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INEQUALITIES FOR ANGLES BETWEEN SUBSPACES WITH APPLICATIONS TO CAUCHY–SCHWARZ INEQUALITY IN INNER PRODUCT SPACES

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Abstract. We show several inequalities for angles between vectors and subspaces in inner product spaces, where concave functions are involved. In specific situations, some of them can be interpreted as triangle inequalities for natural metrics on complex projective spaces. In a consequence, we obtain a few operator generalizations of the famous Cauchy-Schwarz inequality, where powers grater than two occur.

1. Introduction and motivation

Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{F}$ ($\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$). The inequality

$$|\langle x,y\rangle| \leqslant \|x\|\|y\|,\ x,y\in V, \tag{1}$$

is known in the literature as Schwarz's (or Cauchy-Schwarz or Cauchy-Bunyakovsky-Schwarz) inequality, where $\|v\|^2=\langle v,v\rangle$, $v\in V$. The equality holds in Schwarz's inequality if and only if the vectors x and y are linearly dependent. Since A.L. Cauchy (1821) published the first version for sums, V.Y. Bunyakovsky (1859) - for integrals and H.A. Schwarz (1888) derived the first modern proof, it has still attracted mathematicians. Presently, this is one of the fundamental inequalities in all mathematics. Its counterparts are known in functional analysis, in linear, vector, matrix or operator algebra, probability theory, theoretical physics and other areas. A large number of refinements and generalizations can be found in monographs [4, 5, 6, 13, 14], see also the review article [1] and references therein but the list is far from being complete.

The Cauchy-Schwarz inequality is the basis for defining of the angle between vectors or, more general, the angle between subspaces of an inner product space. For example, the angle between the non-zero vectors $x,y\in V$ can be defined in two ways

$$\Psi_{x,y}=\arccos\frac{|\langle x,y\rangle|}{\|x\|\|y\|},\ \Phi_{x,y}=\arccos\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\|y\|}. \tag{2}$$

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The function $\Psi : V \times V \mapsto [0, \pi/2]$ is a natural metric on complex projective space, since $\Psi_{\lambda x, \gamma y} = \Psi_{x,y}$, $x, y \in V \setminus \{0\}$, $\lambda, \gamma \in \mathbb{F} \setminus \{0\}$ and it satisfies the triangle inequality [12]

$$\Psi_{x,y} \leqslant \Psi_{x,z} + \Psi_{z,y} \text{ or, equivalently, } |\Psi_{x,z} - \Psi_{y,z}| \leqslant \Psi_{x,y}, \text{ for any } x,y,z \in V \setminus \{0\}. \quad (3)$$

The triangle inequality, $\Phi_{x,y} \leqslant \Phi_{x,z} + \Phi_{z,y}$, $x,y,z \in V \setminus \{0\}$, for the angle $\Phi_{\cdot,\cdot}$, was proved by M.G. Krein [11]. Other triangle inequalities also hold [12, Prop. 2]), e.g.

$$\left(1 - \frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p} \leqslant \left(1 - \frac{|\langle x,z\rangle|^p}{\|x\|^p\|z\|^p}\right)^{1/p} + \left(1 - \frac{|\langle y,z\rangle|^p}{\|y\|^p\|z\|^p}\right)^{1/p}, \ p \geqslant 2. \quad (4)$$

In case $p = 2$ this is the triangle inequality for the sine function of the angle $\Psi_{\cdot,\cdot}$ and was proved earlier in [16].

Friedrichs [9] introduced the angle $\alpha_{K,L} \in [0, \pi/2]$ between closed subspaces $K, L \subset V$ as follows

$$\cos \alpha_{K,L} = \sup\{|\langle x,y\rangle| : x \in K \cap (K \cap L)^\perp, \ y \in L \cap (K \cap L)^\perp, \ \|x\|, \|y\| \leqslant 1\}.$$

Dixmier [3] defined the minimal angle between subspaces $\Psi_{K,L} \in [0, \pi/2]$ by

$$\cos \Psi_{K,L} = \sup\{|\langle x,y\rangle| : x \in K, \ y \in L, \ \|x\|, \|y\| \leqslant 1\}. \quad (5)$$

These definitions are different unless $K \cap L = \{0\}$. Contrary to the geometric intuition, the first of them treats the angle between equal subspaces as $\pi/2$. The definition of $\Psi x, y$, $x, y \in V$ is compatible with $\Psi_{K,L}$ for $K = \mathrm{span}\{x\}$ and $L = \mathrm{span}\{y\}$. Further basic results which hold for these angles and a few of the many applications are to be found in [2].

In section 2 angles between vectors and subspaces will be considered. We will introduce a specific definition for this case in the spirit of Dixmier's one and prove some inequalities for such angles. Between others, counterparts of inequalities (3) and (4) will be obtained, as well as more general inequalities, where concave functions are involved.

The notion of the orthogonal projection is closely connected with the Cauchy-Schwarz inequality. Recall that the orthogonal projection onto a closed subspace L of V (projection with the range L, in short) is the mapping $P_L : V \mapsto L$ which associates with each $x \in V$ its unique nearest point in L, i.e., $\|x - P_L x\| = \mathrm{dist}(x, L) := \inf_{y \in L} \|x - y\|$. In other words, a bounded linear operator P on V is an orthogonal projection if and only if P is idempotent and self-adjoint (i.e. $P^2 = P = P^*$). Then $P = P_L$, where $L = P(V)$ is the range of P. For example, if $v \in V$ is a non-zero vector, then $P_v x = \frac{\langle x,v\rangle}{\|v\|^2} v$ defines the projection onto 1-dimensional subspace $\mathrm{span}\{v\}$. For an unit vector $e \in V$, $P_e x = \langle x, e\rangle e$.

Now, for the completeness of this article, we recall some relevant results on the Cauchy-Schwarz inequality. The following discrete version of (1) holds [15]

$$\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n x_i y_i\right)^2 \geqslant \left| n \det \begin{bmatrix} |\overline{x}| & S_x \\ |\overline{y}| & S_y \end{bmatrix} \right|^2, \ x_i, y_i \in \mathbb{R}, \ i = 1, 2, \ldots n, \quad (6)$$

where for any n-tuple of real numbers $a = (a_1, a_2, \ldots, a_n)$, $\overline{a} = \frac{1}{n}(a_1 + a_2 + \ldots + a_n)$, $\overline{a^2} = \frac{1}{n}(a_1^2 + a_2^2 + \ldots + a_n^2)$ and $S_a = \sqrt{\overline{a^2} - \overline{a}^2}$.

This inequality has been extended twice so far. The first time as follows [8, Th. 2]

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geqslant \left|\det\begin{bmatrix} |\langle x,e\rangle| & (\|x\|^2 - |\langle x,e\rangle|^2)^{1/2} \\ |\langle y,e\rangle| & (\|y\|^2 - |\langle y,e\rangle|^2)^{1/2}\end{bmatrix}\right|^2, \tag{7}$$

where $x,y,e \in V$ and $\|e\| = 1$. Note, setting $V = \mathbb{R}^n$ with the standard inner product and substituting $e = \frac{1}{\sqrt{n}}(1,1,\ldots,1)$ in (7) we obtain (6) (see [8, sec. 3] for details).

The next extension of (6) (see [7, Th. 1]) is a such generalization of (7), where the norm of projection $\langle P_e x, x\rangle = |\langle x,e\rangle|$ is replaced by $\langle Ax,x\rangle$, A is a selfadjoint operator such that $0 \leqslant \langle Ax,x\rangle \leqslant \langle x,x\rangle, x \in V$.

Taking z as a unit vector e, a simple reformulation of (4) leads to the following generalization of C-S inequality [8, Th. 1]

$$\|x\|^p\|y\|^p - |\langle x,y\rangle|^p \geqslant \left|\det\begin{bmatrix} \|x\| & (\|x\|^p - |\langle x,e\rangle|^p)^{1/p} \\ \|y\| & (\|y\|^p - |\langle y,e\rangle|^p)^{1/p}\end{bmatrix}\right|^p, \; p \geqslant 2, \tag{8}$$

where $x,y,e \in V$ with $\|e\| = 1$.

The section 3 is devoted to applications of inequalities for angles to improving of Cauchy-Schwarz inequality. We obtain a few generalizations of this inequality, where orthogonal projections occur. In this spirit we generalize inequality (8) and derive certain results related to inequalities (6) and (7). In the last section, some specifications for finite dimensional subspaces are studied.

2. Inequalities for angles between vectors and subspaces

Let a closed subspace $L \subset V$ be the range of a projection P and a nonzero vector $v \in V$ be fixed. For any scalar λ with $|\lambda| \leqslant 1/\|v\|$ and any vector $y \in L$ with $\|y\| \leqslant 1$, by Cauchy-Schwarz inequality, we have the estimate $|\langle \lambda v, y\rangle| = |\lambda||\langle v, Py\rangle| = |\lambda||\langle Pv, y\rangle| \leqslant |\lambda|\|Pv\|\|y\| \leqslant \frac{\|Pv\|}{\|v\|}$ with the equality if $Pv = 0$ or, otherwise, if $\lambda = 1/\|v\|$ and $y = Pv/\|Pv\|$. Hence $\sup\{|\langle \lambda v, y\rangle| : |\lambda v| \leqslant 1,\ y \in L,\ \|y\| \leqslant 1\} = \frac{\|Pv\|}{\|v\|}$. In this way, according to Dixmier's definition (5) of the minimal angle between the pair of subspaces $K := \operatorname{span}\{v\}$ and L we determine the angle $\Psi_{v,L} \in [0, \pi/2]$ between a non-zero vector $v \in V$ and the subspace L as follows

$$\cos\Psi_{v,L} = \frac{\|Pv\|}{\|v\|}, \text{ equivalently, } \sin\Psi_{v,L} = \sqrt{1 - \frac{\|Pv\|^2}{\|v\|^2}}. \tag{9}$$

Note, $\Psi_{v,L} = \Psi_{v,Pv}$ if $Pv \neq 0$ and $\Psi_{x,y} = \Psi_{x,P_y x} = \Psi_{y,P_x y}$, if $x,y \neq 0$.

Below, we will establish a few inequalities, where mentioned angles or their functions occur. The first of them extends the triangle inequality (3). Although such theorem is substantially known, we include its simple proof for the readers convenience.

THEOREM 1. *For a closed subspace $L \subset V$ and for non-zero vectors $x, y \in V$ the following inequality holds*

$$|\Psi_{x,L} - \Psi_{y,L}| \leqslant \Psi_{x,y}. \tag{10}$$

Proof. Let P be the orthogonal projection onto L. Applying Schwarz's inequality for the vectors $x - Px$ and $y - Py$ and taking into account that $\langle x - Px, y - Py\rangle = \langle x, y\rangle - \langle Px, Py\rangle$ and $\|x - Px\|^2 = \|x\|^2 - \|Px\|^2$ and $\|y - Py\|^2 = \|y\|^2 - \|Py\|^2$ we have the inequality

$$|\langle x, y\rangle - \langle Px, Py\rangle| \leqslant \sqrt{\|x\|^2 - \|Px\|^2}\sqrt{\|y\|^2 - \|Py\|^2}. \tag{11}$$

Further, division both of sides by $\|x\|\|y\| \neq 0$ and using the continuity condition for the modulus and Schwarz's inequality for Px and Py, consecutively, leads to

$$\frac{|\langle x, y\rangle|}{\|x\|\|y\|} \leqslant \frac{\|Px\|}{\|x\|}\frac{\|Py\|}{\|y\|} + \sqrt{1 - \frac{\|Px\|^2}{\|x\|^2}}\sqrt{1 - \frac{\|Py\|^2}{\|y\|^2}},$$

or, $\cos\Psi_{x,y} \leqslant \cos\Psi_{x,L}\cos\Psi_{y,L} + \sin\Psi_{x,L}\sin\Psi_{y,L} = \cos|\Psi_{x,L} - \Psi_{y,L}|$, utilizing the notion of the angle $\Psi_{\cdot,\cdot}$. Finally, since the function arccos is decreasing we get inequality (10). $\square$

COROLLARY 1. *If $L \subset V$ is a closed subspace, then for non-zero vectors $x, y \in V$ the following inequality holds*

$$|\sin\Psi_{x,L}\cos\Psi_{y,L} - \cos\Psi_{x,L}\sin\Psi_{y,L}| \leqslant \sin\Psi_{x,y}. \tag{12}$$

Proof. By Theorem 1, inequality (10) holds. Thus, since the function $\sin(t)$, $t \in [0, \pi/2]$, is increasing, we have

$$|\sin\Psi_{x,L}\cos\Psi_{y,L} - \cos\Psi_{x,L}\sin\Psi_{y,L}| = |\sin(\Psi_{x,L} - \Psi_{y,L})| = \sin|\Psi_{x,L} - \Psi_{y,L}| \leqslant \sin\Psi_{x,y}.$$

$\square$

The next results require the following known lemma about concave (convex) functions.

LEMMA 1. *If $f : [0, d] \mapsto [0, \infty), (d > 0)$ is a non-decreasing concave (convex) function with $f(0) = 0$ and $0 \leqslant a, b \leqslant d$, then $|f(a) - f(b)| \leqslant (\geqslant) f(|a - b|)$.* $\square$

Consequently, by Lemma 1, Theorem 1 and Corollary 1, we easy obtain the following two inequalities.

THEOREM 2. *Let L be a closed subspace of V and $f : [0, a] \mapsto [0, \infty)$ be a concave non-decreasing function with $f(0) = 0$. Then for non-zero vectors $x, y \in V$ the following inequalities hold*

$$|f(\Psi_{x,L}) - f(\Psi_{y,L})| \leqslant f(\Psi_{x,y}), \tag{13}$$

$$|f(\sin\Psi_{x,L}\cos\Psi_{y,L}) - f(\cos\Psi_{x,L}\sin\Psi_{y,L})| \leqslant f(\sin\Psi_{x,y}), \tag{14}$$

where $a \geqslant \pi/2$ in case of (13) or $a \geqslant 1$ in case of (14).

Applying general inequalities (13) and (14) to specific functions one can obtain next more particular inequalities. Such examples we introduce below.

COROLLARY 2. *Let $L \subset V$ be the range of a projection $P : V \mapsto V$. For non-zero vectors $x, y \in V$ the following inequalities hold*

$$\left|(1-\cos^p \Psi_{x,L})^{1/p} - (1-\cos^p \Psi_{y,L})^{1/p}\right| \leqslant (1-\cos^p \Psi_{x,y})^{1/p}, \ p \geqslant 2, \tag{15}$$

$$\begin{aligned}\left|\sin^{1/p}\Psi_{x,L} - \sin^{1/p}\Psi_{y,L}\right| &\leqslant \left|\sin^{1/p}\Psi_{x,L}\cos^{1/p}\Psi_{y,L} - \cos^{1/p}\Psi_{x,L}\sin^{1/p}\Psi_{y,L}\right| \\ &\leqslant \sin^{1/p}\Psi_{x,y}, \ p \geqslant 1.\end{aligned} \tag{16}$$

Proof. All functions mentioned in the proof are increasing concave with zero value at zero. It can be showed using elementary analysis. We omit these details.

Specifying inequality (13) for functions $f(t) = (1-\cos^p t)^{1/p}$, $t \in [0, \pi/2]$, $p \geqslant 2$, we obtain (15). Inequality (13) applied for functions $f(t) = (\sin t)^{1/p}$, $t \in [0, \pi/2]$, $p \geqslant 1$ establishes also the inequality between the first and the third expression in (16). The second inequality in (16) is a consequence of inequality (14) used for the functions $f(t) = t^{1/p}$, $t \in [0,1]$, $p \geqslant 1$. To prove the first inequality in (16) utilize the same functions and set $\alpha_1 = \sin\Psi_{x,L}\cos\Psi_{y,L}$, $\beta_1 = \sin\Psi_{x,L}$ and $\alpha_2 = \sin\Psi_{y,L}\cos\Psi_{x,L}$, $\beta_2 = \sin\Psi_{y,L}$. The inequality is evident, if $\Psi_{x,L} = \Psi_{y,L}$. Now, let $\Psi_{x,L} \neq \Psi_{y,L}$. Then $\alpha_i \leqslant \beta_i$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\alpha_i, \beta_i \in [0, d]$, $i = 1, 2$. Since f is increasing convex,

$$\frac{|f(\alpha_2) - f(\alpha_1)|}{|f(\beta_2) - f(\beta_1)|} \geqslant \frac{|\alpha_2 - \alpha_1|}{|\beta_2 - \beta_1|},$$

(see e.g. [13, Chap. I, sec. 3]). Moreover,

$$\begin{aligned}|\alpha_2 - \alpha_1| &= |\sin\Psi_{y,L}\cos\Psi_{x,L} - \sin\Psi_{x,L}\cos\Psi_{y,L}| = |\sin(\Psi_{y,L} - \Psi_{x,L})| = \\ &\sin|\Psi_{y,L} - \Psi_{x,L}| \geqslant |\sin\Psi_{y,L} - \sin\Psi_{x,L}| = |\beta_2 - \beta_1|,\end{aligned}$$

by Lemma 1 applied to the sine function. Thus $|f(\alpha_2) - f(\alpha_1)| \geqslant |f(\beta_2) - f(\beta_1)|$. This is exactly what should be proven. $\square$

REMARK 1. For a non-zero vector $z \in V$, if $P = P_z$, i.e. P is the orthogonal projection onto 1-dimensional subspace $\operatorname{span}\{z\}$, then inequalities (10) and (15) weaken to (3) and (4), respectively.

3. Generalizations of the Cauchy-Schwarz inequality

The following generalizations of the Cauchy-Schwarz inequality can be established.

THEOREM 3. *Fix $p \geqslant 2$. For any vectors $x, y \in V$ and any projection P on V,*

$$\|x\|^p\|y\|^p - |\langle x, y\rangle|^p \geqslant \left|\det\begin{bmatrix}\|x\| & (\|x\|^p - \|Px\|^p)^{1/p} \\ \|y\| & (\|y\|^p - \|Py\|^p)^{1/p}\end{bmatrix}\right|^p. \tag{17}$$

Proof. Starting from the definition (9) of the angle $\Psi_{\cdot,\cdot}$, we can express inequality (15) as follows

$$\left|\left(1-\frac{\|Px\|^p}{\|x\|^p}\right)^{1/p}-\left(1-\frac{\|Py\|^p}{\|y\|^p}\right)^{1/p}\right|\leqslant\left(1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p},\ p\geqslant 2.$$

Now, if we take the power p and multiply with $\|x\|^p\|y\|^p$, then we obtain inequality (17). $\square$

REMARK 2. Inequality (17) was obtained by Dragomir [8, Th. 1] for projections $P=P_e$ onto 1-dimensional subspaces, cf. (8).

Similarly, inequality (16) translates to the next generalization of C-S inequality.

COROLLARY 3. *Let $1\leqslant p\leqslant q$. For any vectors $x,y\in V$ and any projection P on V,*

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geqslant\left|\det\begin{bmatrix}\|Px\|^{1/p} & \left(\sqrt{\|x\|^2-\|Px\|^2}\right)^{1/p}\\ \|Py\|^{1/p} & \left(\sqrt{\|y\|^2-\|Py\|^2}\right)^{1/p}\end{bmatrix}\right|^{2p}\geqslant$$
$$\left|\det\begin{bmatrix}\|x\|^{1/p} & \left(\sqrt{\|x\|^2-\|Px\|^2}\right)^{1/p}\\ \|y\|^{1/p} & \left(\sqrt{\|y\|^2-\|Py\|^2}\right)^{1/p}\end{bmatrix}\right|^{2p}. \tag{18}$$

In addition,

$$D_p(x,y)\geqslant D_q(x,y), \tag{19}$$

where $D_p(x,y)=\left|\det\begin{bmatrix}\|Px\|^{1/p} & \left(\sqrt{\|x\|^2-\|Px\|^2}\right)^{1/p}\\ \|Py\|^{1/p} & \left(\sqrt{\|y\|^2-\|Py\|^2}\right)^{1/p}\end{bmatrix}\right|^{2p}.$

Proof. Substituting (9) to (16), taking $2p$-power and multiplying with $\|x\|^2\|y\|^2$ we obtain (18).

To prove (19) let

$$a=\left(\|Px\|\sqrt{\|y\|^2-\|Py\|^2}\right)^{1/q},b=\left(\|Py\|\sqrt{\|x\|^2-\|Px\|^2}\right)^{1/q}.$$

Now, it suffices to apply Lemma 1 for the increasing convex function $u\mapsto u^{q/p},u\geqslant 0$ to get $\left(|a-b|^{q/p}\right)^{2p}\leqslant\left(|a^{q/p}-b^{q/p}|\right)^{2p}$, as desired. $\square$

REMARK 3. A. For $P=\mathrm{id}$ or $P=0$ inequalities (18) become the classic C-S inequality.
B. For $P=P_e$, where $\|e\|=1$ and $p=1$ the first part of (18) takes the form of inequality (7), recently obtained by Dragomir [8].
C. It also corresponds with more general Dragomir's result [7, Th. 1].

The next result specifies situations when we get the equality in inequality (18).

THEOREM 4. *Fix* $p \geqslant 1$. *For any vectors* $x, y \in V$ *and any projection* P *on* V *with the range* L,

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \left|\det\begin{bmatrix} \|Px\|^{1/p} & \left(\sqrt{\|x\|^2-\|Px\|^2}\right)^{1/p} \\ \|Py\|^{1/p} & \left(\sqrt{\|y\|^2-\|Py\|^2}\right)^{1/p}\end{bmatrix}\right|^{2p} \tag{20}$$

if and only if $x = \lambda_1 u + \gamma_1 v$, $y = \lambda_2 u + \gamma_2 v$, *where* $\lambda_r, \gamma_r \in \mathbb{F}$, $r = 1,2$, $u \in L, v \in L^\perp$ *and* $\lambda_1\overline{\lambda_2}\overline{\gamma_1}\gamma_2 \geqslant 0$.

Proof. On account of Corollary 3, if equality (20) holds, then

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 = \left|\det\begin{bmatrix} \|Px\| & \sqrt{\|x\|^2-\|Px\|^2} \\ \|Py\| & \sqrt{\|y\|^2-\|Py\|^2}\end{bmatrix}\right|^{2}. \tag{21}$$

Any vectors $x, y \in V$ admit the representations $x = u_1 + v_1$, $y = u_2 + v_2$, where $u_r \in L$, $v_r \in L^\perp$ so $\langle u_r, v_r\rangle = 0$, $r = 1,2$. It gives

$$\begin{aligned} |\langle x,y\rangle|^2 &= |\langle u_1,u_2\rangle|^2 + 2\mathrm{Re}\langle u_1,u_2\rangle\langle v_2,v_1\rangle + |\langle v_1,v_2\rangle|^2, \\ \|x\|^2 = \|u_1\|^2+\|v_1\|^2 \quad &, \quad \|y\|^2 = \|u_2\|^2+\|v_2\|^2, \\ \|Px\| = \|u_1\| \quad &, \quad \|Py\| = \|u_2\|. \end{aligned}$$

Based on the above, equality (21) holds if and only if

$$\begin{aligned} \left(\|u_1\|^2\|u_2\|^2 - |\langle u_1,u_2\rangle|^2\right) + \left(\|v_1\|^2\|v_2\|^2 - |\langle v_1,v_2\rangle|^2\right) = \\ 2\left(\mathrm{Re}\langle u_1,u_2\rangle\langle v_2,v_1\rangle - \|u_1\|\|u_2\|\|v_1\|\|v_2\|\right). \end{aligned}$$

Note, the left hand side of the above equality is non-negative as the sum of two non-negative components, while the other side is non-positive. Therefore, the equality holds if and only if the both of sides are equal zero. Thus, by C-S inequality (1), we obtain

$$\begin{aligned} |\langle u_1,u_2\rangle|^2 = \|u_1\|^2\|u_2\|^2,\ |\langle v_1,v_2\rangle|^2 = \|v_1\|^2\|v_2\|^2, \\ \|u_1\|\|u_2\|\|v_1\|\|v_2\| = \mathrm{Re}\langle u_1,u_2\rangle\langle v_2,v_1\rangle. \end{aligned}$$

Two first equalities are met if and only if u_1, u_2 and v_1, v_2 are pairs of linearly dependent vectors, i.e. there exist unit vectors $u \in L$, $v \in L^\perp$ and scalars $\lambda_r, \gamma_r \in \mathbb{F}$ such that $u_r = \lambda_r u$, $v_r = \gamma_r v$, $r = 1,2$. In this situation, the third equality is equivalent to $\lambda_1\overline{\lambda_2}\overline{\gamma_1}\gamma_2 \geqslant 0$, what is easily seen. The proof is finished. □

4. More generalization of the Cauchy-Schwarz inequality

Given vectors $x_1, x_2, \ldots, x_p \in V$, the matrix $G(x_1,x_2,\ldots,x_p) := \left[\langle x_i, x_j\rangle\right]_{i,j=1,\ldots,p}$ is called the Gram matrix while its determinant $\Gamma(x_1,x_2,\ldots,x_p)$ is called Gram determinant of the vectors $x_1, x_2, \ldots, x_p$.

Gram's inequality reads

$$0 \leqslant \Gamma(x_1, x_2, \ldots, x_p),\ x_1, x_2, \ldots, x_p \in V$$

with equality if and only if the vectors $x_1, x_2, \ldots, x_p$ are linearly dependent, see [13, Chap. XX].

On the other hand

$$\Gamma(x_1, x_2, \ldots, x_p) \leqslant \|x_1\|^2 \cdot \ldots \cdot \|x_p\|^2,\ x_1, x_2, \ldots, x_p \in V.$$

The equality is met if and only if the vectors $x_1, x_2, \ldots, x_p$ are mutually orthogonal. This is Hadamard's inequality, see [13, Chap. XX],[10].

Let $v_1, v_2, \ldots, v_n \in V$ be a system of linearly independent vectors. The orthogonal projection P onto the subspace $L = \operatorname{span}\{v_1, v_2, \ldots, v_n\}$ takes the form

$$Px = \sum_{r=1}^{n} \frac{\Gamma_r(x, v_1, v_2, \ldots, v_n)}{\Gamma(v_1, v_2, \ldots, v_n)} v_r,\ x \in V,$$

where $\Gamma_r(x, v_1, v_2, \ldots, v_n)$ is the determinant obtained from $\Gamma(v_1, v_2, \ldots, v_n)$ replacing the r-th row by $(\langle x, v_1\rangle,\ \langle x, v_2\rangle,\ \ldots,\ \langle x, v_n\rangle)$. It is well known that $\operatorname{dist}(x, L)$, the distance x to the subspace L, can be expressed as follows

$$\operatorname{dist}(x, L) = \|x - Px\| = \sqrt{\|x\|^2 - \|Px\|^2} = \sqrt{\Gamma(x, v_1, v_2, \ldots, v_n)/\Gamma(v_1, v_2, \ldots, v_n)}.$$

Hence $\|Px\| = \sqrt{\|x\|^2 - \Gamma(x, v_1, v_2, \ldots, v_n)/\Gamma(v_1, v_2, \ldots, v_n)}$. The above yields the following versions of inequalities (17) and (18).

COROLLARY 4. *For linearly independent vectors $v_1, v_2, \ldots, v_n \in V$, any vectors $x, y \in V$ and $p \geqslant 2$, the following generalizations of C-S inequality hold*

$$\|x\|^p \|y\|^p - |\langle x, y\rangle|^p \geqslant \left| \det \begin{bmatrix} \|x\| & (\|x\|^p - (\|x\|^2 - \Gamma_x/\Gamma)^{p/2})^{1/p} \\ \|y\| & (\|y\|^p - (\|y\|^2 - \Gamma_y/\Gamma)^{p/2})^{1/p} \end{bmatrix} \right|^p,$$

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geqslant \left| \det \begin{bmatrix} \left(\|x\|^2 - \Gamma_x/\Gamma\right)^{1/p} & (\Gamma_x/\Gamma)^{1/p} \\ \left(\|y\|^2 - \Gamma_y/\Gamma\right)^{1/p} & (\Gamma_y/\Gamma)^{1/p} \end{bmatrix} \right|^p,$$

where $\Gamma := \Gamma(v_1, v_2, \ldots, v_n)$ and $\Gamma_z := \Gamma(z, v_1, v_2, \ldots, v_n)$, $z \in V$.

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