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ON SOME TRACE INEQUALITIES

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Abstract. In this paper we consider some generalizations of the Ando inequality

$$|||f(A)-f(B)||| \leqslant |||f(|A-B|)|||$$

with the "weight" $(A-B)^p$. More precisely, for $p \geqslant 1$ such that $(-1)^p = -1$ and for a non-negative function f on $[0,\infty)$ such that $f(0)=0$, we study the following inequality:

$$\operatorname{Tr}((A-B)^p(f(A)-f(B))) \geqslant \operatorname{Tr}(|A-B|^p f(|A-B|)),$$

whenever A and B are positive semidefinite matrices. We show that the inequality is true for any operator convex function f and it is reversed whenever f is operator monotone.

1. Introduction

In [1] Ando proved that for $p \geqslant 1$ and for any unitarily invariant norm $|||\cdot|||$,

$$|||A^p - B^p||| \geqslant ||||A-B|^p|||.$$

For the trace norm $||A||_1 = \operatorname{Tr}(|A|)$, the last inequality reduces to the following

$$\operatorname{Tr}(|A^p - B^p|) \geqslant \operatorname{Tr}(|A-B|^p), \quad p \geqslant 1. \tag{1.1}$$

This is a matrix version of the following scalar inequality:

$$|a^p - b^p| \geqslant |a-b|^p, \quad \text{and} \quad p \geqslant 1. \tag{1.2}$$

Interestingly, for $p \geqslant 3$ and $a \geqslant b$ we have the following chain:

$$(a-b)^p \leqslant \cdots \leqslant (a-b)^k(a^{p-k}-b^{p-k}) \leqslant \cdots \leqslant (a-b)^2(a^{p-2}-b^{p-2}) \leqslant (a-b)(a^{p-1}-b^{p-1}),$$

where $p-k \in [1,2]$. Indeed, it is enough to show that

$$(a-b)^p \leqslant (a-b)^k(a^{p-k}-b^{p-k})$$

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and

$$(a-b)^{k+1}(a^{p-k-1}-b^{p-k-1}) \leqslant (a-b)^k(a^{p-k}-b^{p-k}).$$

The first inequality follows from (1.2) since $p-k \geqslant 2$, the second inequality is equivalent to $ab^{p-k}+ba^{p-k} \geqslant 0$ which is obvious.

And for $p \in [2,3]$, we have reverse inequalities as follows:

$$(a-b)^2(a^{p-2}-b^{p-2}) \leqslant (a-b)(a^{p-1}-b^{p-1}) \leqslant (a-b)^p.$$

Naturally, one would ask if there were matrix versions of the above inequalities. Seemingly supporting motivation is given by the following fact (Theorem 3.1): For any positive semidefinite matrices A and B and $p \geqslant 3$,

$$\operatorname{Tr}\left((A-B)^2(A^{p-2}-B^{p-2})\right) \leqslant \operatorname{Tr}\left((A-B)(A^{p-1}-B^{p-1})\right). \tag{1.3}$$

It turns out that (Proposition 3.1) for positive semidefinite matrices A and B, $p \geqslant 1$, and $q \geqslant 0$ is such that $p \geqslant q$ and $0 \leqslant p-q \leqslant 1$,

$$\operatorname{Tr}\left((A-B)^q(A^{p-q}-B^{p-q})\right) \leqslant \operatorname{Tr}\left(|A-B|^p\right). \tag{1.4}$$

For the special case $q=2$, the function t^{p-2} is operator monotone for $p \in [2,3]$. Therefore, it is natural to ask whether inequality (1.4) is true for operator monotone functions.

Let $p \geqslant 1$ such that $(-1)^p = -1$ (p is not necessarily an integer). In this paper, for a non-negative operator convex function $f(t)$ on $[0,\infty)$ such that $f(0)=0$ we show that

$$\operatorname{Tr}\left(|A-B|^p f(|A-B|)\right) \leqslant \operatorname{Tr}\left((A-B)^p(f(A)-f(B))\right) \tag{1.5}$$

whenever positive semidefinite matrices A and B. It is worth noting that the condition $(-1)^p = -1$ is essential. If inequality (1.5) held for any positive number p, we could use a limit process to get that

$$\operatorname{Tr}\left(f(A)-f(B)\right) \geqslant \operatorname{Tr}\left(f(|A-B|)\right)$$

which is not true in general. Mention that Ando [1, Theorem 2] proved a similar inequality for continuous increasing functions whose inverse are operator monotone with some certain conditions.

In addition, a reverse inequality is also studied for operator monotone functions. For $p=1$, inequality (1.5) was recently studied in [8]. For the power functions t^s ($s \in [2,3]$), inequality (1.5) was obtained by E. Ricard [10] for non-commutative L_s-spaces.

The paper is organized as follows. In the next section, we prove two trace inequalities for operator monotone and operator convex functions. In the last section we establish some inequalities for power functions. We also discuss another extensions of (1.2) and establish some trace inequalities for them.

2. Main inequalities

It is well-known that for $s > 0$ the function $f_s(t) = \frac{st}{t+s}$ is operator monotone on $[0,\infty)$. From the proof of [1, Theorem 1] one can see that for any positive semidefinite matrices B and C and for any $s > 0$,

$$|||f_s(B+C) - f_s(B)||| \leqslant |||f_s(C)|||.$$

This inequality is not applicable for inequality (1.5) because we have the weight $(A-B)^p$ in both sides. However, we have the following lemma which is essential for the rest of this section.

LEMMA 2.1. *Let $p \geqslant 1$ and $s > 0$. Then for any positive semidefinite matrices B and C,*

$$\mathrm{Tr}\,(C^p(f_s(B+C) - f_s(B))) \leqslant \mathrm{Tr}\,(C^p f_s(C)).$$

Proof. Since $(B+C+s)^{-1} \leqslant (C+s)^{-1}$, there exists a contraction V such that $(B+C+s)^{-1/2} = V(C+s)^{-1/2}$ and hence $(B+C+s)^{-1} = (C+s)^{-1/2}V(C+s)^{-1/2}$. Put $W = s(B+s)^{-1}$ and $X = (C+s)^{-1/2}$. Then on account of the fact that $XC = CX$ we have

$$\begin{aligned}
\mathrm{Tr}\,(C^p(f_s(B+C) - f_s(B))) &= \mathrm{Tr}\Big(C^p s(B+C+s)^{-1}\big((B+C)(B+s) \\
&\qquad - (B+C+s)B\big)(B+s)^{-1}\Big) \\
&= s^2\mathrm{Tr}\,(C^p(B+s)^{-1}C(B+C+s)^{-1}) \\
&= s\mathrm{Tr}\,(C^pWCXVX) \\
&= s\mathrm{Tr}\,((XC^{p/2}WC^{1/2})(C^{1/2}XVC^{p/2})) \\
&\leqslant s||XC^{p/2}WC^{1/2}||_2 \cdot ||C^{1/2}XVC^{p/2}||_2,
\end{aligned}$$

where we use the Cauchy-Schwarz inequality. On the other hand, $(XC^{p/2}, C^{1/2})$ and $(C^{1/2}X, C^{p/2})$ are monotone pairs in the sense of [5] as $g_1(t) = \sqrt{t^p/(t+s)}$ and $g_2(t) = t^{p/2}$ are non-decreasing for any $p \geqslant 1$. Since W is self-adjoint, by [5, Theorem 1] we have

$$||XC^{p/2}WC^{1/2}||_2^2 \leqslant ||WXC^{(p+1)/2}||_2^2 \leqslant \mathrm{Tr}\,(X^2C^{p+1}). \tag{2.1}$$

Similarly, we also have

$$||C^{1/2}XVC^{p/2}||_2^2 \leqslant \mathrm{Tr}\,(X^2C^{p+1}). \tag{2.2}$$

Thus, the lemma follows from (2.1) and (2.2).

THEOREM 2.1. *Let $f(t)$ be a non-negative operator monotone function on $[0,\infty)$ such that $f(0) = 0$. Then for any positive number $p \geqslant 1$ such that $(-1)^p = -1$ and for any positive semidefinite matrices A and B,*

$$\mathrm{Tr}\,((A-B)^p(f(A) - f(B))) \leqslant \mathrm{Tr}\,(|A-B|^p f(|A-B|)). \tag{2.3}$$

Proof. From the assumption and the integral representation of operator monotone function f [9], we have

$$f(t) = \beta t + \int_0^\infty f_s(t) d\mu(s),$$

where μ is a positive measure on $[0,\infty)$ and $\beta \geqslant 0$. Now, suppose that $A \geqslant B$ and put $C = A - B$. Therefore, on account of Lemma 2.1 we have

$$\begin{aligned}\operatorname{Tr}((A-B)^p(f(A)-f(B))) &= \operatorname{Tr}(\beta C^{p+1}) + \int_0^\infty \operatorname{Tr}(C^p(f_s(A)-f_s(B)))d\mu(s)\\ &\leqslant \operatorname{Tr}(C^p f(C)).\end{aligned}$$

In general, denote by C_- and C_+ the negative and positive parts of C, respectively. Then we have $|A-B| = C_- + C_+$, and $A - B = C_+ - C_-$. Put $Z = A + C_- = B + C_+$. On account of the fact that $(-1)^p = -1$, we have

$$\begin{aligned}\operatorname{Tr}((A-B)^p(f(A)-f(B))) &= \operatorname{Tr}((C_+^p - C_-^p)(f(A)-f(Z)+f(Z)-f(B)))\\ &= \operatorname{Tr}(C_+^p(f(A)-f(Z))) + \operatorname{Tr}(C_+^p(f(Z)-f(B)))\\ &\quad + \operatorname{Tr}(C_-^p(f(Z)-f(A))) - \operatorname{Tr}(C_-^p(f(Z)-f(B))).\end{aligned}$$

Since the function f is operator monotone, and $A,B \leqslant Z$, one can see the first and the the forth terms in the last identity are negative. According to the previous case, we have

$$\begin{aligned}\operatorname{Tr}(C_+^p(f(Z)-f(B))) + \operatorname{Tr}(C_-^p(f(Z)-f(A))) &\leqslant \operatorname{Tr}(C_+^p f(C_+)) + \operatorname{Tr}(C_-^p f(C_-))\\ &= \operatorname{Tr}(C^p f(|C|)).\end{aligned}$$

For operator convex functions the inequality (2.3) is reversed. To prove that we need the following lemma.

LEMMA 2.2. *Let $p \geqslant 1$ and $s > 0$. Then for any positive semidefinite matrices B and C,*

$$\operatorname{Tr}(C^p(h_s(B+C)-h_s(B))) \geqslant \operatorname{Tr}(C^p h_s(C)),$$

where $h_s(t) = t f_s(t)$ is operator convex on $[0,\infty)$.

Proof. Note that $h_s(t) = st - sf_s(t)$. Therefore, from Lemma 2.1, we have

$$\begin{aligned}\operatorname{Tr}(C^p(h_s(B+C)-h_s(B))) &= \operatorname{Tr}(sC^{p+1}) - s\operatorname{Tr}(C^p(f_s(B+C)-f_s(B)))\\ &\geqslant \operatorname{Tr}(C^p(sC - sf_s(C)))\\ &= \operatorname{Tr}(C^p h_s(C)).\end{aligned}$$

THEOREM 2.2. *Let $f(t)$ be a non-negative operator convex function on $[0,\infty)$ such that $f(0) = 0$. Then for any positive number $p \geqslant 1$ such that $(-1)^p = -1$ and for any positive semidefinite matrices A and B,*

$$\operatorname{Tr}((A-B)^p(f(A)-f(B))) \geqslant \operatorname{Tr}(|A-B|^p f(|A-B|)). \tag{2.4}$$

Proof. It is well-known that [9] for any operator convex function f on $[0,\infty)$ there exists a positive measure μ on $[0,\infty)$ such that

$$f(t)=\alpha+\beta t+\gamma t^2+\int_0^\infty h_s(t)d\mu(s),$$

where α and β are real, $\gamma\geqslant 0$ and $h_s(t)$ is defined in Lemma 2.2. By the assumption of the theorem, $\alpha=0$. Now, suppose that $A\geqslant B$ and put $C=A-B$. Therefore,

$$\begin{aligned}\operatorname{Tr}((A-B)^p(f(A)-f(B))) &= \operatorname{Tr}(\beta C^p+\gamma C^p((B+C)^2-B^2))+\\ &\quad+\int_0^\infty \operatorname{Tr}(C^p(h_s(B+C)-h_s(B)))d\mu(s)\\ &= \operatorname{Tr}(\beta C^p+\gamma C^p(C^2+BC+CB))+\\ &\quad+\int_0^\infty \operatorname{Tr}(C^p(h_s(B+C)-h_s(B)))d\mu(s)\\ &\geqslant \operatorname{Tr}(\beta C^p+\gamma C^{p+1})+\int_0^\infty \operatorname{Tr}(C^p h_s(C))d\mu(s)\\ &= \operatorname{Tr}\left(C^p\left(\beta C+\gamma C^2+\int_0^\infty h_s(C)d\mu(s)\right)\right)\\ &= \operatorname{Tr}(C^p f(C)),\end{aligned}$$

where the inequality follows from Lemma 2.2 and that $\operatorname{Tr}(XY)$ is nonnegative for positive semidefinite matrices X and Y. In general, using the same arguments in the proof of Theorem 2.1 we have

$$\begin{aligned}\operatorname{Tr}((A-B)^p(f(A)-f(B))) &= \operatorname{Tr}(C_+^p(f(A)-f(Z)))+\operatorname{Tr}(C_+^p(f(Z)-f(B)))\\ &\quad+\operatorname{Tr}(C_-^p(f(Z)-f(A)))-\operatorname{Tr}(C_-^p(f(Z)-f(B))).\end{aligned}$$

According to the previous case, we have

$$\begin{aligned}\operatorname{Tr}(C_+^p(f(Z)-f(B)))+\operatorname{Tr}(C_-^p(f(Z)-f(A))) &\geqslant \operatorname{Tr}(C_+^p f(C_+))+\operatorname{Tr}(C_-^p f(C_-))\\ &= \operatorname{Tr}(C^p f(|C|)).\end{aligned}$$

To finish the proof, we need to show that the first and the forth terms are positive. We again use the integral representation of operator convex functions and the fact that $C_-C_+=0$. We have

$$\begin{aligned}\operatorname{Tr}(C_+^p(f(A)-f(Z))) &= -\operatorname{Tr}(C_+^p(f(A+C_-)-f(A)))\\ &= -\operatorname{Tr}(\beta C_+^p C_-+\gamma C_+^p((A+C_-)^2-A^2))+\\ &\quad-\int_0^\infty \operatorname{Tr}(C_+^p(sC_--sf_s(A+C_-)+sf_s(A)))d\mu(s)\\ &= \int_0^\infty s\operatorname{Tr}(C_+^p(f_s(A+C_-)-f_s(A)))d\mu(s)\\ &\geqslant 0,\end{aligned}$$

where the inequality follows from the fact that $f_s(t)$ is operator monotone and $A+C_-\geqslant A$. Similarly, we also have that the forth term is positive. Thus, we finish the proof.

REMARK 2.1. To finish this section we would like to note that the essential difference between proofs of Theorems 2.1 and 2.2 and ones in [8] is the using of Lemma 2.1. In [8] we use the fact that

$$(B+s)^{-1}-(B+C+s)^{-1}=(B+s)^{-1}C(B+C+s)^{-1}.$$

Thanks to this, we can rewrite $\operatorname{Tr}(C(B+s)^{-1}C(B+C+s)^{-1})$ as

$$\operatorname{Tr}(C(B+s)^{-1}C(B+C+s)^{-1})=\operatorname{Tr}((C(B+s)^{-1}C)^{1/2}(B+C+s)^{-1}(C(B+s)^{-1}C)^{1/2}),$$

where $X=(C(B+s)^{-1}C)^{1/2}$ is a positive semidefinite matrix. And then, we use the comparisons

$$C(B+C+s)^{-1}C\leqslant C(C+s)^{-1}C \quad \text{and} \quad (B+s)^{-1}\leqslant s^{-1} \tag{2.5}$$

to get the result. But in the proofs of Theorems 2.1 and 2.2 we have

$$\operatorname{Tr}(C^p(B+C+s)^{-1}-(B+s)^{-1}))=\operatorname{Tr}(C^p(B+C+s)^{-1}C(B+s)^{-1}).$$

When $p\neq 1$ there is no way to use the comparisons (2.5), hence, the approach in [8] could not be used. That is why Lemma 2.1 is crucial and make the proofs interesting and different.

3. Trace inequality for power functions

In the following proposition we provide another matrix generalizations of (1.2) for power functions. It is worth mentioning that inequalities in this proposition are not fully covered in Theorem 2.1.

PROPOSITION 3.1. *Assume* $A,B\in M_n(\mathbb{C})$ *are positive semidefinite,* $p\geqslant 1$*, and* $q\geqslant 0$ *is such that* $p\geqslant q$*. Then,*

$$\operatorname{Tr}((A-B)^q(A^{p-q}-B^{p-q})))\leqslant \operatorname{Tr}(|A^p-B^p|). \tag{3.1}$$

Moreover, if $0\leqslant p-q\leqslant 1$ *then*

$$\operatorname{Tr}((A-B)^q(A^{p-q}-B^{p-q}))\leqslant \operatorname{Tr}(|A-B|^p).$$

Proof. Since the proof of the second inequality is similar to the proof of the first, we just prove the first. By Hölder's inequality, we have

$$\operatorname{Tr}((A-B)^q(A^{p-q}-B^{p-q}))\leqslant \|(A-B)^q\|_{p/q}\|A^{p-q}-B^{p-q}\|_{p/(p-q)}.$$

By Ando's theorem for $p\geqslant 1$,

$$\|(A-B)^q\|_{p/q}=\|(A-B)^p\|_1^{q/p}\leqslant \|A^p-B^p\|_1^{q/p}.$$

Using Ando's inequality again for $0 \leqslant \theta = (p-q)/p \leqslant 1$ that says $\|A^\theta - B^\theta\|_\theta \leqslant \|A-B\|_1^\theta$, we obtain

$$\|A^{p-q} - B^{p-q}\|_{p/(p-q)} \leqslant \|A^p - B^p\|_1^{(p-q)/p}.$$

Thus,

$$\begin{aligned}\operatorname{Tr}((A-B)^q(A^{p-q}-B^{p-q})) &\leqslant \|(A-B)^q\|_{p/q}\|A^{p-q}-B^{p-q}\|_{p/(p-q)} \\ &\leqslant \|A^p - B^p\|_1^{q/p}\|A^p - B^p\|_1^{(p-q)/p} \\ &= \operatorname{Tr}(|A^p - B^p|).\end{aligned}$$

From [8, Proposition 1] and Proposition 3.1 we have the following chain of interpolating inequalities between $\|A^p - B^p\|_1$ and $\operatorname{Tr}((A-B)^2(A^{p-2}-B^{p-2}))$.

COROLLARY 3.1. *Let A and B be positive semidefinite matrices and $p \in [2,3]$. Then,*

$$\begin{aligned}\operatorname{Tr}(|A^p - B^p|) &\geqslant \operatorname{Tr}((A-B)(A^{p-1}-B^{p-1})) \\ &\geqslant \operatorname{Tr}(|A-B|^p) \geqslant |\operatorname{Tr}((A-B)^2(A^{p-2}-B^{p-2}))|.\end{aligned}$$

REMARK 3.1. For $p \geqslant 3$ the following inequality

$$\operatorname{Tr}((A-B)^2(A^{p-2}-B^{p-2})) \geqslant \operatorname{Tr}(|A-B|^p), \tag{3.2}$$

does not hold. Both sides are not comparable.

Note that for $p \geqslant 3$ and $a \geqslant b$ we also have another chain of inequalities as follows:

$$a^p - b^p \geqslant (a^2-b^2)(a^{p-2}-b^{p-2}) \geqslant (a-b)^2(a^{p-2}-b^{p-2}) \geqslant (a-b)^p. \tag{3.3}$$

PROPOSITION 3.2. *For any positive semidefinite matrices A and B,*

$$\operatorname{Tr}((A-B)(A^3-B^3)) \geqslant \operatorname{Tr}(|A-B|^4).$$

Proof. Expanding the expressions in the traces and cancelling like terms, the desired inequality is equivalent to,

$$3\operatorname{Tr}(A^3B + AB^3) \geqslant 2\operatorname{Tr}(ABAB + 2A^2B^2).$$

Since $\operatorname{Tr}(ABAB) \leqslant \operatorname{Tr}(A^2B^2)$, the desired result follows if

$$\operatorname{Tr}(A^3B + AB^3) \geqslant 2\operatorname{Tr}(A^2B^2).$$

This result was shown in [3] by using the reduction we have used from the trace to the inequality of scalars by means of the spectral decomposition. However, we give an alternate proof here using the Geometric-Heinz mean inequality [4],

$$|||A^{1/2}XB^{1/2}||| \leqslant \frac{1}{2}|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$$

for positive semidefinite A and B, $\nu \in [0,1]$, arbitrary matrix X, and unitary invariant norm $|||\cdot|||$.

Using $X = I$, $\nu = 1/4$, and replacing A and B with A^2 and B^2, we obtain

$$\|AB\|_2^2 \leqslant \frac{1}{4}\|A^{1/2}B^{3/2} + A^{3/2}B^{1/2}\|_2^2$$

for the 2-Schatten norm. Expanding the norms in terms of traces, we obtain

$$\|AB\|_2^2 = \mathrm{Tr}\,((AB)(AB)^*) = \mathrm{Tr}\,(A^2B^2)$$

and

$$\begin{aligned}\|A^{1/2}B^{3/2} + A^{3/2}B^{1/2}\|_2^2 &= \mathrm{Tr}\,((A^{1/2}B^{3/2} + A^{3/2}B^{1/2})(B^{1/2}A^{3/2} + B^{3/2}A^{1/2})) \\ &= \frac{1}{4}\mathrm{Tr}\,(AB^3 + 2A^2B^2 + BA^3).\end{aligned}$$

Now the result follows by multiplying the inequality times 4 and subtracting $2\mathrm{Tr}\,(A^2B^2)$ on both sides.

To finish this paper, we show a matrix version of inequalities in (3.3).

THEOREM 3.1. *Let $p \geqslant 3$. Then for any positive semidefinite matrices A and B,*

$$\begin{aligned}\mathrm{Tr}\,(|A^p - B^p|) &\geqslant \mathrm{Tr}\,((A^2 - B^2)(A^{p-2} - B^{p-2})) \\ &\geqslant \mathrm{Tr}\,((A - B)(A^{p-1} - B^{p-1})) \\ &\geqslant \mathrm{Tr}\,(|A - B|^p).\end{aligned}$$

Proof. The first inequality actually is true for any $p \geqslant 1$ and follows from [8, Proposition 1]. The last inequality was proved in Ricard's paper [10]. The second inequality is another form of (1.3) and is equivalent to

$$\mathrm{Tr}\,(A^2B^{p-2} + B^2A^{p-2}) \leqslant \mathrm{Tr}\,(AB^{p-1} + AB^{p-1}).$$

If $A = \sum_i a_iA_i$ and $B = \sum_k b_kB_k$ are spectral decomposition of A and B, respectively, then the last inequality is nothing but

$$x^2y^{p-2} + y^2x^{p-2} \leqslant xy^{p-1} + yx^{p-1}$$

which reduces to

$$y^{q-1}(x-y) \leqslant x^{q-1}(x-y), \quad q \geqslant 1.$$

The last inequality is obvious.

REMARK 3.2. The referee pointed out in his/her report that to prove (1.3) it is suffices to use the fact that the function $t \to ||A^tZB^t||_2$ is log-convex (see [6]). Thus, the function $\operatorname{Tr}(A^tB^{p-t})$ is log-convex. Since $(1-\alpha)+\alpha(p-1)$ with $\alpha = 1/(p-2)$, we have

$$\operatorname{Tr}(A^2B^{p-2}) \leqslant (\operatorname{Tr}(AB^{p-1}))^{1-\alpha}(\operatorname{Tr}(A^{p-1}B))^{\alpha} \leqslant (1-\alpha)\operatorname{Tr}(AB^{p-1}) + \alpha\operatorname{Tr}(AB^{p-1}). \tag{3.4}$$

Interchanging A and B in the last inequality, we have

$$\operatorname{Tr}(B^2A^{p-2}) \leqslant (1-\alpha)\operatorname{Tr}(BA^{p-1}) + \alpha\operatorname{Tr}(BA^{p-1}). \tag{3.5}$$

Therefore, inequality (1.3) follows from (3.4) and (3.5).

REMARK 3.3. Inspiring from the second inequality in Theorem 3.1 one may ask if the following inequality holds:

$$\operatorname{Tr}((A-B)^3(A^{p-3}-B^{p-3})) \leqslant \operatorname{Tr}((A-B)^2(A^{p-2}-B^{p-2})). \tag{3.6}$$

Unfortunately, this is not the case. Here we present a counterexample for inequality (3.6). Indeed, consider

$$p = 5, \quad A = \begin{pmatrix} 58 & 13 \\ 13 & 41 \end{pmatrix}, \quad B = \begin{pmatrix} 74 & 31 \\ 31 & 85 \end{pmatrix}.$$

For these values,

$$\begin{aligned} \operatorname{Tr}((A-B)^3(A^{p-3}-B^{p-3})) &= 1{,}258{,}807{,}200, \\ \operatorname{Tr}((A-B)^2(A^{p-2}-B^{p-2})) &= -3{,}067{,}771{,}200 \end{aligned}$$

which contradicts (3.6).

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