EXACT UPPER BOUND ON THE SUM OF SQUARED NEAREST–NEIGHBOR DISTANCES BETWEEN POINTS IN A RECTANGLE

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Abstract. An exact upper bound on the sum of squared nearest-neighbor distances between points in a rectangle is given.

1. Introduction and summary

For any natural $n \geqslant 2$ and any positive real numbers a and b, let $P_1, \ldots, P_n$ be distinct points in an $a \times b$ rectangle R. For each $i \in [n] := \{1, \ldots, n\}$, let

$$d_i := \min \{P_i P_k \colon k \in [n] \setminus \{i\}\}, \tag{1}$$

where PQ denotes the Euclidean distance between points P and Q. So, d_i is the distance from the point P_i to its nearest neighbor among the points $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$.

THEOREM 1.

$$\sum_1^n d_i^2 \leqslant 2a^2 + 2b^2. \tag{2}$$

The upper bound $2a^2 + 2b^2$ on $\sum_1^n d_i^2$ is exact in the following sense: it is attained (i) when $n = 2$ and the points P_1, P_2 are opposite vertices of the rectangle R and (ii) when $n = 4$, $a = b$, and the points $P_1, \ldots, P_4$ are the vertices of the square R.

One might note here that, without further restrictions, the exact lower bound on $\sum_1^n d_i^2$ is the trivial bound 0, "attained in the limit" when the distinct points $P_1, \ldots, P_n$ are arbitrarily close to one another.

The terms used in definition (1) are similar to those in the definition of the cells

$$C_i := \{P \colon PP_i \leqslant \min_{k \in [n] \setminus \{i\}} PP_k\} \tag{3}$$

of the Voronoi diagram/tessellation $\{C_1, \ldots, C_n\}$ generated by the points $P_1, \ldots, P_n$. Voronoi diagrams [3] have very broad applications, not only in mathematics, but also in sciences, engineering, and other fields.

The notion of nearest neighbors is used in various other ways as well, in particular in statistics [5], computational geometry [1], information theory [6], computer science [2], biology [7], and elsewhere.

2. Proof of Theorem 1

If the minimum in the definition (1) of the d_i's were replaced by the maximum, then it would be very easy to give the exact upper bound on $\sum_1^n d_i^2$. Indeed, since the largest distance between two arbitrary points of the rectangle R is $\sqrt{a^2+b^2}$, this exact upper bound on $\sum_1^n d_i^2$ would be $n(a^2+b^2)$, attained when all the points P_i are at two opposite vertices of the rectangle R.

However, despite the great simplicity of the actual statement of Theorem 1, its proof is not at all simple. The reason for this is that the summands d_i^2 in the sum in (2) are the minima, rather than maxima, of a possibly large number of convex functions of the points $P_1, \ldots, P_n$. So, the sum $\sum_1^n d_i^2$ as a function of $P_1, \ldots, P_n$ is in general non-convex, usually with a large number of non-smooth local maxima (some of which may be global), plus a number of false (or quasi-) non-smooth local "maxima". Two of such possible situations are illustrated in Fig. 1, where only one of the many variables on which the sum $\sum_1^n d_i^2$ depends is allowed to actually vary. One can see that the dependence of $\sum_1^n d_i^2$ on just one of the variables may already be quite complicated and exhibit quite different patterns. Of course, the complexity and variety of patterns of dependence of the sum $\sum_1^n d_i^2$ on all the variables involved (that is, on the all the coordinates of all the points $P_1, \ldots, P_n$) is much greater yet.
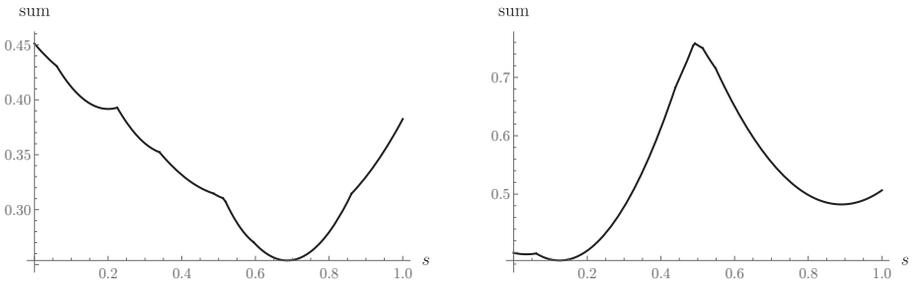

Figure 1: Each of the two graphs here is obtained by selecting $n = 6$ pseudorandom points in the unit square $[0,1]^2$, replacing the abscissa of one of the 6 points by a variable $s \in [0,1]$, and then plotting the resulting varying sum $\sum_1^6 d_i^2$ against s.

To deal with the described difficulties, we will have to consider a variety of cases, in each of the cases carefully constructing convex functions that would be tight enough majorants, at least locally (in an appropriate sense), of the non-convex sum $\sum_1^n d_i^2$. Similar techniques could possibly be used elsewhere.

REMARK 1. One may realize at this point that, for each natural n, finding the exact upper bound on $\sum_1^n d_i^2$ is a problem of real algebraic geometry (also referred to as semialgebraic geometry); see e.g. [4]. In principle, for each given n such a problem can be solved completely algorithmically. However, even for $n = 3$, it takes minutes of computer time, in addition to quite a bit of preparation and manual post-processing of the computer algebra system output, to obtain the following exact upper bound on

$\sum_1^n d_i^2 = \sum_1^3 d_i^2$:

$$S_3(a,b) := \begin{cases} a^2 + \dfrac{9b^2}{4} & \text{if} \quad \dfrac{b}{a} \leqslant \dfrac{1}{\sqrt{3}}, \\[2mm] \dfrac{3a^2}{4} + 3b^2 & \text{if} \quad \dfrac{1}{\sqrt{3}} \leqslant \dfrac{b}{a} \leqslant \dfrac{\sqrt{3}}{2}, \\[2mm] 12(a^2 - \sqrt{3}ab + b^2) & \text{if} \quad \dfrac{\sqrt{3}}{2} \leqslant \dfrac{b}{a} \leqslant \dfrac{2}{\sqrt{3}}, \\[2mm] 3a^2 + \dfrac{3b^2}{4} & \text{if} \quad \dfrac{2}{\sqrt{3}} \leqslant \dfrac{b}{a} \leqslant \sqrt{3}, \\[2mm] \dfrac{9a^2}{4} + b^2 & \text{if} \quad \dfrac{b}{a} \geqslant \sqrt{3}. \end{cases} \tag{4}$$

It follows that

$$S_3(a,b) \leqslant \frac{12}{7}(a^2 + b^2), \tag{5}$$

with the equality if $b/a \in \{2/\sqrt{3}, \sqrt{3}/2\}$, and $\frac{12}{7} \approx 1.71$. Also, $S_3(a,b) = 6(2 - \sqrt{3})(a^2 + b^2) \approx 1.61\,(a^2 + b^2)$ for $a = b$.

For values of n greater than 4 (and also for $n = 4$ with $a \neq b$), the problem of finding the exact upper bound on $\sum_1^n d_i^2$ is likely much more difficult than it is for $n = 3$.

Let us now turn to the actual proof of Theorem 1. For any finite set $\mathscr{S}$ of points on the Euclidean plane, let $\Sigma(\mathscr{S})$ denote the sum of squared nearest-neighbor distances between the points in $\mathscr{S}$. So, (2) can be rewritten as

$$\Sigma(\mathscr{P}) \overset{(?)}{\leqslant} 2a^2 + 2b^2, \tag{6}$$

where

$$\mathscr{P} := \{P_1, \ldots, P_n\}.$$

Let us prove (6) by induction on n. For $n = 2$, (6) is obvious. Suppose now that $n \geqslant 3$.

Without loss of generality (wlog), $R = [0,a] \times [0,b]$. "Partition" R into the four congruent $\frac{a}{2} \times \frac{b}{2}$ rectangles

$$R_1 := [0, \tfrac{a}{2}] \times [0, \tfrac{b}{2}], \; R_2 := [\tfrac{a}{2}, a] \times [0, \tfrac{b}{2}], \; R_3 := [0, \tfrac{a}{2}] \times [\tfrac{b}{2}, b], \; R_4 := [\tfrac{a}{2}, a] \times [\tfrac{b}{2}, b].$$

For each $j \in \{1, \ldots, 4\}$, let

$$\mathscr{P}_j := R_j \cap \mathscr{P} \quad \text{and} \quad n_j := \operatorname{card} \mathscr{P}_j,$$

where card denotes the cardinality.

Case 1: $n_1, \ldots, n_4 \geqslant 2$. Then $2 \leqslant n_j < n$ for all $j \in \{1, \ldots, 4\}$. So, by induction,

$$\Sigma(\mathscr{P}) \leqslant \Sigma(\mathscr{P}_1) + \cdots + \Sigma(\mathscr{P}_4) \leqslant 4[2(\tfrac{a}{2})^2 + 2(\tfrac{b}{2})^2] = 2a^2 + 2b^2,$$

which proves (6) in Case 1.

The consideration of the next two cases is based in part on the following result.

LEMMA 1. *Suppose that $n_1 \geqslant 2$ and $n_2 = 1$, so that there is exactly one point, say Q, in the set $\mathscr{P}_2$. Then*

$$\Sigma(\mathscr{P}_1) + d(Q, \mathscr{P}_1)^2 \leqslant a^2 + b^2.$$

Here and in what follows, for any point P and any set $\mathscr{S}$ of points,

$$d(P, \mathscr{S}) := \inf_{A \in \mathscr{S}} PA,$$

the shortest distance from P to the points in $\mathscr{S}$.

Proof. Let s be the largest abscissa of all the n_1 points in $\mathscr{P}_1$, so that $0 \leqslant s \leqslant \frac{a}{2}$. Let $(u,v) := Q$, so that $\frac{a}{2} \leqslant u \leqslant a$ and $0 \leqslant v \leqslant \frac{b}{2}$. By vertical symmetry, wlog $0 \leqslant v \leqslant \frac{b}{4}$, and hence $d(Q, \mathscr{P}_1)^2 \leqslant (u-s)^2 + (\frac{b}{2} - v)^2$. Also, by induction, $\Sigma(\mathscr{P}_1) \leqslant 2s^2 + 2(\frac{b}{2})^2$. Therefore and in view of the mentioned ranges of the variables s, u, v,

$$\Sigma(\mathscr{P}_1) + d(Q, \mathscr{P}_1)^2 \leqslant 2s^2 + 2(\tfrac{b}{2})^2 + (u-s)^2 + (\tfrac{b}{2} - v)^2$$
$$\leqslant 2s^2 + 2(\tfrac{b}{2})^2 + (a-s)^2 + (\tfrac{b}{2})^2 \leqslant a^2 + \tfrac{3}{4}b^2 \leqslant a^2 + b^2,$$

which proves Lemma 1.

Case 2: One of the n_j's is 1, and the other n_j's are $\geqslant 2$. Here wlog $n_1, n_3, n_4 \geqslant 2$ and $n_2 = 1$. Then, in view of Lemma 1 and by induction,

$$\Sigma(\mathscr{P}) \leqslant \Sigma(\mathscr{P}_1) + d(Q, \mathscr{P}_1)^2 + \Sigma(\mathscr{P}_3) + \Sigma(\mathscr{P}_4) \leqslant a^2 + b^2 + 2[2(\tfrac{a}{2})^2 + 2(\tfrac{b}{2})^2]$$
$$= 2a^2 + 2b^2.$$

Case 3: Two of the n_j's are 1's, and the other n_j's are $\geqslant 2$. Here wlog either $n_1, n_3 \geqslant 2$ and $n_2 = n_4 = 1$ (the "adjacent" subcase) or $n_1, n_4 \geqslant 2$ and $n_2 = n_3 = 1$ (the "non-adjacent" subcase). In the "adjacent" subcase, let Q be the only point in $\mathscr{P}_2$, and let T be the only point in $\mathscr{P}_4$. Then, in view of Lemma 1 and by induction,

$$\Sigma(\mathscr{P}) \leqslant \left(\Sigma(\mathscr{P}_1) + d(Q, \mathscr{P}_1)^2\right) + \left(\Sigma(\mathscr{P}_3) + d(T, \mathscr{P}_3)^2\right) \leqslant (a^2 + b^2) + (a^2 + b^2)$$
$$= 2a^2 + 2b^2.$$

The "non-adjacent" subcase is considered quite similarly, by interchanging R_3 and R_4.

Case 4: Three of the n_j's are 1's, and the other n_j is $\geqslant 2$. Here wlog $n_1 \geqslant 2$ and $n_2 = n_3 = n_4 = 1$. Let $Q_2 = (p,q), Q_3 = (u,v), Q_4 = (x,y)$ be the unique points in $\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4$, respectively, so that $p, x \in [\frac{a}{2}, a]$, $v, y \in [\frac{b}{2}, b]$, $u \in [0, \frac{a}{2}]$, $q \in [0, \frac{b}{2}]$.

Let s and t be, respectively, the largest abscissa and the largest ordinate of all the n_1 points in $\mathscr{P}_1$. Then

$$\Sigma(\mathscr{P}) \leqslant B_4 := \Sigma(\mathscr{P}_1) + Q_2 S^2 + Q_3 T^2 + \frac{Q_4 Q_2^2 + Q_4 Q_3^2}{2}, \tag{7}$$

where S is a point (among the n_1 points in $\mathscr{P}_1$) with the largest abscissa s, and T is a point (among the n_1 points in $\mathscr{P}_1$) with the largest ordinate t, so that $S = (s, \eta)$

for some $\eta \in [0,t]$, and $T = (\xi,t)$ for some $\xi \in [0,s]$. Of course, $Q_2 S^2$ is a convex function of $\eta \in [0,t]$, and so, for $S_0 := (s,0)$ and $S_t := (s,t)$ we have

$$Q_2 S^2 \leqslant \max(Q_2 S_0^2, Q_2 S_t^2) = (p-s)^2 + \max(q^2,(q-t)^2).$$

Similarly,

$$Q_3 T^2 \leqslant \max(u^2,(u-s)^2) + (v-t)^2.$$

Next,

$$Q_4 Q_2^2 + Q_4 Q_3^2 = (x-p)^2 + (y-q)^2 + (x-u)^2 + (y-v)^2,$$

which is convex in (x,y), with the maximum in $(x,y) \in [\frac{a}{2},a] \times [\frac{b}{2},b]$ attained at $(x,y) = (a,b)$, for any given $p \in [\frac{a}{2},a]$, $v \in [\frac{b}{2},b]$, $u \in [0,\frac{a}{2}]$, $q \in [0,\frac{b}{2}]$. Further, by induction, $\Sigma(\mathscr{P}_1) \leqslant 2s^2 + 2t^2$. Thus, by (7),

$$\begin{aligned}
\Sigma(\mathscr{P}) \leqslant B_4 \leqslant \tilde{B}_4 := {} & 2s^2 + 2t^2 + \max\left(q^2,(q-t)^2\right) + \max\left((u-s)^2,u^2\right) \\
& + (p-s)^2 + (v-t)^2 + \tfrac{1}{2}\left((a-p)^2 + (b-q)^2\right) \\
& + \tfrac{1}{2}\left((a-u)^2 + (b-v)^2\right).
\end{aligned}$$

Clearly, $\tilde{B}_4$ is convex in (p,v), and one can see that the maximum of $\tilde{B}_4$ in $(p,v) \in [\frac{a}{2},a] \times [\frac{b}{2},b]$ is attained at $(p,v) = (a,b)$, for any given $s \in [0,\frac{a}{2}]$, $t \in [0,\frac{b}{2}]$, $u \in [0,\frac{a}{2}]$, $q \in [0,\frac{b}{2}]$. So,

$$\tilde{B}_4 \leqslant B_{41} + B_{42} + B_{43},$$

where

$$\begin{aligned}
B_{41} &:= \max\left(q^2,(q-t)^2\right) - bq + q^2/2, \\
B_{42} &:= \max\left(u^2,(u-s)^2\right) - au + u^2/2, \\
B_{43} &:= \tfrac{3}{2}(a^2+b^2) - 2as - 2bt + 3(s^2+t^2).
\end{aligned}$$

Since B_{41} is convex in q and B_{42} is convex in u, one can see that

$$B_{41} \leqslant t^2 \quad \text{and} \quad B_{42} \leqslant s^2$$

for $q,t \in [0,\frac{b}{2}]$ and $u,s \in [0,\frac{a}{2}]$. So,

$$\begin{aligned}
\Sigma(\mathscr{P}) \leqslant \tilde{B}_4 \leqslant B_{41} + B_{42} + B_{43} &\leqslant \tfrac{3}{2}(a^2+b^2) + 4(s-\tfrac{a}{2})s + 4(t-\tfrac{b}{2})t \leqslant \tfrac{3}{2}(a^2+b^2) \\
&\leqslant 2a^2 + 2b^2.
\end{aligned}$$

Case 5: This case obtains from Cases 2, 3, 4 by replacing there some of the conditions of the form $n_j = 1$ by $n_j = 0$. This case immediately follows from Cases 2, 3, 4, because now the nonnegative contributions of the singleton sets $\mathscr{P}_j$ corresponding to $n_j = 1$ will be replaced by 0 — with the only exception occurring when three of the n_j's are 0 and hence the remaining one of the n_j's (say n_1) equals n. Indeed, in the latter exceptional subcase, we cannot use the induction, since $n_1 = n \not< n$. The remedy in this case is to continue the "partitioning" of the smaller rectangles containing

all the n points into yet smaller congruent rectangles until we no longer have such an exceptional situation. This process will stop. Indeed, if it never stopped, then all the n distinct points (with $n \geqslant 3$) would be eventually contained in a singleton set, which is a contradiction.

So far, we have considered all the cases when at least one of the n_j's is $\geqslant 2$. Otherwise, we have $n_j \leqslant 1$ for all $j \in \{1, \ldots, 4\}$ and hence $n \leqslant n_1 + \cdots + n_4 \leqslant 4$. Since $n \geqslant 3$, it remains to consider the following two cases.

Case 6: $n = 3$. By shrinking the rectangle R horizontally and vertically, we can obtain a possibly smaller rectangle $\tilde{R} \subseteq R$, with side lengths $\tilde{a} \leqslant a$ and $\tilde{b} \leqslant b$, such that $\tilde{R}$ still contains all the points P_1, P_2, P_3 and each side of $\tilde{R}$ contains at least one of the three points P_1, P_2, P_3. If we can then show that $\Sigma(\{P_1, P_2, P_3\}) \leqslant 2\tilde{a}^2 + 2\tilde{b}^2$, then the desired inequality $\Sigma(\{P_1, P_2, P_3\}) \leqslant 2a^2 + 2b^2$ will obviously follow. So, wlog we may assume that each side of R contains at least one of the three points P_1, P_2, P_3. Then, by the pigeonhole principle, at least two sides of the rectangle R must share one of the three points. Also, wlog none of the points P_1, P_2, P_3 is in the interior of R. Indeed, if e.g. P_3 is in the interior of R, then we can move P_3 away from the line $\ell(P_1, P_2)$ (through P_1, P_2) in the direction perpendicular to $\ell(P_1, P_2)$ (till we hit the boundary of R), so that all the pairwise distances between the points P_1, P_2, P_3 may only increase, and then $\Sigma(\mathscr{P})$ may only increase.

Hence, wlog $P_1 = (0,0), P_2 = (u,b), P_3 = (a,v)$ for some $u \in [0,a]$ and $v \in [0,b]$. So,

$$\Sigma(\mathscr{P}) \leqslant B_6 := \frac{P_1P_2^2 + P_1P_3^2}{2} + \frac{P_2P_1^2 + P_2P_3^2}{2} + \frac{P_3P_2^2 + P_3P_1^2}{2} = P_1P_2^2 + P_1P_3^2 + P_2P_3^2.$$

Since B_6 is convex in (u,v), its maximum in $(u,v) \in [0,a] \times [0,b]$ is attained when $(u,v) \in \{0,a\} \times \{0,b\}$, and this maximum is $2a^2 + 2b^2$. More explicitly, one may also note that

$$\tfrac{1}{2} B_6 - (a^2 + b^2) = -(a-u)u - (b-v)v \leqslant 0.$$

This proves Case 6.

Case 7: $n = 4$. Again, by shrinking the rectangle R, wlog we may assume that each side of R contains at least one of the four points P_1, P_2, P_3, P_4. Also, wlog each of the four points P_1, P_2, P_3, P_4 is either (i) in the convex hull of the other three points or (ii) on the boundary of R. Indeed, otherwise wlog P_4 is in the interior of R, but not in the convex hull of P_1, P_2, P_3. Then there is a closed half-plane, say H, containing the points P_1, P_2, P_3 but not containing the point P_4. So, then we can move P_4 away from the half-plane H in the direction perpendicular to the boundary line of H (till we hit the boundary of R), so that all the pairwise distances between the points P_1, P_2, P_3, P_4 may only increase, and then $\Sigma(\mathscr{P})$ may only increase. Therefore, wlog we have one of the following two subcases.

Subcase 7.1: P_4 is in the convex hull of P_1, P_2, P_3. Then $P_4 = (1 - s - t)P_1 + sP_2 + tP_3$ for some s,t such that $s,t \geqslant 0$ and $s + t \leqslant 1$. Also, here, as in Case 6, wlog $P_1 = (0,0), P_2 = (u,b), P_3 = (a,v)$ for some $u \in [0,a]$ and $v \in [0,b]$. Hence

$$\Sigma(\mathscr{P}) \leqslant B_{71} := P_1P_4^2 + P_2P_4^2 + P_3P_4^2 + [(1 - s - t)P_4P_1^2 + sP_4P_2^2 + tP_4P_3^2].$$

Clearly, B_{71} is convex in (u,v) and hence attains its maximum in $(u,v) \in [0,a] \times [0,b]$ at one of the four vertices of the rectangle $[0,a] \times [0,b]$. To complete the consideration of Subcase 7.1, it remains to note that, given the above conditions on s and t,

$$B_{71}|_{u=0,\,v=0} - (2a^2 + 2b^2) = a^2(t-1)(2t+1) + b^2(s-1)(2s+1) \leqslant 0,$$
$$B_{71}|_{u=a,\,v=0} - (2a^2 + 2b^2) = a^2(s+t)(2(s+t)-3) + b^2(s-1)(2s+1) \leqslant 0,$$
$$B_{71}|_{u=0,\,v=b} - (2a^2 + 2b^2) = a^2(t-1)(2t+1) + b^2(s+t)(2(s+t)-3) \leqslant 0,$$
$$B_{71}|_{u=a,\,v=b} - (2a^2 + 2b^2) = (a^2+b^2)(s+t)(2(s+t)-3) \leqslant 0.$$

Alternatively, one may note that the Hessian matrix of B_{71} with respect to s and t equals $4G$, where G is the Gram matrix of the vectors $P_2[= \overrightarrow{P_1 P_2} = (u,b)]$ and $P_3[= \overrightarrow{P_1 P_3} = (a,v)]$. So, B_{71} is convex in (s,t), and hence attains its maximum in (s,t) at one of the points $(0,0),(0,1),(1,0)$. To complete the consideration of Subcase 7.1 this other way, it remains to note that

$$B_{71}|_{s=0,t=0} - (2a^2 + 2b^2) = (u^2 - a^2) + (v^2 - b^2) \leqslant 0,$$
$$B_{71}|_{s=0,t=1} - (2a^2 + 2b^2) = -(2a-u)u - 2(b-v)v - b^2 \leqslant 0,$$
$$B_{71}|_{s=1,t=0} - (2a^2 + 2b^2) = -(2b-v)v - 2(a-u)u - a^2 \leqslant 0.$$

Subcase 7.2: All the four points P_1,P_2,P_3,P_4 are on the boundary of R.

Subsubcase 7.2.1: None of the points P_1,P_2,P_3,P_4 is shared by any two sides of the rectangle R. So, wlog $P_1 = (s,0), P_2 = (0,t), P_3 = (u,b), P_4 = (a,v)$ for some $s,u \in [0,a]$ and $t,v \in [0,b]$. Then, similarly to the case of $n=3$ (that is, Case 6), here

$$\Sigma(\mathscr{P}) \leqslant B_{721} := \frac{P_1 P_2^2 + P_1 P_4^2}{2} + \frac{P_2 P_1^2 + P_2 P_3^2}{2} + \frac{P_3 P_2^2 + P_3 P_4^2}{2} + \frac{P_4 P_1^2 + P_4 P_3^2}{2}$$
$$= P_1 P_2^2 + P_2 P_3^2 + P_3 P_4^2 + P_4 P_1^2 \leqslant 2a^2 + 2b^2,$$

by convexity. More explicitly, one may also note that

$$\tfrac{1}{2} B_{721} - (a^2 + b^2) = -(a-s)s - (b-t)t - (a-u)u - (b-v)v \leqslant 0.$$

Subsubcase 7.2.1 is done.

Subsubcase 7.2.2: One of the points P_1,P_2,P_3,P_4 (say P_1) is shared by two sides of the rectangle R. So, wlog $P_1 = (0,0)$. Suppose that one of the two sides of R (say S_1 and S_2) sharing the point P_1 contains one of the points P_2,P_3,P_4; let us say this side is S_1. Then we can move P_1 slightly along the side S_2 out of its position at $(0,0)$. In view of the continuity of $\Sigma(\mathscr{P})$ in P_1, we can thereby get rid of the sharing and thus reduce Subsubcase 7.2.2 to Subsubcase 7.2.1 – provided that at least one of the two sides of R sharing the point P_1 contains one of the points P_2,P_3,P_4. So, wlog $P_1 = (0,0)$ and none of the two sides of R sharing the point P_1 contains any of the points P_2,P_3,P_4. So, one of the sides of R not sharing the point P_1 contains two of the points P_2,P_3,P_4. Therefore and by the interchangeability of the horizontal and vertical directions, wlog

$P_1 = (0,0), P_2 = (u,b), P_3 = (a,v), P_4 = (a,w)$ for some $u \in [0,a]$ and $v, w \in [0,b]$ such that $v > w$. So,

$$\Sigma(\mathscr{P}) \leqslant B_{722} := P_1 P_2^2 + P_2 P_3^2 + P_3 P_4^2 + P_4 P_1^2 \leqslant 2a^2 + 2b^2,$$

again by convexity. More explicitly, one may also note that

$$\tfrac{1}{2} B_{722} - (a^2 + b^2) = -(a-u)u - (b-v)v - (v-w)w \leqslant 0.$$

Subsubcase 7.2.2 is done, as well as the entire proof of Theorem 1.

The special case of Theorem 1 with $a = b = 1$ was conjectured by T. Amdeberhan [8].

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