect to the Lebesgue measure (see Lemma 5.3 in [18]), as $0 < \delta < 1$ and using Kolmogorov's inequality (see Lemma 5.16 in [12]) we get

$$II \leqslant \frac{C}{|B|}\int_{\mathbb{R}^n}|f_1(y)|dy = \sum_{i=1}^{2}\frac{C}{|B|}\int_{\tilde{B}_i}|f_1(y)|dy \leqslant C\sum_{i=1}^{2}Mf(A_i^{-1}f(x)),$$

and the theorem follows in this case.

For the case $m > 2$, the estimates for terms I and II holds as the case $m = 2$. For III, we define Z^l, $l = 1,2,\dots,m$, as

$$Z^l = (\cup_{i=1}^{m}\tilde{B}_i)^c \cap \{z : |c_B - A_l z| \leqslant |c_B - A_r z|, r \neq l, r = 1,2,\dots,m\}.$$

For $y \in B$ and $z \in Z^l$, the inequality (4.4) in this case is

$$|K(y,z) - K(c_B,z)| \leqslant \sum_{i=1}^{m}|k_i(y - A_i z) - k_i(c_B - A_i z)|\prod_{j\neq i, j=1}^{m}|k_j(y - A_j z)|.$$

The estimate

$$\int_{Z^l}|K(y,z) - K(c_B,z)||b(z) - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}||f(z)|dz \leqslant C\|b\|_{BMO}M_{\alpha,\varphi}f(A_l^{-1}x)$$

is prove in an analogous way as above.

For the case $k > 1$, suppose that $a := T_\alpha((b - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})^k f_2)(c_B) < \infty$.

We write

$$T_{\alpha,b}^k f(x) = \sum_{l=0}^{k-l}C(b(x) - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})^{k-l}T_{\alpha,b}^l f(x) + T_\alpha((b - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})^k f)(x)$$

and the term I, II, III are

$$I = \sum_{l=0}^{k-l}\left(\frac{1}{|B|}\int_B|b(y) - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}|^{(k-l)\delta}|T_{\alpha,b}^l f(y)|^\delta dy\right)^{1/\delta},$$

$$II = \left(\frac{1}{|B|}\int_B|T_\alpha((b - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})^k f_1)(y)|^\delta dy\right)^{1/\delta},$$

$$III = \left(\frac{1}{|B|}\int_B|T_\alpha((b - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})^k f_2)(y) - T_\alpha((b - b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2})^k f_2)(c_B)|^\delta dy\right)^{1/\delta}.$$

The estimate for I is analogous for the case $k=1$. To obtain estimates II and III, we observe that, by Lemma 4.1, we have $|b-b_{\tilde{B}\cup\tilde{B}_1\cup\tilde{B}_2}|^k \lesssim |b-b_X|^k + j^k\|b\|^k_{BMO}$ with $X=B$ or $\tilde{B}_{l,j}$, the rest follows as above.

Proof of Theorem 1.1. By the extrapolation result Theorem 1.1 in [9], estimate (1.6) holds for all $0<p<\infty$ and all $w\in A_\infty$ if, and only if, it holds for some $0<p_0<\infty$ and all $w\in A_\infty$. Therefore, we will show that (1.6) is true for p_0, which is taken such that $\frac{n-\alpha}{n}<p_0<\infty$. This will make some computations cleaner and avoid some technicalities. We first consider the case on which w and $b\in L^\infty$. By homogeneity, we assume that $\|b\|_{BMO}=1$. We proceed by induction.

When $k=0$, then $T^0_{\alpha,m,b}=T_{\alpha,m}$. As $k_i\in H_{n-\alpha_i,\Psi_i,0}=H_{n-\alpha_i,\Psi_i}$, Theorem 3.3 in [18] implies that

$$\int_{\mathbb{R}^n}|T_{\alpha,m}f(x)|^p w(x)dx \leqslant C\sum_{i=1}^m\int_{\mathbb{R}^n}|M_{\alpha,\phi}f(x)|^p w(A_ix)dx.$$

Next, we assume that the result holds for all $0\leqslant j\leqslant k-1$ and let us see how to derive the case k. We fix $\Psi_1,\dots,\Psi_m$ and ϕ so that $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t)\lesssim t$ for $t\geqslant t_0$, for some $t_0>0$, with $\varphi_k(t)=t\log(e+t)^k$ and $k_i\in S_{n-\alpha_i,\Psi_i}\cap H_{n-\alpha_i,\Psi_i,k}$.

Let $f\in L^\infty_c$. Without loss of generality, we assume that $\|M_{\alpha,\phi}f\|_{L^{p_0}(w_{A_i})}$, $i=1,\dots,m$ and $\|T^k_{\alpha,m,b}f\|_{L^{p_0}(w)}$ are finite. Let $w\in A_\infty$, then there exists $r>1$ such that $w\in A_r$. Let $0<\delta<1$ such that $1<r<p_0/\delta$, thus $w\in A_{p_0/\delta}$. We want to use the Fefferman-Stein's inequality. To do so we need to check that $\|M_\delta(T^k_{\alpha,m,b}f)\|_{L^{p_0}(w)}$ is finite. Notice that since $w\in A_{p_0/\delta}$ with $p_0/\delta>1$ we have

$$\left\|M_\delta(T^k_{\alpha,m,b}f)\right\|_{L^{p_0}(w)}=\left\|M(T^k_{\alpha,m,b}f)^\delta\right\|^{\frac{1}{\delta}}_{L^{\frac{p_0}{\delta}}(w)}\leqslant C\left\|T^k_{\alpha,m,b}f\right\|_{L^{p_0}(w)}<\infty,$$

by assumption. Then, by Fefferman-Stein's inequality and Lemma 4.3, for all ε with $\delta<\varepsilon<1$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|T^k_{\alpha,m,b}f(x)\right|^{p_0}w(x)dx\\ \leqslant&\int_{\mathbb{R}^n}\left|M\left(T^k_{\alpha,m,b}f\right)^\delta(x)\right|^{p_0/\delta}w(x)dx\leqslant\int_{\mathbb{R}^n}\left(M^\sharp_\delta\left(T^k_{\alpha,m,b}f\right)(x)\right)^{p_0}w(x)dx\\ \leqslant&C\sum_{l=0}^{k-1}\left\|M_\varepsilon\left(T^l_{\alpha,m,b}f\right)\right\|^{p_0}_{L^{p_0}(w)}+C\sum_{i=1}^m\int_{\mathbb{R}^n}\left(M_{\alpha,\phi}f\left(A_i^{-1}x\right)\right)^{p_0}w(x)dx.\end{aligned}\tag{4.6}$$

Since $\delta<q/r<1$, we can take $\varepsilon>0$ such that $\delta<\varepsilon<p_0/r<1$, and so $w\in A_{p_0/\varepsilon}$. Hence,

$$\left\|M_\varepsilon\left(T^l_{\alpha,m,b}f\right)\right\|_{L^{p_0}(w)}=\left\|M\left(\left|T^l_{\alpha,m,b}f\right|^\varepsilon\right)\right\|^{1/\varepsilon}_{L^{p_0/\varepsilon}(w)}\leqslant c\left\|T^l_{\alpha,m,b}f\right\|_{L^{p_0}(w)}.$$

Notice that for $0\leqslant l\leqslant k-1$ and for all $t\geqslant e$, we have

$$\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_l}^{-1}(t)\phi^{-1}(t)\leqslant\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\overline{\varphi_k}^{-1}(t)\phi^{-1}(t)\lesssim t.$$

Besides, $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k} \subset S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,l}$. Thus, the induction hypothesis implies that, for any $0 \leqslant l \leqslant k-1$,

$$\left\| M_\varepsilon \left(T^l_{\alpha,m,b} f \right) \right\|^{p_0}_{L^{p_0}(w)} \leqslant c \left\| T^l_{\alpha,m,b} f \right\|^{p_0}_{L^{p_0}(w)} \leqslant c \sum_{i=1}^m \int_{\mathbb{R}^n} \left(M_{\alpha,\phi} f \left(A_i^{-1} x \right) \right)^{p_0} w(x) dx,$$

provided the middle term is finite. Assume for the moment that this is the case. Plugging the last estimate into (4.6) it follows that

$$\int_{\mathbb{R}^n} \left| T^k_{\alpha,m,b} f(x) \right|^{p_0} w(x) dx \leqslant C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{\alpha,\phi} f(x))^{p_0} w(A_i x) dx.$$

Observe that we have not used that w and $b \in L^\infty$, this will be needed in the following argument to show that some quantities are finite.

We still have to see that $\|T^l_{\alpha,m,b}\|_{L^{p_0}(w)} < \infty$ for all $0 \leqslant l \leqslant k-1$. As $w \in L^\infty$ and $T_{\alpha,m} : L^q(dx) \to L^{p_0}(dx)$, with $\frac{1}{p_0} = \frac{1}{q} - \frac{\alpha}{n}$,

$$\begin{aligned} \left\| T^l_{\alpha,m,b} f \right\|_{L^{p_0}(w)} &= \left\| \sum_{j=1}^l c_{l,j} b^{l-j} T_{\alpha,m}(b^j f) \right\|_{L^{p_0}(w)} \leqslant \|w\|_\infty \left\| \sum_{j=1}^l c_{l,j} b^{l-j} T_{\alpha,m}(b^j f) \right\|_{L^{p_0}} < \infty, \\ &\leqslant C \|w\|_\infty \|b\|^l_\infty \|f\|_{L^q} < \infty, \end{aligned}$$

since $f \in L^\infty_c$. Hence, for w and $b \in L^\infty$, (1.6) holds, that is

$$\int_{\mathbb{R}^n} \left| T^k_{\alpha,m,b} f(x) \right|^{p_0} w(x) dx \leqslant C \|b\|^k_{BMO} \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{\alpha,\phi} f(x))^{p_0} w(A_i x) dx,$$

where C does no depend on $\|b\|_{L^\infty}$ and $\|w\|_{L^\infty}$ (C only depends on the A_∞ constant of w, p_0, k, T).

For any weight $w \in A_\infty$, we define $w_N = \min\{w, N\}$, then $w_N \in A_\infty$ and $[w_N]_{A_\infty} \leqslant C[w]_{A_\infty}$ with C independent of N. Since $w_N \in L^\infty$ then (1.6) holds with C not depending on N. Letting $N \to \infty$ and using the monotone convergence theorem we conclude that (1.6) holds for any $w \in A_\infty$.

For the general case, if $b \in BMO$, for any $N \in \mathbb{N}$ we define $b_N = b\chi_{[-N,N]} + N\chi_{(N,\infty)} - N\chi_{(-\infty,-N)}$, then $\|b_N\|_\infty = \|b_N\|_{BMO} \leqslant 2\|b\|_{BMO}$. Now, using convergence theorems, for details see [21], we conclude that (1.6) holds for any $b \in BMO$.

Thus, as mentioned, using the extrapolation results obtained in [9], (1.6) holds for all $0 < p < \infty$, $b \in BMO$ and $w \in A_\infty$.

If w satisfies (1.4), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left| T^k_{\alpha,m,b} f(x) \right|^{p} w(x) dx &\leqslant C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{\alpha,\phi} f(x))^{p} w(A_i x) dx \\ &\leqslant C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{\alpha,\phi} f(x))^{p} w(x) dx. \end{aligned}$$

5. Proof of one weight norm inequalities

For the proof of Theorem 3.1 $a)$ and $b)$, we need the Coifman inequality (1.6) and the boundedness of the maximal operator, given in [3] (see Theorem 2.6). In the case of the classical Lebesgue spaces the theorem is the following.

THEOREM 5.1. [3] *Let $0 \leqslant \alpha < n$, w be a weight, $1 \leqslant \beta < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let η be a Young function such that $\eta^{1+\frac{\rho\alpha}{n-\alpha}} \in B_{\frac{\rho n}{n-\alpha}}$ for every $\rho > \beta(n-\alpha)/(n-\alpha\beta)$, and let ϕ be a Young function such that $\phi^{-1}(t)t^{\alpha/n} \lesssim \eta^{-1}(t)$ for every $t > 0$. If $w^{\beta} \in A_{\frac{p}{\beta},\frac{q}{\beta}}$, then $M_{\alpha,\eta}$ is bounded form $L^p(w^p)$ into $L^q(w^q)$.*

The boundedness of the $M_{\alpha,\phi}$ from $L^p(w^p)$ into $L^q(w^q)$ with bump conditions, given in [11] (see Theorem 5.37), is the following,

THEOREM 5.2. [11] *Let $0 \leqslant \alpha < n$, $1 < p < n/\alpha$, let $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let ϕ, B and C be Young functions such that $B^{-1}(t)C^{-1}(t) \leqslant c\phi^{-1}(t)$, $t \geqslant t_0 > 0$. If $C \in B_p^{\alpha}$ and $w \in A_{q,B}$, then for every $f \in L^p(w^p)$,*

$$\int (M_{\alpha,\phi} f)^q w^q \leqslant C \int |f|^p w^p.$$

Now we prove part (a) and (b) of Theorem 3.1,

Proof of Theorem 3.1 a) and b). From the previous Theorems, hypothesis (a) or (b) implies that $M_{\alpha,\phi}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$.Then, by Theorem 1.1 and w satisfies (1.4),

$$\left\| T^k_{\alpha,m,b} f \right\|_{L^q(w^q)} \leqslant c\|b\|^k_{BMO} \|M_{\alpha,\phi} f\|_{L^q(w^q)} \leqslant c\|b\|^k_{BMO} \|f\|_{L^p(w^p)}.$$

For the proof of Theorem 3.1 c) we use a Cauchy integral formula technique, see [7],[5] and [2]. This technique is as follows, let T be a linear operator, we can write T^k_b as a complex integral operator

$$T^k_b f = \frac{d^k}{dz^k} e^{zb} T(fe^{-zb}) \Big|_{z=0} = \frac{k!}{2\pi i} \int_{|z|=\varepsilon} \frac{T_z(f)}{z^{k+1}} dz,$$

where $\varepsilon > 0$ and $T_z(f) = e^{zb} T(fe^{-zb})$, $z \in \mathbb{C}$. This is called the "conjugation" of T by e^{zb}. Now, if $\|\cdot\|$ is a norm we can apply Minkowski inequality,

$$\left\| T^k_b f \right\| \leqslant \frac{1}{2\pi\varepsilon^k} \sup_{|z|=\varepsilon} \|T_z(f)\| \qquad \varepsilon > 0.$$

Observe that using this technique we can obtain the boundedness of the commutator using the boundedness of the conjugation of the operator.

LEMMA 5.3. [2] *Fix* $1 < r, \eta < \infty$. *If* $w^{\eta} \in A_r$ *and* $b \in BMO$. *Then* $we^{\lambda b} \in A_r$ *for every* $\lambda \in \mathbb{R}$ *verifying*

$$|\lambda| \leqslant \frac{\min\{1, p-1\}}{\eta' \|b\|_{BMO}}.$$

Proof of Theorem 3.1 *(c).* Let $T = T_{\alpha,m}$. Let $w \in A_{p,q}$ and $v = we^{Re(z)b}$, where $Re(z)$ is the real part of the complex number z. If $v \in A_{p,q}$, then

$$\|T_z f\|_{L^q(w^q)} = \|T(fe^{-zb})\|_{L^q(v^q)} \leqslant c\|fe^{-zb}\|_{L^p(v^p)} = c\|f\|_{L^p(w^p)},$$

since T is boundedness from $L^p(v^p)$ into $L^q(v^q)$.

Let us prove that $v \in A_{p,q}$. If $w \in A_{p,q}$ then $w^q \in A_{1+\frac{q}{p'}}$ and exists $r > 1$ such that $w^{qr} \in A_{1+\frac{q}{p'}}$. Let $\varepsilon_0 = \frac{\min\left\{1, \frac{p'}{q}\right\}}{qr'\|b\|_{BMO}}$, if $|z| = \varepsilon_0$ then

$$|qRe(z)| \leqslant q|z| = \frac{\min\left\{1, \frac{p'}{q}\right\}}{r'\|b\|_{BMO}}.$$

By Lemma 5.3, $v^q \in A_{1+\frac{q}{p'}}$ and $v \in A_{p,q}$.

Hence,

$$\left\|T_b^k f\right\|_{L^p(w^p)} \leqslant \frac{1}{2\pi\varepsilon_0^k} \sup_{|z|=\varepsilon_0} \|T_z(f)\|_{L^p(w^p)} \leqslant \frac{1}{2\pi c_{p,q}^k} \|b\|_{BMO}^k \|f\|_{L^q(w^q)}.$$

6. Proof of two weight norm inequalities

For the proof of the two weight norm inequality we need the following auxiliary results.

LEMMA 6.1. *(a)* [24] *Let* Φ *be a Young function. If* $\Phi \in B_p$ *then for every weight* v *we have*

$$\int |M_\Phi f(x)|^p v(x)dx \leqslant c \int |f(x)|^p Mv(x)dx.$$

(b) [20] *If* $r > 1$, *then*

$$M(M_r) \approx M_r.$$

Proof of Theorem 3.2. Let u a weight and $v(x) = M_{\alpha p,\mathscr{D}} u(x)$. By duality, (3.2) turns out to be equivalent to

$$\int_{\mathbb{R}^n} |T^* f(x)|^{p'} v(x)^{1-p'} dx \leqslant c \int_{\mathbb{R}^n} \sum_{i=1}^{m} |f(A_i x)|^{p'} u(x)^{1-p'} dx.$$

Since $\nu = M_{\alpha p,\mathscr{D}} w^{1-p'} \in A_\infty$, see [4], then by Proposition 2.2 and the fact that $\mathscr{E} \in B_{p'}$ we get for all $x \in \mathbb{R}^n$,

$$\begin{aligned} M_{\alpha,\mathscr{F}}(w_{A_i^{-1}}^{1/p})(A_i x)^{p'} \nu(x)^{1-p'} &= M_{\alpha p,\mathscr{D}}(w_{A_i^{-1}})(A_i x)^{p'/p} \nu(x)^{1-p'} \\ &\leqslant M_{\alpha p,\mathscr{D}}(w)(x)^{p'/p} \nu(x)^{1-p'} \end{aligned}$$

and using the Coifman inequality settled earlier, Theorem 1.1,

$$\begin{aligned} \int_{\mathbb{R}^n} |T^* f(x)|^{p'} \nu(x)^{1-p'} dx &\leqslant c \int_{\mathbb{R}^n} M_{\alpha,\phi} f(A_i x)^{p'} \nu(x)^{1-p'} dx \\ &\leqslant c \int_{\mathbb{R}^n} M_{\mathscr{E}}(f w_{A_i^{-1}}^{-1/p})(A_i x)^{p'} M_{\alpha,\mathscr{F}}(w_{A_i^{-1}}^{1/p})(A_i x)^{p'} \nu(x)^{1-p'} dx \\ &\leqslant c \int_{\mathbb{R}^n} M_{\mathscr{E}}(f w_{A_i^{-1}}^{-1/p})(A_i x)^{p'} dx \\ &\leqslant c \int_{\mathbb{R}^n} |f(A_i x) w_{A_i^{-1}}^{-1/p}(A_i x)|^{p'} dx = c \int_{\mathbb{R}^n} |f(A_i x)|^{p'} w(x)^{1-p'} dx. \end{aligned}$$

Proof of Theorem 3.3. We proceed by induction on k. We consider $m = 2$, $T_b^k = T_{0,2,b}^k$. The general case is analogous.

We assume that the cases $l = 0, 1, \ldots, k-1$ are proved and we show the desired estimate for T_b^k. Let u be a weight, suppose that $u \in L_c^\infty$ (otherwise consider $u_N = \min\{u, N\}\chi_{B(0,N)}$ and use monotone converge theorem). Let $0 \leqslant f \in L_c^\infty$. By homogeneity we can also assume that $\|b\|_{BMO} = 1$.

By the standard Calderón-Zygmund decomposition of f at height λ, there exist dyadic cubes $\{Q_j\}_j$ such that

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f \leqslant 2^n \lambda,$$

and we can write $f = g + h$ where

$$g = f\chi_{\mathbb{R}^n \setminus \cup_j Q_j} + \sum_j f_{Q_j} \chi_{Q_j}, \qquad h = \sum_j h_j = \sum_j (f - f_{Q_j}) \chi_{Q_j},$$

where f_{Q_j} denotes the average of f over Q_j. Let us recall that $0 \leqslant g \leqslant 2^n \lambda$ a.e. and also that each h_j has vanishing integral. We set $\tilde{Q}_{j,i}$, $i = 1, 2$, the cube with center $A_i c_j$ with length $2\sqrt{n} M l(Q_j)$, where $M = \max_{1 \leqslant i \leqslant 2} \|A_i\|$, $\tilde{\Omega} = \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2})$ and $\tilde{u} = u\chi_{\mathbb{R}^n \setminus \tilde{\Omega}}$. Then

$$\begin{aligned} u\{x \in \mathbb{R}^n : |T_b^k f(x)| > \lambda\} \leqslant & u(\tilde{\Omega}) + u\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |T_b^k h(x)| > \lambda/2\} \\ & + u\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |T_b^k g(x)| > \lambda/2\} = I + II + III. \end{aligned}$$

We estimate each term separately. For I, observe that $|\tilde{Q}_{j,i}| = (2\sqrt{n}M)^n |Q_j|$. Then, we have

$$I = u\left(\bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2})\right) \leqslant \sum_j \left[u(\tilde{Q}_{j,1}) + u(\tilde{Q}_{j,2})\right]$$

$$= (2\sqrt{n}M)^n \sum_j \left[\frac{u(\tilde{Q}_{j,1})}{|\tilde{Q}_{j,1}|} + \frac{u(\tilde{Q}_{j,2})}{|\tilde{Q}_{j,2}|} \right] |Q_j| \leqslant \frac{c}{\lambda} \sum_j \left[\frac{u(\tilde{Q}_{j,1})}{|\tilde{Q}_{j,1}|} + \frac{u(\tilde{Q}_{j,2})}{|\tilde{Q}_{j,2}|} \right] \int_{Q_j} f$$
$$\leqslant \frac{c}{\lambda} \sum_j \int_{Q_j} \big[Mu(A_1x) + Mu(A_2x) \big] f(x)dx,$$

where the last inequality follows since $x \in Q_j$ then $A_i x \in \tilde{Q}_{j,i}$. Then,

$$I \leqslant \frac{c}{\lambda} \sum_j \int_{Q_j} \big[Mu(A_1x) + Mu(A_2x) \big] f(x)dx$$
$$\leqslant c \sum_j \int_{Q_j} \varphi_k \left(\frac{f(x)}{\lambda} \right) \big[Mu(A_1x) + Mu(A_2x) \big] dx,$$

and we observe that Mu is pointwise controlled by either $M_\phi u$ or $M_{\mathscr{D}} u$. So the desired estimate follows in all cases.

To estimate II, we write

$$T_b^k h(x) = \sum_j T_b^k h_j(x)$$
$$= \sum_{l=0}^{k-1} c_{k,l} T_b^l \left(\sum_j (b - b_{Q_j \cup \tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}})^{k-l} h_j \right)(x) + \sum_j (b(x) - b_{Q_j \cup \tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}})^k T h_j(x)$$
$$= F_1(x) + F_2(x).$$

Then,

$$II = u\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |Th(x)| > \lambda/2\}$$
$$\leqslant u\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_1(x)| > \lambda/4\} + u\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_2(x)| > \lambda/4\}.$$

For F_1, we would like to use the induction hypothesis. We consider the case (a). If $0 \leqslant l \leqslant k-1$ then $H_{n-\alpha_i,\Psi_i,k} \subset H_{n-\alpha_i,\Psi_i,l}$ and so $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,l}$. Also, as $\overline{\varphi}_k(t) \leqslant \overline{\varphi}_l(t)$ we have

$$\Psi_1^{-1}(t) \cdots \Psi_m^{-1}(t) \overline{\varphi}_l(t) \phi(t) \lesssim \Psi_1^{-1}(t) \cdots \Psi_m^{-1}(t) \overline{\varphi}_k^{-1}(t) \phi^{-1}(t) \lesssim t,$$

for $t \geqslant t_0$, for some $t_0 > 0$. Thus the hypothesis on (a) are satisfied for every $0 \leqslant l \leqslant k-1$ and therefore

$$u\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_1(x)| > \lambda/4\}$$
$$\leqslant \sum_{l=0}^{k-1} \tilde{u} \left\{ x : \left| T_b^l \left(\sum_j (b - b_{Q_j \cup \tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}})^{k-l} h_j \right)(x) \right| > \lambda/C \right\}$$
$$\lesssim \sum_{l=0}^{k-1} \int_{\mathbb{R}^n} \varphi_l \left(\frac{|\sum_j (b - b_{Q_j \cup \tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}})^{k-l} h_j|}{\lambda} \right) (M_\phi \tilde{u}(A_1(x)) + M_\phi \tilde{u}(A_2(x)))dx$$
$$\lesssim \sum_{l=0}^{k-1} \sum_j \int_{Q_j} \varphi_l \left(\frac{|b - b_{Q_j \cup \tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}}|^{k-l} |h_j|}{\lambda} \right) (M_\phi \tilde{u}(A_1(x)) + M_\phi \tilde{u}(A_2(x)))dx$$

$$\lesssim \sum_{l=0}^{k-1}\sum_{j}\left(\sum_{i=1}^{2}\operatorname*{ess\,inf}_{Q_j} M_\phi\tilde{u}(A_i(x))\right)\int_{Q_j}\varphi_l\left(\frac{|b-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|^{k-l}|h_j|}{\lambda}\right)dx,$$

where the last inequality holds by $M_\phi\tilde{u}\simeq \operatorname*{ess\,inf}_{Q_j} M_\phi\tilde{u}$, see [20]. Let us observe that $C_k^{-1}(t)\overline{C_{k-l}}^{-1}(t)\lesssim C_l^{-1}(t)$. Then, Young inequality implies

$$\begin{aligned}
&\int_{Q_j}\varphi_l\left(\frac{|b-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|^{k-l}|h_j|}{\lambda}\right)dx\\
&\lesssim\int_{Q_j}\varphi_k\left(\frac{|h_j|}{c\lambda}\right)dx+\int_{Q_j}\overline{\varphi_{k-l}}\left(c|b-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|^{k-l}\right)dx\\
&=\int_{Q_j}\varphi_k\left(\frac{|h_j|}{\lambda}\right)dx+\int_{Q_j}e^{c|b-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|}dx\\
&\leqslant\int_{Q_j}\varphi_k\left(\frac{|h_j|}{\lambda}\right)dx+\int_{Q_j}e^{c|b-b_{Q_j}|+|b_{Q_j}-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|}dx\\
&\leqslant\int_{Q_j}\varphi_k\left(\frac{|h_j|}{\lambda}\right)dx+c\int_{Q_j}e^{c|b-b_{Q_j}|}dx\lesssim\int_{Q_j}\varphi_k\left(\frac{|h_j|}{\lambda}\right)dx+|Q_j|.
\end{aligned}$$

As $\|b\|_{BMO}=1$, using John-Nirenberg theorem, we get that $\|b-b_{Q_j}\|_{\exp L,Q_j}\leqslant c$ and $|b_{Q_j}-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|\leqslant c$. Besides, using that φ_k^δ, $0<\delta<1$, is concave, therefore subadditive, it follows that φ_k is quasi-subadditive, this is $\varphi_k(t_1+t_2)\lesssim\varphi_k(t_1)+\varphi_k(t_2)$. Then, by Jensen inequality

$$\int_{Q_j}\varphi_k\left(\frac{|h_j|}{\lambda}\right)dx\leqslant\int_{Q_j}\varphi_k\left(\frac{f}{\lambda}\right)dx+|Q_j|\varphi_k\left(\frac{f_{Q_j}}{\lambda}\right)\leqslant 2\int_{Q_j}\varphi_k\left(\frac{f}{\lambda}\right)dx.$$

Also, by Calderón-Zygmund descomposition

$$|Q_j|\leqslant\frac{1}{\lambda}\int_{Q_j}f\,dx\leqslant\int_{Q_j}\varphi_k\left(\frac{f}{\lambda}\right)dx.$$

Then, we obtain

$$\begin{aligned}
u\{x\in\mathbb{R}^n\setminus\tilde{\Omega}:|F_1(x)|>\lambda/4\}&\lesssim\sum_{l=0}^{k-1}\sum_{j}\left(\sum_{i=1}^{2}\operatorname*{ess\,inf}_{Q_j} M_\phi\tilde{u}(A_i(x))\right)\int_{Q_j}\varphi_k\left(\frac{f}{\lambda}\right)dx\\
&\lesssim\int_{\mathbb{R}^n}\varphi_k\left(\frac{f}{\lambda}\right)(M_\phi\tilde{u}(A_1(x))+M_\phi\tilde{u}(A_2(x)))dx.
\end{aligned}$$

This gives the desired estimate for F_1 in case (a). Notice that the same computations hold in case (b) replacing everywhere M_ϕ by $M_{\mathscr{D}}$. Next, we estimate F_2,

$$\begin{aligned}
&u\{x\in\mathbb{R}^n\setminus\tilde{\Omega}:|F_2(x)|>\lambda/4\}\\
&\leqslant\frac{4}{\lambda}\sum_{j}\int_{\mathbb{R}^n\setminus\tilde{\Omega}}|b(x)-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|^k|Th_j(x)|u(x)dx
\end{aligned}$$

$$\leqslant \frac{4}{\lambda}\sum_j \int_{\mathbb{R}^n\setminus\tilde{\Omega}} |b(x)-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|^k \left|\int_{Q_j}(K(x,y)-K(x,c_j))h_j(y)dy\right| u(x)dx$$
$$\leqslant \frac{4}{\lambda}\sum_j \int_{Q_j}|h_j(y)|\int_{\mathbb{R}^n\setminus\tilde{\Omega}}|b(x)-b_{Q_j\cup\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}}|^k|(K(x,y)-K(x,c_j))|u(x)dxdy.$$

We claim that for every Q, with center c_Q, and for every $y\in Q$ we have

$$\int_{\mathbb{R}^n\setminus(\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2})} |b(x)-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k|(K(x,y)-K(x,c_Q))|u(x)dx \leqslant c\sum_{i=1}^{2}\operatorname*{ess\,inf}_{x\in Q_j} M_\Phi u(A_i x), \tag{6.1}$$

where Q_i, $i=1,2$, is the cube with center $A_i c_Q$ with length $2\sqrt{n}Ml(Q)$, and $M=\max_{1\leqslant i\leqslant 2}\|A_i\|$. This estimate applied to each Q_j drives us to

$$\begin{aligned} u\{x\in\mathbb{R}^n\setminus\tilde{\Omega}:|F_2(x)|>\lambda/4\} &\leqslant \frac{c}{\lambda}\sum_j\sum_{i=1}^{2}\operatorname*{ess\,inf}_{x\in Q_j} M_\Phi u(A_i x).\int_{Q_j}|h_j(y)|dy \\ &\leqslant \frac{c}{\lambda}\sum_j\int_{Q_j} f(y)\left[M_\Phi u(A_1 y)+M_\Phi u(A_2 y)\right]dy \\ &\lesssim \int_{\mathbb{R}^n}\varphi_k\left(\frac{f}{\lambda}\right)(M_\phi\tilde{u}(A_1(x))+M_\phi\tilde{u}(A_2(x)))dx. \end{aligned}$$

Observe that this leads to the desired estimate in (a) and also in (b), since $M_\phi\leqslant M_{\mathscr{D}}$, see Remark 3.4. Collecting the obtained inequalities for F_1 and F_2 we complete the estimate of II.

Let us proof (6.1). Let Q be a cube with center c_Q, and Q_i, $i=1,2$, be the cubes with center $A_i c_Q$ with length $2\sqrt{n}Ml(Q)$, where $M=\max_{1\leqslant i\leqslant 2}\|A_i\|$. Using (4.4), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus(\tilde{Q}_1\cup\tilde{Q}_2)} |b(x)-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k|(K(x,y)-K(x,c_Q))|u(x)dx \\ &\leqslant \int_{Z^1\cup Z^2}|b(x)-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k|k_1(x-A_1y)-k_1(x-A_1c_Q)||k_2(x-A_2y)|u(x)dx \\ &\quad+\int_{Z^1\cup Z^2}|b(x)-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k|k_1(x-A_1c_Q)||k_2(x-A_2y)-k_2(x-A_2c_Q)|u(x)dx, \end{aligned}$$

where $Z^i=\mathbb{R}^n\setminus\left(\tilde{Q}_1\cup\tilde{Q}_2\right)\cap\{x:|x-A_iy|\leqslant|x-A_ry|, r\neq i\}$.

We only estimate the first summand, the other follows in an analogous way. Using generalized Hölder's inequality and observing that $|\tilde{Q}_i|=(2\sqrt{n}M)^n|Q|$, we have

$$\begin{aligned} &\int_{Z^1}|b(x)-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k|k_1(x-A_1y)-k_1(x-A_1c_j)||k_2(x-A_2y)|u(x)dx \\ &\lesssim\int_{Z^1}\left(|b(x)-b_{Q^{t+1}}|^k+|b_{Q^{t+1}}-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k\right)|k_1(x-A_1y)-k_1(x-A_1c_j)||k_2(x-A_2y)|u(x)dx \end{aligned}$$

$$\leqslant c\sum_{t=1}^{\infty}|Q^t|\left(\left\|(b-b_{Q^{t+1}})^k\right\|_{\varphi_k,Q^{t+1}}+t^k\right)\left\|k_1(\cdot-A_1y)-k_1(\cdot-A_1c_j)\chi_{Q^{t+1}\setminus Q^t}\right\|_{\Psi_1,Q^{t+1}}$$
$$\left\|k_2(\cdot-A_2y)\chi_{Q^{t+1}\setminus Q^t}\right\|_{\Psi_2,Q^{t+1}}\|u\|_{\Phi,Q^{t+1}},$$

the last inequality holds using Lemma 4.1 and the fact that Q^t is the cube with center A_1c_B and length $2^t\sqrt{n}Ml(Q)$. Observe $Q^1=\tilde{Q}_1$.

Since $\|b\|_{BMO}=1$, then $\|(b-b_{Q^{t+1}})^k\|_{\varphi_k,Q^{t+1}}\leqslant C$. Now as $k_2\in S_{n-\alpha_2,\Phi_2}$, we obtain

$$\left\|k_2(\cdot-A_2y)\chi_{Q^{t+1}\setminus Q^t}\right\|_{\Psi_2,Q^{t+1}}\leqslant c|Q^t|^{-\alpha_2/n}.$$

Also, if $x\in Q_j$ then for all $t\in\mathbb{N}$ we get $A_1x\in\tilde{Q}_{j,1}\subset Q^t$ and

$$|Q^t|^{\frac{\alpha}{n}}\|u\|_{\Phi,Q^{t+1}}\leqslant c\operatorname*{ess\,inf}_{Q_j}M_\Phi u(A_1\cdot).$$

Then,

$$\int_{Z^1}|k_1(x-A_1y)-k_1(x-A_1c_j)||k_2(x-A_2y)|u(x)dx$$
$$\leqslant c\operatorname*{ess\,inf}_{Q_j}M_\Phi u(A_1\cdot)\sum_{t=1}^{\infty}|Q^t|^{\frac{\alpha_1}{n}}t^k\left\|k_1(\cdot-A_1y)-k_1(\cdot-A_1c_j)\chi_{Q^{t+1}\setminus Q^t}\right\|_{\Psi_1,Q^{t+1}}$$
$$\leqslant c\operatorname*{ess\,inf}_{Q_j}M_\Phi u(A_1\cdot),$$

where the last inequality holds since $k_1\in H_{n-\alpha_1,\Psi_1,k}$.

In an analogous way, we obtain

$$\int_{Z^2}|b(x)-b_{Q\cup\tilde{Q}_1\cup\tilde{Q}_2}|^k|k_1(x-A_1y)-k_1(x-A_1c_Q)||k_2(x-A_2y)|u(x)dx$$
$$\leqslant c\operatorname*{ess\,inf}_{Q_j}M_\Phi u(A_2\cdot).$$

The estimate III is different in each case. We start with (a). For $p>1$, using Theorem 1.1, the fact that $M_ru\in A_1$ and Lemma 6.1, we get

$$III=u\{x\in\mathbb{R}^n\setminus\tilde{\Omega}:|Tg(x)|>\lambda/2\}\leqslant\frac{2^p}{\lambda^p}\int_{\mathbb{R}^n}|Tg(x)|^p\tilde{u}(x)dx$$
$$\leqslant\frac{2^p}{\lambda^p}\int_{\mathbb{R}^n}|Tg(x)|^pM_r\tilde{u}(x)dx\leqslant\frac{c}{\lambda^p}\sum_{i=1}^{2}\int_{\mathbb{R}^n}|M_\Phi g(A_i^{-1}x)|^pM_r\tilde{u}(x)dx$$
$$\leqslant\frac{c}{\lambda^p}\sum_{i=1}^{2}\int_{\mathbb{R}^n}|g(A_i^{-1}x)|^pM(M_r\tilde{u})(x)dx\leqslant\frac{c}{\lambda^p}\sum_{i=1}^{2}\int_{\mathbb{R}^n}|g(A_i^{-1}x)|^pM_r\tilde{u}(x)dx$$
$$\leqslant\frac{c}{\lambda^p}\int_{\mathbb{R}^n}|g(x)|^p\sum_{i=1}^{2}M_r\tilde{u}(A_ix)dx\leqslant\frac{c}{\lambda^p}\int_{\mathbb{R}^n}|g(x)|^p\sum_{i=1}^{2}M_\Phi\tilde{u}(A_ix)dx,$$

where the last inequality holds using that $t^r \leqslant \Phi(t)$ for $t \geqslant t_0 > 0$. Since $g \leqslant 2^n\lambda$,

$$\begin{aligned} III &\leqslant \frac{c}{\lambda^p}\int_{\mathbb{R}^n}|g(x)|^p\sum_{i=1}^{2}M_\Phi\tilde{u}(A_ix)dx \leqslant \frac{c}{\lambda}\int_{\mathbb{R}^n}|g(x)|\sum_{i=1}^{2}M_\Phi\tilde{u}(A_ix)dx \\ &\leqslant \frac{c}{\lambda}\int_{\mathbb{R}^n}f(x)\sum_{i=1}^{2}M_\Phi\tilde{u}(A_ix)dx \lesssim \int_{\mathbb{R}^n}\varphi_k\left(\frac{f}{\lambda}\right)(M_\phi\tilde{u}(A_1(x))+M_\phi\tilde{u}(A_2(x)))dx, \end{aligned}$$

which completes the proof of (a). To show (b), we only have to estimate III. We can apply Theorem 3.2, to the adjoint of T_b^k. Observe that $(T_b^k)^* = (T^*)_{-b}^k$, where T^* is the integral operator with kernel

$$\tilde{K}(y,x) = \tilde{k_1}(y-A_1^{-1}x)\tilde{k_2}(y-A_2^{-1}x),$$

and $\tilde{k_i}(x) = k_i(-A_ix)$. Since $k_i \in S_{n-\alpha_i,\Psi_i}\cap H_{n-\alpha_i,\Psi_i,k}$, we have $\tilde{k_i} \in S_{n-\alpha_i,\Psi_i}\cap H_{n-\alpha_i,\Psi_i,k}$, then we can apply Theorem 1.1 to $(T_b^k)^*$. Hence, by Theorem 3.2 we obtain

$$\begin{aligned} III &= u\{x\in\mathbb{R}^n\setminus\tilde{\Omega} : |T_b^kg(x)| > \lambda/2\} \leqslant \frac{2^p}{\lambda^p}\int_{\mathbb{R}^n}|T_b^kg(x)|^p\tilde{u}(x)dx \\ &\leqslant \frac{c}{\lambda^p}\int_{\mathbb{R}^n}g(x)^p\sum_{i=1}^{2}M_{\mathscr{D}}u(A_ix)dx \leqslant \frac{c}{\lambda}\int_{\mathbb{R}^n}f(x)\sum_{i=1}^{2}M_{\mathscr{D}}u(A_ix)dx \\ &\lesssim \int_{\mathbb{R}^n}\varphi_k\left(\frac{f}{\lambda}\right)(M_{\mathscr{D}}\tilde{u}(A_1(x))+M_{\mathscr{D}}\tilde{u}(A_2(x)))dx, \end{aligned}$$

which completes the Theorem.

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