WEIGHTED INEQUALITIES OF FEFFERMAN–STEIN
TYPE FOR RIESZ–SCHRÖDINGER TRANSFORMS

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Abstract. In this work we are concerned with Fefferman-Stein type inequalities. More precisely, given an operator T and some p, $1 < p < \infty$, we look for operators $\mathcal{M}$ such that the inequality

$$\int |Tf|^p w \leqslant C \int |f|^p \mathcal{M} w,$$

holds true for any weight w. Specifically, we are interested in the case of T being any first or second order Riesz transform associated to the Schrödinger operator $L = -\Delta + V$, with V a non-negative function satisfying an appropriate reverse-Hölder condition. For the Riesz-Schrödinger transforms $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$ we make use of a result due to C. Pérez where this problem is solved for classical Calderón-Zygmund operators.

1. Introduction

In the theory of weighted L^p-inequalities a relevant question is the following: given an operator T and $1 < p < \infty$, to find a positive operator $\mathcal{M}$ such that inequalities of the form

$$\int |Tf|^p w \leqslant \int |f|^p \mathcal{M} w, \qquad (1)$$

hold for some reasonable set of functions f defined on $\mathbb{R}^d$, $d \geqslant 1$, and a general weight w, i.e. $w \in L^1_{\text{loc}}(\mathbb{R}^d)$, $w \geqslant 0$. However, the above inequality becomes more interesting when $\mathcal{M} w$ is finite a.e. and to that end it is desirable to get the operator $\mathcal{M}$ as small as possible.

The first appearance of such inequality goes back to the classical result of Fefferman-Stein ([7]) for $T = \mathcal{M} = M$, the Hardy-Littlewood maximal operator, namely

$$\int_{\mathbb{R}^d} |Mf|^p w \leqslant \int_{\mathbb{R}^d} |f|^p Mw,$$

for $1 < p < \infty$.

When T is a singular integral operator, Córdoba and Fefferman showed in [4] that inequality (1) holds taking $\mathcal{M} = M_r = (M(w^r))^{1/r}$, for any $1 < r < \infty$. However, it is known that for the Hilbert transform that inequality fails for $r = 1$.

Later, Wilson in [11] obtained inequalities for $1 < p < 2$ and $\mathcal{M} = M \circ M$ improving the result in [4] since $M \circ M(w) \leqslant (M(w^r))^{1/r}$, for all $r > 1$.

In 1995, C. Pérez provided a full answer to this question with different techniques including weak type inequalities for $p = 1$. He deals with maximal operators associated to averages with respect to a Young function which can be smaller than M_r.

Below, we state the precise statements since they are essential to our work.

By a Young function A we mean $A : [0, \infty) \to [0, \infty)$ continuous, convex, increasing and such that $A(0) = 0$. To define a maximal operator associated to a Young function A we introduce the A-average of a function f over a ball B as

$$\|f\|_{A,B} = \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B A\left(\frac{|f(t)|}{\lambda}\right) dt \leqslant 1 \right\}.$$

Then, the maximal operator associated to a Young function A is

$$M_A f(x) = \sup_{B \ni x} \|f\|_{A,B}.$$

For $1 < p < \infty$, we define $\mathscr{D}_p$ as the class of Young functions such that

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty \tag{2}$$

for some $c > 0$.

The following theorem appears as Theorem 1.5 in [9]. There it is stated for singular integral operators. But according to the comment in Section 3 there, it also holds for Calderón-Zygmund operators as it is stated next.

THEOREM 1. *Let $1 < p < \infty$, and let T be a Calderón-Zygmund operator. Suppose that $A \in \mathscr{D}_p$. Then there exists a constant C such that for each weight w*

$$\int |Tf|^p w \leqslant C \int |f|^p M_A w.$$

The following theorem deals with the endpoint case $p = 1$ and it is also due to C. Pérez. Here we state a version that can be found in [5] as Theorem 9.31.

THEOREM 2. *Let T be a Calderón-Zygmund operator and let $A \in \bigcup_{p>1} \mathscr{D}_p$. Then there exists a constant C such that for each weight w and for all $\lambda > 0$ we have*

$$w(\{y \in \mathbb{R}^d : |Tf(y)| > \lambda\}) \leqslant \frac{C}{\lambda} \int |f(y)| M_A w(y) dy.$$

Some examples of functions on the class $\mathscr{D}_p$ are $A(t) = t \log^{p-1+\varepsilon}(1+t)$ or $A(t) = t \log^{p-1}(1+t) \log^{p-1+\varepsilon}(\log(1+t))$ for any $\varepsilon > 0$. As for the class $\bigcup_{p>1} \mathscr{D}_p$, we can take $A(t) = t \log^\varepsilon(1+t)$ for any $\varepsilon > 0$.

In this work we attempt to provide results of this type for the first and second order Riesz transforms associated to the Schrödinger differential operator $L = -\Delta + V$ on

$\mathbb{R}^d$, $d \geqslant 3$ and with V satisfying a reverse Hölder inequality of order q, $q > d/2$, that is, there exists C such that

$$\left(\frac{1}{|B|} \int_B V^q \right)^{1/q} \leqslant C \frac{1}{|B|} \int_B V, \tag{3}$$

holds for every ball B in $\mathbb{R}^d$. From now on, if a function V satisfies (3) above we will say that $V \in RH_q$.

The study of these operators under such assumptions on V, was started by Shen in [10], where he proves L^p boundedness for most of the operators we will be concerned with. As he observed, when $q > d$, the first order Riesz transforms $\nabla L^{-1/2}$ are standard Calderón-Zygmund operators. Otherwise, they are not necessarily bounded on L^p for all p, $1 < p < \infty$. The case of the second order Riesz transforms given by $\nabla^2 L^{-1}$ is even worse since one can assure boundedness only for $1 < p < q$. However, even in the case that they are Calderón-Zygmund operators, we may expect in inequality (1) a smaller operator $\mathcal{M}$ than those given by Pérez, since Schrödinger Riesz transforms have kernels with a better decay at infinity. Also, in this context, kernels may have no symmetry and hence we might obtain different results for T and its adjoint.

Essentially, we will consider two types of first and second order Riesz transforms: one involving only derivatives $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$, and the others involving the potential V, as $V^{1/2} L^{-1/2}$, $V L^{-1}$ and $V^{1/2} \nabla L^{-1}$. In the first case we will get our results by locally comparing with the classical Riesz transforms, allowing us to apply the results of C. Pérez. Let us point out that for $\nabla L^{-1/2}$ such comparison estimate appeared already in [10] but that is not the case for $\nabla^2 L^{-1}$, so it must be provided. We do that in Lemma 7 and we believe that it might be useful for other purposes. As for those operators involving V we shall require only estimates on the size of their kernels.

We would like to make a remark about the values of p for which inequalities like (1) will be obtained. In all instances the operator $\mathcal{M}$ on the right hand side satisfies $\mathcal{M}(1) \leqslant 1$ and therefore our results would imply boundedness on L^p, so the range of p should be limited as in the original work of Shen.

The paper is organized as follows. In the next section we state some general theorems in a somehow abstract framework but having in mind the Schrödinger Riesz transforms mentioned above, leaving all the proofs and technical lemmas to Section 3.

The results include strong type (p, p) inequalities like (1) as well as weak type $(1, 1)$ estimates for a suitable class of operators and their adjoints. Let us remark that inequalities for the adjoint operators are not obtained by duality. In fact, if we proceed in that way, we would not arrive to an inequality with an arbitrary weight on the left hand side as we wanted.

Section 4 is devoted to apply the general theorems of Section 2 to specific operators associated to Schrödinger semigroup: $\nabla L^{-1/2}$, $\nabla^2 L^{-1}$, $V^\alpha L^{-\alpha}$, $V^{\alpha-1/2} \nabla L^{-\alpha}$, with α in a range depending on the operator. In order to check that their kernels satisfy the required assumptions, sometimes we make use of known estimates but in other occasions we must prove them. In particular we prove a local comparison between the kernels of $\nabla^2 (-\Delta)^{-1}$ and $\nabla^2 L^{-1}$ stated in Lemma 7.

Finally in the last section we use the above results to get sufficient conditions on a

function f to ensure local L^p integrability of Tf, where T is any of the operators of Section 4. Consequently we obtain a large class of functions f such that Tf is finite a.e.. In fact, f is allowed to increase polynomially. When these results are applied to the Riesz-Schrödinger transforms they provide qualitative information about some solutions of differential equations involving L.

2. General results

In this section we will consider the space $\mathbb{R}^d$ equipped with a *critical radius function* $\rho : \mathbb{R}^d \to (0,\infty)$, that is, a function whose variation is controlled by the existence of C_0 and $N_0 \geqslant 1$ such that

$$C_0^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N_0} \leqslant \rho(y) \leqslant C_0 \rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{\frac{N_0}{N_0+1}}. \tag{4}$$

It is worth noting that if ρ is a critical radius function , then for any $\gamma > 0$ the function $\gamma\rho$ is also a critical radius function. Moreover, if $0 < \gamma \leqslant 1$ then $\gamma\rho$ satisfies (4) with the same constants as ρ.

Let us remark that in [10] a function ρ satisfying (4) was introduced related with the potential V. But, once such a function ρ is defined, there is not need of any further reference to V. So we choose to work in this frame in order to emphasize that fact.

Very often we will refer to critical balls, meaning balls of the type $B(x_0, \rho(x_0))$, and we shall call subcritical balls to those $B(x_0, r)$ with $r \leqslant \rho(x_0)$. Observe that from (4), $\rho(y) \simeq \rho(x_0)$ whenever $y \in B(x_0, \rho(x_0))$.

The next lemma is a useful consequence of (4).

LEMMA 1. (see [3], Corollary 1) *Let $y \in B(x_0, R)$. Then, there exists a constant $C > 0$ such that*

$$1 + \frac{R}{\rho(y)} \leqslant C \left(1 + \frac{R}{\rho(x_0)} \right)^{N_0+1}.$$

Proof. From (4) and the fact that $y \in B(x_0, R)$ we have

$$\frac{1}{\rho(y)} \leqslant \frac{C_0}{\rho(x_0)} \left(1 + \frac{R}{\rho(x_0)} \right)^{N_0}.$$

Multiplying by R and adding 1, we get

$$1 + \frac{R}{\rho(y)} \leqslant C_0 \left(\frac{R}{\rho(x_0)} \left(1 + \frac{R}{\rho(x_0)} \right)^{N_0} + 1 \right) \leqslant C \left(1 + \frac{R}{\rho(x_0)} \right)^{N_0+1},$$

where we used that $C_0 \geqslant 1$. $\quad\square$

Associated to a critical radius function ρ we can define the following maximal operators. First, let us denote $\mathscr{F}_\rho$ the set of all balls $B(x, r)$ such that $r \leqslant \rho(x)$. Then,

for f a locally integrable function, and A a Young function, we set

$$M_A^{\text{loc}} f(x) = \sup_{\substack{B \ni x \\ B \in \mathscr{F}_\rho}} \|f\|_{A,B},$$

and for $\theta \geqslant 0$,

$$M_A^\theta f(x) = \sup_{B(x_0,r_0) \ni x} \left(1 + \frac{r_0}{\rho(x_0)}\right)^{-\theta} \|f\|_{A,B}.$$

As usual, when $A(t) = t^r$ we use the notation M_r^{loc} and M_r^θ respectively.

Now, we are in position to state our main theorems.

THEOREM 3. *Let T be a linear operator with associated kernel K. Suppose that for some $s > 1$, K satisfies the following estimates.*

(a_s) *For each $N > 0$ there exists C_N such that*

$$\left(\int_{R < |x_0 - x| < 2R} |K(x,y)|^s dx\right)^{1/s} \leqslant C_N R^{-d/s'} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N},$$

whenever $|y - x_0| < R/2$.

(b_s) *There exists a Calderón-Zygmund operator T_0 with kernel K_0 such that, for some C and $\delta > 0$,*

$$\left(\int_{R < |x_0 - x| < 2R} |K(x,y) - K_0(x,y)|^s dx\right)^{1/s} \leqslant C R^{-d/s'} \left(\frac{R}{\rho(x_0)}\right)^\delta,$$

whenever $|y - x_0| < R/2$ with $R \leqslant \rho(x_0)$.

Then, for each $\theta \geqslant 0$, the operator T and its adjoint $T^\star$ satisfy the following inequalities for any weight w,

$$\int |Tf|^p w \leqslant C_\theta \int |f|^p M_r^\theta w, \tag{5}$$

for $1 < p < s$ and $r = (s/p)'$,

$$\int |T^\star f|^p w \leqslant C_\theta \int |f|^p (M_A^{\text{loc}} + M^\theta) w, \tag{6}$$

for $s' < p < \infty$ and any Young function $A \in \mathscr{D}_p$.

REMARK 1. Assumption (a_s) can be seen as a size condition with a kind of "decay at infinity", while condition (b_s) tells us that K has the same singularity as a Calderón-Zygmund kernel. Nevertheless, both conditions on K are not symmetric since integration is always made in the first variable. Consequently we do not get the same kind of estimates for T and $T^\star$.

If the kernel K satisfies point-wise estimates we obtain a sharper result for T, as a corollary of the previous theorem.

COROLLARY 1. *Let T be a linear operator with associated kernel K and T_0 be a Calderón-Zygmund operator with kernel K_0. Suppose that K and K_0 satisfy the following estimates.*

(a_∞) *For each $N > 0$ there exists C_N such that*

$$|K(x,y)| \leqslant \frac{C_N}{|x-y|^d} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}.$$

(b_∞) *There exist C and $\delta > 0$ such that*

$$|K(x,y) - K_0(x,y)| \leqslant \frac{C}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)}\right)^{\delta}.$$

Then, T and its adjoint $T^\star$ satisfy (6) for $1 < p < \infty$ and any Young function $A \in \mathscr{D}_p$.

Corollary 1 follows immediately from Theorem 3 since conditions (a_∞) and (b_∞) imply conditions (a_s) and (b_s) for all $1 < s < \infty$, and they can be made symmetric in x and y due to Lemma 1. For the limiting case $p = 1$ we obtain the following weak-type inequalities.

THEOREM 4. *Let T be a linear operator with associated kernel K and let T_0 be a Calderón Zygmund operator with kernel K_0. Suppose that for some $s > 1$, K satisfies conditions (a_s) and (b_s). Then, for $\theta \geqslant 0$ and $w \in L^1_{loc}$, $w \geqslant 0$, T satisfies*

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C_\theta}{\lambda} \int |f| M_{s'}^\theta(w), \text{ for } \lambda > 0. \tag{7}$$

Further, if T satisfies (a_∞) and (b_∞), then, for any Young function $A \in \bigcup_{p>1} \mathscr{D}_p$,

$$w(\{|Tf| > \lambda\}) \leqslant \frac{C_\theta}{\lambda} \int |f| \left(M_A^{\text{loc}} + M^\theta\right) w, \text{ for } \lambda > 0. \tag{8}$$

Moreover, inequality (8) also holds for $T^\star$.

The associated kernels of some operators related to L satisfy condition (b_s) without subtracting K_0 and hence condition (a_s) and (b_s) can be unified. For this type of operators we can get sharper inequalities stated in the following theorem.

THEOREM 5. *Let T be a linear operator with associated kernel K. Suppose that for some $s > 1$ and $\delta > 0$, K satisfies the following condition.*

(c_s) *For each $N > 0$, there exists C_N such that for any $x_0 \in \mathbb{R}^d$ and $R > 0$,*

$$\left(\int_{R<|x_0-x|<2R} |K(x,y)|^s dx\right)^{1/s} \leqslant C_N R^{-d/s'} \left(1 + \frac{\rho(x_0)}{R}\right)^{-\delta} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N},$$

whenever $|y - x_0| < R/2$.

Then, for any $\theta \geqslant 0$ and any weight w, there exists C_θ such that T satisfies (5) for $1 \leqslant p < s$ and

$$\int |T^\star f|^p w \leqslant C_\theta \int |f|^p M^\theta w, \tag{9}$$

for $s' < p < \infty$.

3. Proofs

Before giving the proofs of the theorems above we need to state some technical lemmas that will be useful in the sequel. The first one gives a nice covering of $\mathbb{R}^d$ with critical balls. It is a consequence of inequality (4) and can be found in [6].

In some proofs we will use the notation $\lesssim$ instead of $\leqslant$ to denote that the right hand side of the inequality is greater up to multiplicative constants that may depend on some parameters specified when necessary.

PROPOSITION 1. *There exists a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ such that the family of critical balls $Q_j = B(x_j, \rho(x_j))$ satisfies:*

i) $\displaystyle\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^d.$

ii) *There exist constants C and N_1 such that for any $\sigma \geqslant 1$, $\displaystyle\sum_{j\in\mathbb{N}} \chi_{\sigma Q_j} \leqslant C\sigma^{N_1}.$*

In general, maximal operators can not be controlled point-wisely by localized ones. Nevertheless, this is possible if we are considering functions supported on sub-critical balls and for points close enough to the support. In the next lemma we determine how much a critical ball must be contracted in order to have that kind of control. Such contraction of critical balls is needed to arrive to inequality (6) of Theorem 3.

LEMMA 2. *Let A be a Young function and B_0 any critical ball. There exists $\gamma_0 > 0$ such that if $0 < \gamma \leqslant \gamma_0$ then for any measurable function f,*

$$M_A(f\chi_{\gamma B_0})(x) \leqslant CM_A^{\mathrm{loc}}(f)(x),$$

for all $x \in 2\gamma B_0$. Here, the constant C only depends on the dimension d and the Young function A.

Proof. Assume $x \in 2\gamma B_0$ with γ to be determined later. It is enough to consider balls centered at x; in fact, it is not difficult to see that if M_A^c is the centered maximal function, then $M_A(f)(x) \leqslant CM_A^c(f)(x)$ for any function f with C that only depends on d and A. Let x_0 be the center of B_0 and suppose first that $r > 3\gamma\rho(x_0)$. Therefore $B(x,r) \supset B(x, 3\gamma\rho(x_0)) \supset \gamma B_0$ and thus, for any non-negative function g,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\gamma B_0} g \;\leqslant\; \frac{1}{|B(x,3\gamma\rho(x_0))|}\int_{\gamma B_0} g \leqslant \frac{1}{|B(x,3\gamma\rho(x_0))|}\int_{B(x,3\gamma\rho(x_0))} g.$$

Now, if $\lambda > 0$, applying the above inequality to $g = A(|f|/\lambda)$ we have, for $r \geqslant 3\gamma\rho(x_0)$,

$$\|f\chi_{\gamma B_0}\|_{A,B(x,r)} \leqslant \|f\|_{A,B(x,3\gamma B_0)}.$$

Therefore, if $x \in 2\gamma B_0$,

$$M_A^c(f\chi_{\gamma B_0})(x) \leqslant \sup_{r \leqslant 3\gamma\rho(x_0)} \|f\|_{A,B(x,r)}.$$

To complete the proof, it is enough to take γ such that $3\gamma\rho(x_0) \leqslant \rho(x)$ for all $x \in 2\gamma B_0$.

From inequality (4), we have $\rho(x_0) \leqslant \rho(x)C_0(1+2\gamma)^{N_0}$ and thus γ should be taken such that

$$3\gamma C_0(1+2\gamma)^{N_0} \leqslant 1. \tag{10}$$

Since the left hand side goes to 0 when γ goes to 0, there exists γ_0 such that for $0 < \gamma \leqslant \gamma_0$ the above inequality holds. $\square$

Conditions (a_s) and (b_s) are written in a suitable way to prove inequalities concerning T^*. To prove the inequalities for T it will be easier to use the following equivalent conditions.

LEMMA 3. *For any $s > 1$, conditions (a_s) and (b_s) are equivalent, respectively, to the following conditions.*

(a_s') *For each $N > 0$ there exists C_N such that*

$$\left(\int_{B(x_0,R/2)} |K(x,y)|^s dx\right)^{1/s} \leqslant C_N R^{-d/s'}\left(1 + \frac{R}{\rho(x_0)}\right)^{-N},$$

   whenever $R < |y - x_0| < 2R$.

(b_s') *There exist C and $\delta > 0$ such that*

$$\left(\int_{B(x_0,R/2)} |K(x,y) - K_0(x,y)|^s dx\right)^{1/s} \leqslant CR^{-d/s'}\left(\frac{R}{\rho(x_0)}\right)^{\delta}$$

   whenever $R < |y - x_0| < 2R$ and $R \leqslant \rho(x_0)$.

REMARK 2. Observe that (a_s) holds true replacing the ring, $R < |x - x_0| < 2R$ with $R < |x - x_0| < c_0 R$ for any constant $c_0 > 1$, with the constant C_N depending on c_0. Similarly in (a_s') the ring $R < |y - x_0| < 2R$ may be replaced by $R < |y - x_0| < c_0 R$. In fact, it is only a matter of applying (a_s) or (a_s') a finite number of times depending on c_0.

The same comment applies to (b_s) and (b_s').

Proof of Lemma 3. We will show first that (a_s) implies (a_s'). Let K be a kernel satisfying (a_s) for some $s > 1$, and let $x_0 \in \mathbb{R}^d$, $R > 0$ and y such that $R < |x_0 - y| <$

$2R$. It is easy to check that $B(x_0, R/2) \subset \{x : R/2 < |x - y| < 4R\}$. So, applying condition (a_s) and the previous remark we get that

$$\left(\int_{B(x_0, R/2)} |K(x,y)|^s dx\right)^{1/s} \leqslant \left(\int_{R/2 < |y-x| < 4R} |K(x,y)|^s dx\right)^{1/s}$$

$$\leqslant C_N R^{-d/s'} \left(1 + \frac{R}{\rho(y)}\right)^{-N} \leqslant C_N R^{-d/s'} \left(1 + \frac{R}{\rho(x_0)}\right)^{-\bar{N}},$$

where in the last inequality we used Lemma 1.

To see that (a'_s) implies (a_s) let $x_0 \in \mathbb{R}^d$, $R > 0$ and $y \in B(x_0, R/2)$. The ring $\{x : R < |x - x_0| < 2R\}$ can be covered by M balls (depending on d), of radius $R/4$ and centres x_i, with $R < |x_i - x_0| < 2R$, for $i = 1, \dots, M$. For each of these balls we can check that $R/2 < |x_i - y| < 5R/2$. Applying condition (a'_s) and Remark 2 on each ball,

$$\left(\int_{R < |y-x| < 2R} |K(x,y)|^s dx\right)^{1/s} \leqslant \sum_{i=1}^{M} \left(\int_{B(x_i, R/4)} |K(x,y)|^s dx\right)^{1/s}$$

$$\leqslant \sum_{i=1}^{M} C_N R^{-d/s'} \left(1 + \frac{R}{\rho(x_i)}\right)^{-N}$$

$$\leqslant C_N R^{-d/s'} \left(1 + \frac{R}{\rho(x_0)}\right)^{-\bar{N}},$$

where we used again Lemma 1 in the last inequality.

We can omit the proof of the equivalence of (b_s) and (b'_s) since it follows the same lines as above. $\square$

Proof of Theorem 3. Let T be a linear operator with kernel K satisfying (a_s) and (b_s), for some $s > 1$ and some Calderón-Zygmund operator T_0 with kernel K_0. Let $w \geqslant 0$, $w \in L^1_{loc}$, $\theta \geqslant 0$, $1 < p < s$ and let A be a Young function satisfying (2).

We will prove first inequality (5). Let γ_0 be as in Lemma 2. For some $\gamma \leqslant \gamma_0$, to be chosen later, let $\{Q_n\}$ be the decomposition of the space given in Proposition 1 for the critical radius function $\gamma\rho$. Then we write

$$\int |Tf|^p w \leqslant \sum_{n \in \mathbb{N}} \int_{Q_n} |Tf|^p w$$

$$= \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) + T(f\chi_{(2Q_n)^c}) \pm T_0(f\chi_{2Q_n})|^p w$$

$$\lesssim \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) - T_0(f\chi_{2Q_n})|^p w + \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|^p w$$

$$+ \sum_{n \in \mathbb{N}} \int_{Q_n} |T_0(f\chi_{2Q_n})|^p w = I + II + III.$$

(11)

For III, since T_0 is a Calderón-Zygmund operator, we apply Theorem 1 and Lemma 2 to get

$$III = \sum_{n \in \mathbb{N}} \int |T_0(f\chi_{2Q_n})|^p w \chi_{Q_n} \lesssim \sum_{n \in \mathbb{N}} \int |f\chi_{2Q_n}| M_A(w\chi_{Q_n}) \lesssim \sum_{n \in \mathbb{N}} \int_{2Q_n} |f|^p M_A^{\text{loc}} w$$

$$\lesssim \int |f|^p M_A^{\text{loc}} w,$$

for any Young function $A \in \mathcal{D}_p$.

For $k \in \mathbb{Z}$ we denote $Q_n^k = 2^k Q_n$. To estimate II we use Minkowski's and Hölder's inequalities to obtain

$$II = \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|^p w = \sum_{n \in \mathbb{N}} \int_{Q_n} \left[\int_{(2Q_n)^c} |K(x,y)| |f(y)| dy \right]^p w(x) dx$$

$$\leqslant \sum_{n \in \mathbb{N}} \left[\int_{(2Q_n)^c} |f(y)| \left(\int_{Q_n} |K(x,y)|^p w(x) dx \right)^{1/p} dy \right]^p$$

$$\leqslant \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{N}} \int_{Q_n^{k+1} \setminus Q_n^k} |f(y)| \left(\int_{Q_n} |K(x,y)|^s dx \right)^{1/s} \left(\int_{Q_n} w^r(x) dx \right)^{1/rp} dy \right]^p,$$

where $r = (s/p)'$.

Next we apply condition (a_s') for K since by Lemma 3 is equivalent to (a_s). Then for each N we have

$$II \lesssim \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{N}} |Q_n^k|^{-1/s'} 2^{-kN} \int_{Q_n^k} |f(y)| \left(\int_{Q_n} w^r \right)^{1/rp} dy \right]^p$$

$$\lesssim \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{N}} |Q_n^k|^{-1/s'+1/p'} 2^{-kN} \left(\int_{Q_n^k} |f(y)|^p \left(\int_{Q_n^k} w^r \right)^{1/r} dy \right)^{1/p} \right]^p$$

$$\lesssim \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{N}} 2^{-k(N-\theta/p)} \left(\int_{Q_n^k} |f(y)|^p 2^{-k\theta} \left(\frac{1}{|Q_n^k|} \int_{Q_n^k} w^r \right)^{1/r} dy \right)^{1/p} \right]^p$$

$$\lesssim \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{N}} 2^{-k(N-\theta/p)} \left(\int_{Q_n^k} |f(y)|^p M_r^\theta w(y) dy \right)^{1/p} \right]^p,$$

with constants that may depend on N.

Finally, using Hölder's inequality in the sum over k and choosing $N = N_1 + \theta/p + 1$, where N_1 is the constant appearing in Proposition 1, we arrive to

$$II \lesssim \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{N}} 2^{-k(N_1+1)} \int_{Q_n^k} |f|^p M_r^\theta w \right] \left[\sum_{k \in \mathbb{N}} 2^{-k(N_1+1)} \right]^{p/p'}$$

$$\lesssim \sum_{k \in \mathbb{N}} 2^{-k(N_1+1)} \int_{\mathbb{R}^d} \left(\sum_{n \in \mathbb{N}} \chi_{Q_n^k} \right) |f|^p M_r^\theta w \lesssim \int_{\mathbb{R}^d} |f|^p M_r^\theta w,$$

with constants depending on N_1 and θ and p.

It only remains to estimate I. Let $D(x,y) = K(x,y) - K_0(x,y)$. For $x \in Q_n$, we have $2Q_n \subset B(x, \rho(x))$ due to our choice of γ (see inequality (10)), therefore we may write

$$
\begin{aligned}
I &= \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) - T_0(f\chi_{2Q_n})|^p w \\
&\leqslant \sum_{n \in \mathbb{N}} \int_{Q_n} \left[\int_{2Q_n} |D(x,y)||f(y)| dy \right]^p w(x) dx \\
&\leqslant \sum_{n \in \mathbb{N}} \int_{Q_n} \left[\int_{B(x,\rho(x))} |D(x,y)||f(y)| dy \right]^p w(x) dx \\
&\leqslant \int_{\mathbb{R}^d} |h(x)|^p w(x) dx = \|h\|^p_{L^p(\mathbb{R}^d, w)},
\end{aligned}
\tag{12}
$$

where

$$
h(x) = \int_{B(x,\rho(x))} |D(x,y)||f(y)| dy.
$$

For a fixed k and for any n we can take 2^{dk} disjoint balls of the form $B^l_{n,k} = B(x^l_{n,k}, 2^{-k}\gamma\rho(x_n))$ such that for $\sigma > \sqrt{d}$,

$$
Q_n \subset \bigcup_{l=1}^{2^{dk}} \sigma B^l_{n,k} \subset 2\sigma Q_n.
$$

Moreover, there exists a constant depending only on σ and d such that,

$$
\sum_{l=1}^{2^{dk}} \chi_{\sigma B^l_{n,k}} \leqslant C_{d,\sigma} \chi_{2\sigma Q_n}.
$$

Therefore, from Proposition 1, the family of balls $\{\sigma B^l_{n,k}\}_{l,n}$ covers $\mathbb{R}^d$ and

$$
\sum_{l,n} \chi_{\sigma B^l_{n,k}} \leqslant C_{d,\sigma,\rho}.
$$

Let us fix $\sigma = 2\sqrt{d}$. It is possible to choose γ small enough such that if $x \in \sigma B^l_{n,k}$ and $2^{-k-1}\rho(x) \leqslant |y - x| \leqslant 2^{-k}\rho(x)$ then

$$
y \in E^l_{n,k} = \{y : 4\sqrt{d}\gamma 2^{-k}\rho(x_n) \leqslant |y - x^l_{n,k}| \leqslant \beta\gamma 2^{-k}\rho(x_n)\}.
$$

for some constant $\beta > 4\sqrt{d}$ depending only on ρ and d.[1]

Now, we write h in the following way

$$
h(x) = \sum_{k=0}^{\infty} h_k(x) = \sum_{k=0}^{\infty} \int_{B(x,2^{-k}\rho(x)) \setminus B(x,2^{-k-1}\rho(x))} |D(x,y)||f(y)| dy.
$$

---

[1] For example, it works taking $\gamma = \frac{1}{2C_0(5\sqrt{d})^{N_0+1}}$ and $\beta = 2C_0^2(5\sqrt{d})^{N_0+2}$

So, for the covering of the space described above, we may write

$$\|h_k\|_{L^p(w)}^p \leqslant \sum_{n,l} \int_{2\sqrt{d}B_{n,k}^l} \left[\int_{B(x,2^{-k}\rho(x))\setminus B(x,2^{-k-1}\rho(x))} |D(x,y)||f(y)|dy \right]^p w(x)dx$$

$$\leqslant \sum_{n,l} \int_{2\sqrt{d}B_{n,k}^l} \left[\int_{E_{n,k}^l} |D(x,y)||f(y)|dy \right]^p w(x)dx$$

$$\leqslant \sum_{n,l} \left[\int_{E_{n,k}^l} |f(y)| \left(\int_{2\sqrt{d}B_{n,k}^l} |D(x,y)|^p w(x)dx \right)^{1/p} dy \right]^p$$

$$\leqslant \sum_{n,l} \left[\int_{E_{n,k}^l} |f(y)| \left(\int_{2\sqrt{d}B_{n,k}^l} |D(x,y)|^s dx \right)^{1/s} \left(\int_{2\sqrt{d}B_{n,k}^l} w^r(x)dx \right)^{1/(rp)} dy \right]^p,$$

where we have used Minkowski's and Hölder's inequalities in the last two steps. Now, using condition (b_s') for $D(x,y)$ (see Remark 2), we arrive to

$$\|h_k\|_{L^p(w)}^p \lesssim \sum_{n,l} (2^{-k}\rho(x_n))^{-dp/s'} 2^{-k\delta p} \left[\int_{\beta B_{n,k}^l} |f(y)| \left(\int_{\beta B_{n,k}^l} w^r(x)dx \right)^{1/(rp)} dy \right]^p$$

$$\lesssim 2^{-k\delta p} \sum_{n,l} \int_{\beta B_{n,k}^l} |f(y)|^p \left(\frac{1}{|\beta B_{n,k}^l|} \int_{\beta B_{n,k}^l} w^r(x)dx \right)^{1/r} dy$$

$$\lesssim 2^{-k\delta p} \sum_{n,l} \int_{\beta B_{n,k}^l} |f(y)|^p M_r^\theta w(y)dy dy \lesssim 2^{-k\delta p} \|f\|_{L^p(M_r^\theta w)}^p.$$

Finally,

$$\|h\|_{L^p(w)} \leqslant \sum_{k\geqslant 0} \|h_k\|_{L^p(w)} \lesssim \sum_{k\geqslant 0} 2^{-k\delta} \|f\|_{L^p(M_r^\theta w)} \lesssim \|f\|_{L^p(M_r^\theta w)}. \tag{13}$$

Using the estimates obtained for I, II and III we arrive to inequality (5).

Now, let us prove inequality (6). Proceeding as in (11) we get

$$\int |T^\star f|^p w \lesssim \sum_{n\in\mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n}) - T_0^\star(f\chi_{2Q_n})|^p w + \sum_{n\in\mathbb{N}} \int_{Q_n} |T^\star(f\chi_{(2Q_n)^c})|^p w$$

$$+ \sum_{n\in\mathbb{N}} \int_{Q_n} |T_0^\star(f\chi_{2Q_n})|^p w = I^\star + II^\star + III^\star,$$

and we can estimate $III^\star$ in the same way as III, since $T_0^\star$ is also a Calderón-Zygmund operator.

For $II^\star$, we write

$$II^\star = \sum_{n\in\mathbb{N}} \int_{Q_n} |T^\star(f\chi_{(2Q_n)^c})|^p w = \sum_{n\in\mathbb{N}} \int_{Q_n} \left(\int_{(2Q_n)^c} |K(x,y)||f(x)|dx \right)^p w(y)dy.$$

If $y \in Q_n$ we may use Hölder inequality and condition (a_s) to obtain

$$\int_{(2Q_n)^c} |K(x,y)||f(x)|dx \leqslant \sum_{k \geqslant 1} \left(\int_{Q_n^{k+1} \setminus Q_n^k} |K(x,y)|^{p'} dx \right)^{1/p'} \left(\int_{Q_n^{k+1}} |f|^p \right)^{1/p}$$

$$\lesssim \sum_{k \geqslant 1} \left(\int_{Q_n^{k+1} \setminus Q_n^k} |K(x,y)|^s dx \right)^{1/s} \left(\int_{Q_n^{k+1}} |f|^p \right)^{1/p} |Q_n^k|^{1/s' - 1/p}$$

$$\lesssim \sum_{k \geqslant 1} 2^{-kN} \left(\frac{1}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \right)^{1/p}$$

$$\lesssim \left[\sum_{k \geqslant 1} \frac{2^{-kN}}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \right]^{1/p} \left[\sum_{k \geqslant 1} 2^{-kN} \right]^{1/p'}$$

$$\lesssim \left[\sum_{k \geqslant 1} \frac{2^{-kN}}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \right]^{1/p},$$

with constants independent of n.

Therefore,

$$II^\star \lesssim \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-kN} \frac{1}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \int_{Q_n^{k+1}} w(y) dy \lesssim \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-k(N-\theta)} \int_{Q_n^{k+1}} |f|^p M^\theta w$$

$$\lesssim \sum_{k \in \mathbb{N}} 2^{-k(N-\theta)} \int_{\mathbb{R}^d} \left(\sum_{n \in \mathbb{N}} \chi_{Q_n^{k+1}} \right) |f|^p M^\theta w \lesssim \int_{\mathbb{R}^d} |f|^p M^\theta w,$$

choosing $N = N_1 + \theta + 1$, where N_1 is the exponent given in Proposition 1.

It only remains to estimate $I^\star$. Proceeding as in (12), we have

$$I^\star = \sum_{n \in \mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n}) - T_0^\star(f\chi_{2Q_n})|^p w \leqslant \|h^\star\|_{L^p(w)}^p,$$

where

$$h^\star(y) = \int_{B(y,\rho(y))} |D(x,y)||f(x)|dx,$$

and write

$$h^\star(y) = \sum_{k=0}^\infty h_k^\star(y) = \sum_{k=0}^\infty \int_{B(y,2^{-k}\rho(y)) \setminus B(y,2^{-k-1}\rho(y))} |D(x,y)||f(x)|dx.$$

Now, for a fixed k, using Hölder's inequality and denoting $B(y,2^{-k}\rho(y)) = B_y^k$, we have

$$h_k^\star(y) \leqslant C \left(\int_{B_y^k \setminus B_y^{k-1}} |D(x,y)|^s dx \right)^{1/s} \left(\int_{B_y^k} |f|^p \right)^{1/p} (2^{-k}\rho(y))^{d/((s/p')'p')}.$$

Again for a fixed k, we consider the covering $\{B(x_{n,k}^l, 2\sqrt{d}\gamma 2^{-k}\rho(x_n))\}_{n,l}$. Using condition (b_s), we obtain

$$\|h_k^\star\|_{L^p(w)}^p \leqslant \sum_{n,l} \int_{B_{n,k}^l} |h_k^\star(y)|^p w(y) dy$$

$$\lesssim \sum_{n,l} \left(\int_{\beta B_{n,k}^l} |f|^p \right) (2^{-k} \rho(y_j))^{dp(1/s'-1/p)} \int_{B_{n,k}^l} \left(\int_{E_{n,k}^l} |Q(x,y)|^s dx \right)^{p/s} w(y) dy$$

$$\lesssim \sum_{n,l} \left(\int_{\beta B_{n,k}^l} |f|^p \right) \frac{2^{-kp\delta}}{(2^{-k} \rho(y_n))^d} \int_{\beta B_{n,k}^l} w$$

$$\lesssim \sum_{n,l} 2^{-kp\delta} \int_{\beta B_{n,k}^l} |f(x)|^p \left(\frac{1}{|\beta B_{n,k}^l|} \int_{\beta B_{n,k}^l} w \right) dx \lesssim 2^{-kp\delta} \sum_{n,l} \int_{\beta B_{n,k}^l} |f|^p M^\theta w$$

$$\lesssim 2^{-kp\delta} \|f\|_{L^p(M^\theta w)}^p.$$

So, as it was done in (13),

$$\|h^\star\|_{L^p(w)} \leqslant C_\theta \sum_k \|h_k^\star\|_{L^p(w)} \lesssim \|f\|_{L^p(M^\theta w)}.$$

Using the estimates obtained for $I^\star$, $II^\star$ and $III^\star$ we arrive to inequality (6).   □

REMARK 3. It is worth noting that the estimates obtained for I and II also hold for the case $p = 1$. Following the same ideas as above we arrive to

$$\sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|w \leqslant C_\theta \int_{\mathbb{R}^d} |f| M_{s'}^\theta(w),$$

$$\sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) - T_0(f\chi_{2Q_n})|w \leqslant C_\theta \int_{\mathbb{R}^d} |f| M_{s'}^\theta(w).$$

Now we prove the weak-type inequalities stated in Theorem 4.

Proof of Theorem 4. Let T be a linear operator with kernel K and $w \in L_{loc}^1$, $w \geqslant 0$. Suppose first that K satisfies conditions (a_s) and (b_s) for some $1 < s < \infty$ and $\delta > 0$. Consider again $\{Q_n\}_{n\in\mathbb{N}}$, the partition of the space associated to $\gamma\rho$, with γ chosen as in the proof of Theorem 3. For $\lambda > 0$, we may write

$$w(\{|Tf| > \lambda\}) \leqslant \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |Tf(x)| > \lambda\})$$

$$\leqslant \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T(f\chi_{2Q_n})(x) - T_0(f\chi_{2Q_n})(x)| > \lambda/3\})$$

$$+ \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T(f\chi_{(2Q_n)^c})(x)| > \lambda/3\}) \qquad (14)$$

$$+ \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T_0(f\chi_{2Q_n})(x)| > \lambda/3\})$$

$$= I + II + III.$$

To estimate *III* we can use this time Theorem 2 together with Lemma 2 to get

$$III = \sum_{n \in \mathbb{N}} w(\{x \in Q_n : |T_0(f\chi_{2Q_n})(x)| > \lambda/3\})$$

$$\leqslant \sum_{n \in \mathbb{N}} w\chi_{Q_n}(\{x : |T_0(f\chi_{2Q_n})(x)| > \lambda/3\}) \lesssim \frac{1}{\lambda} \sum_{n \in \mathbb{N}} \int_{2Q_n} |f| M_A(w\chi_{Q_n})$$

$$\lesssim \frac{1}{\lambda} \sum_{n \in \mathbb{N}} \int_{2Q_n} |f| M_A^{loc}(w) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| M_A^{loc}(w),$$

for any Young function $A \in \bigcup_{p>1} \mathscr{D}_p$. In particular we can take $A(t) = t^{s'}$ since we will not get anything better for the other terms.

As for *I* and *II* we use the strong type inequalities for $p = 1$ stated on Remark 3. In this way we obtain (7).

Now, suppose that the kernels K and K_0 satisfy conditions (a_∞) and (b_∞). Let $\lambda > 0$, we use the same decomposition as in (14) to get

$$w(\{|Tf > \lambda\}) \leqslant I + II + III.$$

We deal with *III* in the same way, obtaining

$$III \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| M_A^{loc}(w),$$

for any $A \in \bigcup_{p>1} \mathscr{D}_p$.

For $k \in \mathbb{Z}$ we set $Q_n^k = B(x_j, \gamma 2^k \rho(x_j))$. To estimate the term *II* by the Tchebysheff's inequality we may write

$$II = \sum_{n \in \mathbb{N}} w(\{x \in Q_n : |T(f\chi_{(2Q_n)^c})(x)| > \lambda/3\}) \leqslant \sum_{n \in \mathbb{N}} \frac{3}{\lambda} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|(x)w(x)dx$$

$$\leqslant \sum_{n \in \mathbb{N}} \frac{3}{\lambda} \int_{Q_n} \left(\sum_{k \in \mathbb{N}} \int_{Q_n^{k+1} \setminus Q_n^k} |K(x,y)||f(y)|dy \right) w(x)dx.$$

Now, using condition (a_∞),

$$II \lesssim \frac{1}{\lambda} \sum_{n \in \mathbb{N}} \int_{Q_n} \sum_{k \in \mathbb{N}} \frac{2^{-kN}}{(2^k \rho(x_n))^d} \left(\int_{Q_n^{k+1}} |f(y)|dy \right) w(x)dx$$

$$\lesssim \frac{1}{\lambda} \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-kN} \int_{Q_n^{k+1}} |f(y)| \left(\frac{1}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} w(x)dx \right) dy$$

$$\lesssim \frac{1}{\lambda} \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} 2^{-k(N-\theta)} \int_{Q_n^{k+1}} |f(y)| M^\theta(w)dy$$

$$\lesssim \sum_{k \in \mathbb{N}} 2^{-k(N-\theta)} \int_{\mathbb{R}^d} \left(\sum_{n \in \mathbb{N}} \chi_{Q_n^{k+1}} \right) |f(y)| M^\theta(w)dy \lesssim \int_{\mathbb{R}^d} |f(y)| M^\theta(w)dy,$$

choosing $N = N_1 + \theta + 1$.

Next, to estimate I we use the Tchebysheff's inequality and condition (b_∞).

$$I = \sum_{j \in \mathbb{N}} w(\{x \in Q_n : |(T - T_0)(f\chi_{2Q_n})(x)| > \lambda/3\})$$

$$\leqslant \sum_{n \in \mathbb{N}} \frac{3}{\lambda} \int_{Q_n} \left(\int_{2Q_n} |K(x,y) - K_0(x,y)||f(y)|dy \right) w(x)dx$$

$$\lesssim \frac{1}{\lambda} \sum_{j \in \mathbb{N}} \int_{Q_n} \left(\int_{2Q_n} \frac{|f(y)|}{|x-y|^d} \left(\frac{|x-y|}{\rho(x)} \right)^\delta dy \right) w(x)dx$$

$$\lesssim \frac{1}{\lambda} \sum_{n \in \mathbb{N}} \rho(x_n)^{-\delta} \int_{2Q_n} |f(y)| \int_{Q_n} |x-y|^{\delta-d} w(x)dxdy.$$

Now, if $y \in 2Q_n$, and calling $B_n^y = B(y, 3\gamma\rho(x_n))$, then $Q_n \subset B_n^y$ and hence

$$\int_{Q_n} |x-y|^{\delta-d} w(x)dx \leqslant \sum_{k \in \mathbb{N}} \int_{2^{-k+1}B_n^y \setminus 2^{-k}B_n^y} |x-y|^{\delta-d} w(x)dx$$

$$\leqslant \rho(x_n)^\delta \sum_{k \in \mathbb{N}} \frac{2^{-k\delta}}{(2^{-k}\rho(x_n))^d} \int_{2^{-k+1}B_n^y} w \leqslant \rho(x_n)^\delta M^{\mathrm{loc}} w(y),$$

since $\rho(y) \simeq \rho(x_n)$. Therefore, we obtain

$$I \lesssim \frac{1}{\lambda} \sum_n \int_{2Q_n} |f| M^{\mathrm{loc}} w \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f(y)| M^{\mathrm{loc}} w.$$

Altogether we obtain inequality (8). The same estimate is obtained for $T^\star$ since conditions (a_∞) and (b_∞) are symmetric on x and y. □

Finally, we end this section with the proof of Theorem 5.

Proof of Theorem 5. Let T be a linear operator with associated kernel K satisfying (c_s). First, observe that condition (c_s) implies both conditions (a_s) and (b_s) with $K_0 = 0$. Then, proceeding as in equation (11) we can write

$$\int |Tf|^p w = \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) + T(f\chi_{2Q_n^c})|^p w$$

$$\lesssim \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n})|^p w + \sum_{n \in \mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n^c})|^p w = I + II.$$

Then, inequality (5) holds for $1 \leqslant p < s$ following the same lines as in the proof of Theorem 3 and taking into account Remark 3 for $p = 1$.

To obtain estimate (9) we proceed as above to get

$$\int |T^\star f|^p w \leqslant \sum_{n \in \mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n})|^p w + \sum_{n \in \mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n^c})|^p w = I^\star + II^\star,$$

and we deal with $I^\star$ and $II^\star$ as in the proof of Theorem 3. □

4. Application to Schrödinger operators

In this section we apply our general results to operators associated to the semigroup generated by the Schrödinger differential operator $L = -\Delta + V$ on $\mathbb{R}^d$ with $d \geqslant 3$. We will always suppose that the potential V is a non-negative function, non-identically zero, satisfying a reverse Hölder condition of order $q > d/2$. Under these assumptions the function ρ defined by

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V \leqslant 1\right\}, \; x \in \mathbb{R}^d,$$

is a critical radius function, that is, property (4) is satisfied for some constants C_0 and N_0.

It is known that $V \in RH_q$, $q > 1$ implies that V is a doubling measure, i.e. there exists C_1 such that

$$\int_{B(x,2r)} V \leqslant C_1 \int_{B(x,r)} V.$$

In fact, if $V \in RH_q$, $q > 1$, then V belongs to the A_∞ class of Muckenhoupt.

The following is an useful inequality for $V \in RH_q$ with $q > d/2$ that follows easily from Lemma 1.2 and Lemma 1.8 in [10].

LEMMA 4. *Let* $V \in RH_q$ *for some* $q > d/2$. *Let* $N_2 = \log_2 C_1 + 2 - d$, *where* C_1 *is the doubling constant of* V. *Then, for any* $x_0 \in \mathbb{R}^d$, $R > 0$,

$$\frac{1}{R^{d-2}} \int_{B(x_0,R)} V(y)dy \leqslant C\left(1 + \frac{R}{\rho(x_0)}\right)^{N_2}\left(1 + \frac{\rho(x_0)}{R}\right)^{d/q-2}.$$

REMARK 4. Observe that when $R < \rho(x_0)$ the leading term is the second factor while the latter is bounded by a constant when $R \geqslant \rho(x_0)$.

For the fundamental solution of L, the following estimate was shown in [10].

LEMMA 5. *Let* $V \in RH_q$, *with* $q > d/2$ *and* Γ *the fundamental solution of the operator* L *in* $\mathbb{R}^d$. *Then for each* $N > 0$, *there exists a constant* C_N *such that*

$$|\Gamma(x,y)| \leqslant C_N \frac{1}{|x-y|^{d-2}}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}.$$

4.1. Riesz-Schrödinger transforms

We consider the operators $\mathscr{R}_1 = \nabla L^{-1/2}$ and $\mathscr{R}_2 = \nabla^2 L^{-1}$, the Riesz-Schrödinger transforms of order 1 and 2 respectively. Let K_1 and K_2 be their associated kernels.

The size condition (a_s) can be found in [1], for both K_1 and K_2. In fact, (a_s) holds for K_1 with $s = p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{d}$, in the case $d/2 < q < d$ (see page 28 in [1]), and $s = \infty$ for $q \geqslant d$ (see inequality (6.5) in [10]). Regarding K_2, it satisfies (a_s) for $s = q$ (see Proposition 8 in [1]).

To prove that these kernels satisfy also condition (b_s) we will compare them with the classical Riesz transforms $R_1 = \nabla(-\Delta)^{-1/2}$ and $R_2 = \nabla^2(-\Delta)^{-1}$ and their associated kernels $K_{0,1}$ and $K_{0,2}$.

Before proving condition (b_s) we present the following lemmas that provide us point-wise estimates for the difference between the kernels associated to the Riesz-Schrödinger transforms and the classical ones. For the Riesz-Schrödinger transform of order 1 such result was already obtained by Shen. On the other hand, the estimate corresponding to the second order operator is new and we believe is interesting in its own right.

LEMMA 6. *[See* [10], *inequality* (5.9)*] Let* $V \in RH_q$ *for* $d/2 < q < d$. *There exists* C *such that*

$$|K_1(x,y) - K_{0,1}(x,y)| \leqslant \frac{C}{|x-y|^{d-1}} \left(G(x,y) + \frac{1}{|x-y|} \left(\frac{|x-y|}{\rho(x)} \right)^{2-d/q} \right),$$

where

$$G(x,y) = \int_{B(x,|x-y|/4)} \frac{V(u)}{|u-x|^{d-1}} du. \tag{15}$$

LEMMA 7. *Let* x, $y_0 \in \mathbb{R}^d$ *and* $R > 0$ *such that* $R \leqslant |y-x_0| \leqslant \rho(x_0)$. *Let* $x \in B(x_0, R/8)$. *Then there exists a constant* C *such that*

$$|K_2(x,y) - K_{0,2}(x,y)| \leqslant C|R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)| + \frac{C}{R^d} \left(\frac{R}{\rho(x_0)} \right)^{\delta},$$

with $\delta = \min\{1, 2 - d/q\}$.

Proof. Let Γ and Γ_0 be the fundamental solution of L and $-\Delta$ respectively. As it was shown in [10], page 540,

$$\Gamma(x,y) - \Gamma_0(x,y) = -\int_{\mathbb{R}^d} \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi.$$

From this we get the following expression for the difference of the kernels.

$$K_2(x,y) - K_{0,2}(x,y) = \nabla_1^2 \Gamma(x,y) - \nabla_1^2 \Gamma_0(x,y) = -\nabla_1^2 \int_{\mathbb{R}^d} \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi.$$

Next, we define the domains $J_1 = B(x_0, R/4)$, $J_2 = B(y, R/4)$ and $J_3 = (J_1 \cup J_2)^c$. The term corresponding to the integral over J_1 is, upon a sign, the classical second order Riesz transform applied to function in L^q with compact support, that is

$$\left| \nabla_1^2 \int_{J_1} \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi \right| = |R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)|.$$

On J_2, since we are away from the singularity of Γ_0, we can use the size estimate for Γ given in Lemma 5, together with Hölder's inequality to obtain

$$\left| \int_{J_2} \nabla_1^2 \Gamma_0(x,\xi) V(\xi) \Gamma(y,\xi) d\xi \right| \leqslant \frac{C}{R^d} \int_{B(y,R/4)} \frac{V(\xi)}{|y-\xi|^{d-2}} d\xi$$

$$\leqslant \frac{C}{R^d} \left(\int_{B(y,R/4)} V^q(\xi) d\xi \right)^{\frac{1}{q}} \left(\int_{B(y,R/4)} \frac{d\xi}{|y-\xi|^{(d-2)q'}} \right)^{\frac{1}{q'}}.$$

For the first integral we can use the reverse Hölder condition for V together with Lemma 4, while on the second integral $q > d/2$ implies that $(d-2)q' < d$. Then

$$\left| \int_{J_2} \nabla_1^2 \Gamma_0(x,\xi) V(\xi) \Gamma(y,\xi) d\xi \right| \lesssim \frac{1}{R^d} \left(\frac{R}{\rho(x_0)} \right)^{2-d/q},$$

since $y \in B(x_0, \rho(x_0))$.

To estimate the integral on J_3 we divide $J_3 = J_{31} \cup J_{32}$, where $J_{31} = \{\xi \in \mathbb{R}^d : R/4 \leqslant |y-\xi| < 2R \wedge |x_0-\xi| \geqslant R/4\}$ and $J_{32} = \{\xi \in \mathbb{R}^d : |y-\xi| \geqslant 2R\}$. On J_{31} we are away from the singularities of both Γ y Γ_0, then

$$\left| \int_{J_{31}} \nabla_1^2 \Gamma_0(x,\xi) V(\xi) \Gamma(y,\xi) d\xi \right| \lesssim \int_{J_{31}} \frac{V(\xi)}{|x-\xi|^d |y-\xi|^{d-2}} d\xi \lesssim \frac{1}{R^{2d-2}} \int_{B(y,2R)} V(\xi) d\xi$$

$$\lesssim \frac{1}{R^d} \left(\frac{R}{\rho(x_0)} \right)^{2-d/q},$$

where in the last inequality we have used again Lemma 4.

Regarding J_{32} it is easy to check that $|x-\xi| \geqslant 3|y-\xi|/8$. Then, using Lemma 5 again,

$$\left| \int_{J_{32}} \nabla_1^2 \Gamma_0(x,\xi) V(\xi) \Gamma(y,\xi) d\xi \right| \leqslant C_N \int_{J_{32}} \frac{V(\xi)}{|x-\xi|^d |y-\xi|^{d-2}} \left(1 + \frac{|y-\xi|}{\rho(y)} \right)^{-N} d\xi$$

$$\leqslant C_N \int_{J_{32}} \frac{V(\xi)}{|y-\xi|^{2d-2}} \left(1 + \frac{|y-\xi|}{\rho(y)} \right)^{-N} d\xi.$$

$$(16)$$

Assume first that $2R < \rho(y)$. We split the integral in $J_{321} = \{\xi \in \mathbb{R}^d : 2R \leqslant |y-\xi| < \rho(y)\}$ and $J_{322} = \{\xi \in \mathbb{R}^d : |y-\xi| \geqslant \rho(y)\}$. For the integral on J_{321}, let $k_0 \in \mathbb{N}$ such that $2^{k_0-1}R \leqslant \rho(y) \leqslant 2^{k_0}R$. Then using again Lemma 4 and that $d > 2 - d/q$,

$$\int_{2R \leqslant |y-\xi| < \rho(y)} \frac{V(\xi)}{|y-\xi|^{2d-2}} d\xi \leqslant \sum_{k=2}^{k_0} \int_{2^{k-1}R \leqslant |y-\xi| < 2^k R} \frac{V(\xi)}{|y-\xi|^{2d-2}}$$

$$\lesssim \sum_{k=1}^{k_0} \frac{1}{(2^k R)^d} \frac{1}{(2^k R)^{d-2}} \int_{B(y,2^k R)} V(\xi) d\xi$$

$$\lesssim \frac{1}{R^d} \sum_{k=1}^{k_0} 2^{-kd} \left(\frac{2^k R}{\rho(y)} \right)^{2-d/q} \lesssim \frac{1}{R^d} \left(\frac{R}{\rho(x_0)} \right)^{2-d/q},$$

since $y \in B(x_0, \rho(x_0))$, and hence $\rho(y) \simeq \rho(x_0)$.

On J_{322}, let $\mu = \log_2 C_1$, where C_1 is the doubling constant of the potential V. Then we bound the right hand side of (16) by a constant (that may depend on N) times

$$\int_{|x-\xi| \geqslant \rho(y)} \frac{V(\xi)}{|y-\xi|^{2d-2}} \left(\frac{\rho(y)}{|y-\xi|} \right)^N d\xi \lesssim \sum_{k=1}^{\infty} \frac{1}{2^{kN}} \int_{2^{k-1}\rho(y)|y-\xi|<2^{k+}\rho(y)} \frac{V(\xi)}{|y-\xi|^{2d-2}}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{2^{k(2d-2+N)}\rho(y)^{2d-2}} \int_{B(y,2^k\rho(y))} V(\xi) d\xi$$

$$\lesssim \frac{1}{\rho(y)^d} \sum_{k=1}^{\infty} \frac{1}{2^{k(2d-2+N-\mu)}\rho(y)^{d-2}} \int_{B(y,\rho(y))} V(\xi) d\xi \lesssim \frac{1}{\rho(y)^d} \leqslant \frac{1}{R^d} \left(\frac{R}{\rho(x_0)} \right)^{2-d/q},$$

choosing N big enough and using that $\rho(y) \simeq \rho(x_0)$, $R < \rho(x_0)$ and $2 - d/q < d$.   □

For $\mathscr{R}_1$ different inequalities hold true depending on q. For $q > d$, Shen showed in [10] that $\mathscr{R}_1$ and $\mathscr{R}_1^\star$ are Calderón-Zygmund operators. Moreover, their associated kernels satisfy the stronger size condition (a_∞) (see inequality (6.5) there). Later on, condition (b_∞) was proved for the difference between K_1 and $K_{1,0}$ (see [2] Lemma 3).

Therefore, as an application of Theorem 3, Corollary 1 and Theorem 4 we obtain the following result.

THEOREM 6. *Let $V \in RH_q$ with $q > d/2$, $\theta \geqslant 0$ and p_0 such that $1/p_0 = (1/q - 1/d)^+$. Then for any weight w the following inequalities hold.*

$$\int |\mathscr{R}_1 f|^p w \leqslant C_\theta \int |f|^p M_r^\theta w, \tag{17}$$

when $d/2 < q < d$, for $1 < p < p_0$ and $r = (p_0/p)'$.

$$\int |\mathscr{R}_1^\star f|^p w \leqslant C_\theta \int |f|^p (M_A^{loc} w + M^\theta w), \tag{18}$$

when $q > d/2$, for $p_0' < p < \infty$, and any Young function $A \in \mathscr{D}_p$, and

$$w(\{|\mathscr{R}_1 f| > \lambda\}) \leqslant \frac{C_\theta}{\lambda} \int |f| M_{p_0'}^\theta w, \tag{19}$$

for any $q > d/2$.

Moreover, if $q > d$, we have

$$\int |\mathscr{R}_1 f|^p w \leqslant C_\theta \int |f|^p (M_A^{loc} w + M^\theta w), \tag{20}$$

for $1 < p < \infty$ and any Young function $A \in \mathscr{D}_p$. Also

$$w(\{|\mathscr{R}_1 f| > \lambda\}) \leqslant \frac{C_\theta}{\lambda} \int |f| M_A^\theta w, \tag{21}$$

and

$$w(\{|\mathscr{R}_1^\star f| > \lambda\}) \leqslant \frac{C_\theta}{\lambda} \int |f| M_A^\theta w, \tag{22}$$

for any Young function $A \in \bigcup_{p>1} \mathscr{D}_p$.

Proof. Let $V \in RH_q$ for $q > d/2$. Suppose first that $q < d$. As we said, condition (a_s) was proved in [1] for $s = p_0$ with $1/p_0 = (1/q - 1/d)$. Therefore it is enough to check (b'_{p_0}) which is equivalent to (b_{p_0}).

Let $x_0 \in \mathbb{R}^d$, $0 < R \leqslant \rho(x_0)$ and $R < |y - x_0| < 2R$. First, we make use of Lemma 6. Due to the boundedness of the classical fractional integral operator I_1 and the reverse Hölder property of V we get that, for G defined in (15),

$$
\begin{aligned}
\left(\int_{B(x_0, R/2)} \left(\frac{G(x,y)}{|x-y|^{d-1}} \right)^{p_0} dx \right)^{1/p_0} &\leqslant \frac{C}{R^{d-1}} \left(\int_{B(x_0, R/2)} \left(\int_{B(x_0,R)} \frac{V(u)}{|u-x|} du \right)^{p_0} dx \right)^{1/p_0} \\
&\leqslant \frac{C}{R^{d-1}} \left(\int_{\mathbb{R}^d} |I_1(\chi_{B(x_0,R)} V)|^{p_0} \right)^{1/p_0} \\
&\leqslant \frac{C}{R^{d-1}} \left(\int_{B(x_0,R)} V^q \right)^{1/q} \leqslant C \frac{R^{d/q-d}}{R^{d-1}} \int_{B(x_0,R)} V \\
&\leqslant C R^{-d/p'_0} \left(\frac{R}{\rho(x_0)} \right)^{2-d/q},
\end{aligned}
\tag{23}
$$

where, in the last inequality, we have used Lemma 4. As for the second term appearing in Lemma 6, the same estimate holds easily. Therefore, inequalities (17), (18) and (19) follow as an application of Theorems 3 and 4.

Next, suppose that $q > d$. In this case, as we mentioned it is known that K_1 satisfy the point-wise estimates (a_∞) and (b_∞). For the size condition we refer to inequality (6.5) in [10]. Condition (b_∞) was stated and proved in [2], Lemma 3. Thus, applying now Corollary 1 and Theorem 4 we obtain inequalities (20), (21) and (22). $\quad\square$

As an application of Lemma 7, Theorem 3 and Theorem 4 we obtain the following inequalities for $\mathscr{R}_2$.

THEOREM 7. *Let $V \in RH_q$ for $q > d/2$, and $\theta \geqslant 0$. Then, for any weight w the following inequalities hold.*

$$
\int |\mathscr{R}_2 f|^p w \leqslant C_\theta \int |f|^p M_r^\theta w,
\tag{24}
$$

for $1 < p < q$ and $r = (q/p)'$,

$$
\int |\mathscr{R}_2^\star f|^p w \leqslant C_{\theta,A} \int |f|^p (M_A^{loc} + M^\theta) w,
\tag{25}
$$

for $q' < p < \infty$ and any Young function $A \in \mathscr{D}_p$,

$$
w(\{|\mathscr{R}_2 f| > \lambda\}) \leqslant \frac{C_\theta}{\lambda} \int |f| M_{q'}^\theta w.
\tag{26}
$$

Proof. As we said before, it only remains to check condition (b'_s) for the kernel K_2. Let x_0, $y \in \mathbb{R}^d$ and $R > 0$ such that $R < |y - x_0| < 2R$ and $R \leqslant \rho(x_0)$. We are

going to check condition (b'_s) with $s = q$ using Lemma 7,

$$\left(\int_{B(x_0,R/2)} |K_2(x,y) - K_{2,0}(x,y)|^q dx \right)^{1/q}$$

$$\leqslant \left(\int_{B(x_0,R/2)} \left(|R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)| + \frac{C}{R^d} \left(\frac{R}{\rho(x_0)} \right)^\delta \right)^q dx \right)^{1/q}.$$

Dividing the integral in two terms it is straightforward that the second one gives us the desired estimate. For the first one, recalling that R_2 is a bounded operator on L^q for $1 < q < \infty$, and applying Lemma 4,

$$\left(\int_{B(x_0,R/2)} |R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)|^q dx \right)^{1/q} \lesssim \left(\int_{B(x_0,R/4)} V^q(x)|\Gamma(y,x)|^q dx \right)^{1/q}$$

$$\lesssim \frac{1}{R^{d-2}} \left(\int_{B(x_0,R/4)} V^q \right)^{1/q}$$

$$\lesssim R^{-d/q'} \left(\frac{R}{\rho(x_0)} \right)^{2-d/q}. \qquad \square$$

REMARK 5. Observe that except for $\mathscr{R}_1$ in the case $q > d$, the maximal operators on the right hand side are better for the adjoints $\mathscr{R}_1^\star$, $\mathscr{R}_2^\star$, even for common values of p. Also, the maximal operators appearing in (17) and (24), get closer to those in (18) and (25) as q goes to d or infinity, respectively.

4.2. Operators $V^\gamma L^{-\gamma}$

We consider, for $V \in RH_q$, $q > d/2$, the family of operators of type $V^\gamma L^{-\gamma}$ for $0 < \gamma < d/2$. For each γ, we can write K_γ, the kernel of $V^\gamma L^{-\gamma}$, as

$$K_\gamma(x,y) = V^\gamma(x)J_\gamma(x,y),$$

where J_γ is the corresponding kernel of the fractional integral operator $L^{-\gamma}$. For J_γ we have the following estimate that can be found in [8], page 587. For each $N > 0$ there exists C_N such that

$$|J_\gamma(x,y)| \leqslant \frac{1}{|x-y|^{d-2\gamma}} C_N \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N}. \tag{27}$$

We will show next that the size estimate for J_γ gives us condition (c_s) for K_γ with $s = q/\gamma$. In fact, let x_0, $y \in \mathbb{R}^d$ and R such that $|y - x_0| < R/2$. Applying Lemma 1

and Lemma 4 we get

$$\left(\int_{R<|x-x_0|<2R}|K_\gamma(x,y)|^{q/\gamma}dx\right)^{\gamma/q}$$

$$\leqslant \frac{C_N}{R^{d-2\gamma}}\left(1+\frac{R}{\rho(x_0)}\right)^{-N/(N_0+1)}\left(\int_{B(x_0,2R)}V^q\right)^{\gamma/q} \tag{28}$$

$$\lesssim R^{-d/(q/\gamma)'}\left(1+\frac{R}{\rho(x_0)}\right)^{-N/(N_0+1)+\gamma N2}\left(1+\frac{\rho(x_0)}{R}\right)^{-\gamma(2-d/q)}.$$

The above estimate together with Theorem 5 give us the following result.

THEOREM 8. *Let $V \in RH_q$ for $q > d/2$, $0 < \gamma < d/2$ and $\theta \geqslant 0$. Then, for any weight w,*

$$\int |V^\gamma L^{-\gamma}f|^p w \leqslant C_\theta \int |f|^p M_r^\theta w,$$

for $1 \leqslant p < q/\gamma$, $r = (q/(\gamma p))'$ and

$$\int |L^{-\gamma}V^\gamma f|^p w \leqslant C_\theta \int |f|^p M^\theta w,$$

for $(q/\gamma)' < p < \infty$.

4.3. Operators $V^{\gamma-1/2}\nabla L^{-\gamma}$

We consider the family of operators $V^{\gamma-1/2}\nabla L^{-\gamma}$ for $1/2 < \gamma \leqslant 1$ that includes the operator $L^{-1}\nabla V^{1/2}$ which appeared first in [10]. As a consequence of the results in Section 4.2 in [1], the associated kernel $\mathscr{K}^\gamma$ can be written as the product $\mathscr{K}^\gamma(x,y) = V^{\gamma-1/2}(x)K_{2\gamma-1}(x,y)$, with $K_{2\gamma-1}$ a fractional kernel of order $2\gamma - 1$, satisfying for each N,

$$|K_{2\gamma-1}(x,y)| \leqslant \frac{C_N}{|x-y|^{d-2\gamma+1}}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-N}, \tag{29}$$

as long as $V \in RH_q$ with $q > d$, and

$$\left(\int_{R<|x-y|<2R}|K_{2\gamma-1}(x,y)|^{p_0}dx\right)^{1/p_0} \leqslant CR^{-d/p_0'+2\gamma-1}\left(1+\frac{R}{\rho(y)}\right)^{-N}, \tag{30}$$

when $d/2 < q < d$, with p_0 such that $1/p_0 = 1/q - 1/d$. In fact, estimates of this type were shown in [1] (see estimates (66) and (67) there) for the kernel of the adjoint operator $L^{-\gamma}\nabla V^{\gamma-1/2}$.

We will show now that these inequalities imply condition (c_s) for $s = p_\gamma$ such that

$$\frac{1}{p_\gamma} = \left(\frac{1}{q}-\frac{1}{d}\right)^+ + \frac{2\gamma-1}{2q}. \tag{31}$$

Let x_0, $y \in \mathbb{R}^d$ and $R > 0$ such that $|y - x_0| < R/2$. If $q > d$, and hence $\frac{1}{p_\gamma} = \frac{2\gamma - 1}{2q}$, we may use estimate (29), Lemma 4 and the reverse Hölder inequality to get, as in (28),

$$\left(\int_{R < |x - x_0| < 2R} |K_\gamma(x,y) V^{\gamma - 1/2}(x)|^{\frac{2q}{2\gamma - 1}} dx \right)^{\frac{2\gamma - 1}{2q}}$$

$$\leqslant \frac{C_N}{R^{d - 2\gamma + 1}} \left(\int_{B(x_0, 2R)} V^q \right)^{\frac{2\gamma - 1}{2q}} \left(1 + \frac{R}{\rho(x_0)} \right)^{-N/(N_0 + 1)}$$

$$\lesssim R^{-d/p_\gamma'} \left(1 + \frac{\rho(x_0)}{R} \right)^{-(\gamma - 1/2)(2 - d/q)} \left(1 + \frac{R}{\rho(x_0)} \right)^{-N/(N_0 + 1) + N_2(\gamma - 1/2)}.$$

If $d/2 < q < d$, now we have $\frac{1}{p_\gamma} = \frac{1}{p_0} + \frac{2\gamma - 1}{2q}$. Then, by Holder's inequality together with (30) and Lemma 4 as above we obtain

$$\left(\int_{R < |x - x_0| < 2R} |K_\gamma(x,y) V^{\gamma - 1/2}(x)|^{p_\gamma} dx \right)^{1/p_\gamma}$$

$$\leqslant \left(\int_{R < |x - x_0| < 2R} |K_\gamma(x,y)|^{p_0} dx \right)^{1/p_0} \left(\int_{B(x_0, 2R)} V^q \right)^{\frac{2\gamma - 1}{2q}}$$

$$\lesssim R^{-d/p_\gamma'} \left(1 + \frac{\rho(x_0)}{R} \right)^{-(\gamma - 1/2)(2 - d/q)} \left(1 + \frac{R}{\rho(x_0)} \right)^{-N + N_1(\gamma - 1/2)}.$$

Applying the above estimates and Theorem 5 we obtain the following result.

THEOREM 9. *Let* $V \in RH_q$ *for* $q > d/2$, $1/2 < \gamma \leqslant 1$, *and* $\theta \geqslant 0$. *Then if* p_γ *is given by* (31), *for any weight* w *we have*

$$\int |V^{\gamma - 1/2} \nabla L^{-\gamma} f|^p w \leqslant C_\theta \int |f|^p M_r^\theta w,$$

for $1 \leqslant p < p_\gamma$ *with* $r = (p_\gamma/p)'$, *and*

$$\int |L^{-\gamma} \nabla V^{\gamma - 1/2} f|^p w \leqslant C_\theta \int |f|^p M^\theta w,$$

for $p_\gamma' < p < \infty$.

5. On local integrability of Tf and $T^\star f$

In this section we are going to apply the general results of Section 2 to weights of the form $w = \chi_B$. Studying maximal operators like M_ϕ^θ acting on such weights we are going to get sufficient conditions on f to assume some local integrability of Tf. We do that in the next lemma.

LEMMA 8. *Let $\theta \geqslant 0$, ϕ a Young function and $Q = B(x_0, \rho(x_0))$ a critical ball. Then there exist positive constants c_1, c_2, such that*

$$c_1 \left(1 + \frac{|x - x_0|}{\rho(x_0)} \right)^{-\theta} \leqslant M_\phi^\theta \chi_Q(x) \leqslant c_2 \left(1 + \frac{|x - x_0|}{\rho(x_0)} \right)^{-\theta/(N_0+1)} \tag{32}$$

where N_0 is the exponent appearing in (4).

Proof. Let $Q = B(x_0, \rho(x_0))$ be a critical ball, $\theta \geqslant 0$ and ϕ a Young function. We may suppose without loss of generality that $\phi(1) = 1$. Recalling that

$$M_\phi^\theta \chi_Q(x) = \sup_{B(x_B, r_B) \ni x} \left(1 + \frac{r_B}{\rho(x_B)} \right)^{-\theta} \|\chi_Q\|_{\phi, B},$$

it is enough to consider B such that $Q \cap B \neq \emptyset$, otherwise $\|\chi_Q\|_{\phi, B} = 0$, since

$$\|\chi_Q\|_{\phi, B} = \inf \left\{ \lambda : \frac{1}{|B|} \int_B \phi \left(\frac{\chi_Q}{\lambda} \right) \leqslant 1 \right\} = \inf \left\{ \lambda : \frac{1}{|B|} \int_{B \cap Q} \phi \left(\frac{1}{\lambda} \right) \leqslant 1 \right\}.$$

Let us consider first a ball $B = B(x_B, r_B)$ with $r_B \leqslant \rho(x_B)$, and take $x \in B$. Then choosing $y \in B \cap Q$,

$$|x - x_0| \leqslant |x - y| + |y - x_0| \leqslant 2r_B + \rho(x_0) \leqslant 2\rho(x_B) + \rho(x_0).$$

Also, since B is sub-critical, Q is critical and $B \cap Q \neq \emptyset$ we have that $\rho(x_B) \simeq \rho(y) \simeq \rho(x_0)$. Then,

$$|x - x_0| \leqslant \tilde{C} \rho(x_0),$$

for some $\tilde{C} > 0$. Then if $x \notin \tilde{Q} = B(x_0, \tilde{C}\rho(x_0))$ we have

$$M_\phi^{\mathrm{loc}}(\chi_Q)(x) = \sup_{\substack{B \ni x \\ r_B \leqslant \rho(x_B)}} \|\chi_Q\|_{\phi, B} = 0.$$

Now if $x \in \tilde{Q}$ and $B \cap Q \neq \emptyset$,

$$\|\chi_Q\|_{\phi, B} = \inf \left\{ \lambda : \frac{|B \cap Q|}{|B|} \phi \left(\frac{1}{\lambda} \right) \leqslant 1 \right\} \leqslant \inf \left\{ \lambda : \phi \left(\frac{1}{\lambda} \right) \leqslant 1 \right\} = 1/\phi^{-1}(1) = 1.$$

So, taking the supremum over all balls we have for $x \in \tilde{Q}$,

$$M_\phi^{\mathrm{loc}}(\chi_Q)(x) \leqslant \left(1 + \frac{r_B}{\rho(x_B)} \right)^{-\sigma},$$

for any $\sigma > 0$.

Next, we consider the operator

$$M_\phi^{\theta, \mathrm{glob}}(\chi_Q)(x) = \sup_{\substack{B \ni x \\ r_B \geqslant \rho(x_B)}} \left(1 + \frac{r_B}{\rho(x_B)} \right)^{-\theta} \|\chi_Q\|_{\phi, B}.$$

We take a ball B with $r_B \geqslant \rho(x_B)$ and such that $Q \cap B \neq \emptyset$. For $y \in Q \cap B$, we have $\rho(y) \simeq \rho(x_0)$. Using also Lemma 1,

$$\left(1 + \frac{r_B}{\rho(x_B)}\right)^{-\theta} \leqslant C \left(1 + \frac{r_B}{\rho(y)}\right)^{-\theta/(N_0+1)} \leqslant C \left(1 + \frac{r_B}{\rho(x_0)}\right)^{-\theta/(N_0+1)}.$$

Let $x \in B$ and suppose first that $x \notin 2Q$, then

$$|x - x_0| \leqslant |x - y| + |y - x_0| \leqslant 2r_B + \rho(x_0) \leqslant 2r_B + |x - x_0|/2,$$

and hence $|x - x_0| \leqslant 4r_B$. Therefore,

$$\left(1 + \frac{r_B}{\rho(x_B)}\right)^{-\theta} \leqslant C \left(1 + \frac{|x - x_0|}{\rho(x_0)}\right)^{-\theta/(N_0+1)}.$$

As before, we have $\|\chi_Q\|_{\phi,B} \leqslant 1$. Then, if $x \notin 2Q$

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) \leqslant C \left(1 + \frac{|x - x_0|}{\rho(x_0)}\right)^{-\sigma},$$

where $\sigma = \theta/(N_0 + 1)$.

On the other hand, if $x \in 2Q$,

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) \leqslant M_\phi(\chi_Q)(x) \leqslant 1.$$

Then, since $|x - x_0|/\rho(x_0) \leqslant 2$

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) \leqslant C \left(1 + \frac{|x - x_0|}{\rho(x_0)}\right)^{-\sigma}.$$

Using that $M_\phi^\theta \leqslant M_\phi^{\mathrm{loc}} + M^{\theta,\mathrm{glob}}$ and collecting the obtained estimates we arrive to the right hand side of (32). For the boundedness by below, given x we consider $B_x = B(x, |x - x_0| + \rho(x_0))$. Then $x \in B_x$ and $\|\chi_Q\|_{\phi,B_x} = 1$. Therefore,

$$M_\phi^\theta(x) \geqslant \left(1 + \frac{|x - x_0| + \rho(x_0)}{\rho(x_0)}\right)^{-\theta} \|\chi_Q\|_{\phi,B_x} \geqslant 2^\theta \left(1 + \frac{|x - x_0|}{\rho(x_0)}\right)^{-\theta}. \quad \square$$

REMARK 6. We observe that in particular Lemma 8 holds for all maximal operators appearing in Theorem 3 and Theorem 4. Hence they satisfy inequality (32) for some constants c_1, c_2, σ_1 and σ_2 when applied to the function χ_B, with B a critical ball.

PROPOSITION 2. *Let $p \geqslant 1$ and ϕ a Young function. There exists $\theta \geqslant 0$ such that for any ball $Q = B(x_0, \rho(x_0))$*

$$\int |f|^p M_\phi^\theta(\chi_Q) < \infty \tag{33}$$

if and only if there exists $\sigma > 0$ *such that*

$$\int \frac{|f|^p}{(1+|x|)^\sigma} < \infty. \tag{34}$$

Proof. Let $p \geqslant 1$ and ϕ a Young function. Let $Q = B(x_0, \rho(x_0))$ a critical ball. It is a straightforward verification that there are constants c and $\tilde{c}$ depending on x_0 and ρ such that

$$\frac{c}{1+\frac{|x-x_0|}{\rho(x_0)}} \leqslant \frac{1}{1+|x|} \leqslant \frac{\tilde{c}}{1+\frac{|x-x_0|}{\rho(x_0)}}. \tag{35}$$

Then, the equivalence between conditions (33) and (34) follows from equation (35) above and Lemma 8. $\square$

THEOREM 10. *Let* $1 \leqslant p < \infty$ *and* T *an operator such that for some Young function* ϕ *and for all* θ *there exists a constant* C *such that*

$$\int |Tf|^p w \leqslant C \int |f|^p M_\phi^\theta w, \tag{36}$$

for any weight w. *Then, if a function* f *satisfies* (34), $Tf \in L_{loc}^p$. *In particular* Tf *is finite almost everywhere.*

Proof. Let $1 \leqslant p < \infty$ and T as stated. Let f be a function satisfying (34) for some $\sigma > 0$. Then, applying Proposition 2, there exists some $\theta \geqslant 0$ such that (33) holds for any critical ball Q.

Let B be a ball in $\mathbb{R}^d$. According to Proposition 1 we can cover B by a finite number of critical balls $B_1, \dots B_N$. Using the hypothesis on the operator for such θ,

$$\int_B |Tf|^p \leqslant \sum_{i=1}^N \int |Tf|^p \chi_{B_i} \leqslant C \sum_{i=1}^N \int |f|^p M_\phi^\theta \chi_{B_i} < \infty. \quad \square$$

For operators that satisfy a weak type inequality for $p = 1$ we obtain an analogous result following the same lines as in the proof of Theorem 10.

THEOREM 11. *Let* T *be an operator such that for some Young function* ϕ *and for all* θ *there exists a constant* C *such that*

$$w(\{|Tf| > \lambda\}) \leqslant C \int |f| M_\phi^\theta w, \text{ for all } \lambda > 0,$$

for all weight w. *Then, if a function* f *satisfy* (34) *with* $p = 1$, $Tf \in L_{loc}^{1,\infty}$. *In particular* Tf *is finite almost everywhere.*

The above results can be applied to all operators considered in Section 4 since, as it was shown there, theorems of Section 2 hold in those cases. In particular we point out that for $\mathscr{R}_1$ and $\mathscr{R}_1^*$ we can apply Theorem 10, for $1 < p < \infty$, and Theorem 11, if

$V \in RH_q$ with $q > d$. As for the case $d/2 < q < d$, the conclusion holds for $1 < p < p_0$ and $p > p_0'$ respectively. On the other hand, Theorem 10 and Theorem 11 can be applied to $\mathscr{R}_2$ for $1 < p < q$, when $q > d/2$.

Similarly VL^{-1}, $V^{1/2}\nabla L^{-1}$ and $V^{1/2}L^{-1/2}$ fall under the scope of Theorem 10 for $1 \leqslant p < q$, $1 \leqslant p \leqslant p_1$ and $1 \leqslant p < 2q$, respectively (see Theorem 8 and Theorem 9).

In [10], Shen obtained L^p-estimates for derivatives of solutions of differential equations related to Schrödinger operator as a consequence of L^p-continuity of Riesz-Schrödinger Transforms (see Corollary 0.9 and Corollary 0.10). Here, with our results, we can give qualitative information on their local integrability.

COROLLARY 2. *Suppose $V \in RH_q$ for some $q > d/2$. Assume that $-\Delta u + Vu = f$ in $\mathbb{R}^d$, with f satisfying* (34) *for some $\sigma > 0$ and some $p \geqslant 1$. Then,*

1. *if $1 < p < q$, $\nabla^2 u \in L_{loc}^p$,*

2. *if $1 \leqslant p < q$, $Vu \in L_{loc}^p$,*

3. *if $1 \leqslant p < p_1$, $V^{1/2}\nabla u \in L_{loc}^p$,*

with p_1 such that $1/p_1 = (1/q - 1/d)^+ + 1/2q$.

Proof. If we set $u = L^{-1}f$ we have $\nabla^2 u = \nabla^2 L^{-1}f$, $Vu = VL^{-1}f$ and $V^{1/2}\nabla u = V^{1/2}\nabla L^{-1}f$. Therefore we only have to apply Theorem 10 to the operators $\nabla^2 L^{-1}$, VL^{-1} and $V^{1/2}\nabla L^{-1}$ to get the result. □

COROLLARY 3. *Suppose $V \in RH_q$ for some $q > d/2$ and let $p_0' < p < p_0$, with p_0 such that $1/p_0 = (1/q - 1/d)^+$. Assume that $-\Delta u + Vu = \nabla \cdot F$ in $\mathbb{R}^d$, for a vector field F with $|F|$ satisfying* (34) *for some $\sigma > 0$.*

1. *If $p_0' < p < p_0$, then $u \in L_{loc}^p$.*

2. *If $p_0' < p < 2q$, then $V^{1/2}u \in L_{loc}^p$.*

Proof. We will show only item 1. The proof of 2 is similar. Let $u = L^{-1}\nabla \dot{F}$. Then $\nabla u = \mathscr{R}_1(\mathscr{R}_1^\star \cdot F)$. Then in order to get that $\nabla u \in L_{loc}^p$, due to Theorem 10, it will be enough to check that the operators $T_j = \mathscr{R}_1 \circ (\mathscr{R}_1^\star)_j$ satisfy inequality (36). In fact, if $p_0' < p < p_0$, then

$$\int |T_j f|^p w \lesssim \int |(\mathscr{R}_1^\star)_j f|^p M_r^\theta w \lesssim \int |f|^p M_\nu^\theta M_r^\theta w,$$

for any $\nu > 1$. Choosing $\nu > r$, it follows easily $M_\nu^\theta(M_r^\theta w) \leqslant M_\nu^\theta w$, and then (36) holds. □

REFERENCES

[1] B. BONGIOANNI, E. HARBOURE AND P. QUIJANO, *Weighted inequalities for Schrödinger type singular integrals*, Journal of Fourier Analysis and Applications, 25(3):595–632, 2019.

[2] B. BONGIOANNI, E. HARBOURE AND O. SALINAS, *Riesz transforms related to Schrödinger operators acting on BMO type spaces*, J. Math. Anal. Appl., 357(1):115–131, 2009.

[3] B. BONGIOANNI, E. HARBOURE AND O. SALINAS, *Weighted inequalities for commutators of Schrödinger-Riesz transforms*, J. Math. Anal. Appl., 392(1):6–22, 2012.

[4] A. CORDOBA AND C. FEFFERMAN, *A weighted norm inequality for singular integrals*, Studia Math., 57(1):97–101, 1976.

[5] DAVID V. CRUZ-URIBE, JOSÉ MARIA MARTELL AND CARLOS PÉREZ, *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of Operator Theory: Advances and Applications, Birkhäuser/Springer Basel AG, Basel, 2011.

[6] J. DZIUBAŃSKI AND J. ZIENKIEWICZ, *Hardy spaces H^1 associated to Schrödinger operators with potential satisfying reverse Hölder inequality*, Revista Matemática Iberoamericana, 15(2):279–296, 1999.

[7] C. FEFFERMAN AND E. M. STEIN, *Some maximal inequalities*, Amer. J. Math., 93:107–115, 1971.

[8] TAO MA, PABLO RAÚL STINGA, JOSÉ L. TORREA AND CHAO ZHANG, *Regularity estimates in Hölder spaces for Schrödinger operators via a $T1$ theorem*, Ann. Mat. Pura Appl. (4), 193(2):561–589, 2014.

[9] C. PÉREZ, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. (2), 49(2):296–308, 1994.

[10] Z. SHEN, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble), 45(2):513–546, 1995.

[11] J. MICHAEL WILSON, *Weighted norm inequalities for the continuous square function*, Trans. Amer. Math. Soc., 314(2):661–692, 1989.

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