EMBEDDINGS BETWEEN WEIGHTED CESÀRO FUNCTION SPACES

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Abstract. In this paper, we give the characterization of the embeddings between weighted Cesàro function spaces. The proof is based on the duality technique, which reduces this problem to the characterizations of some direct and reverse Hardy-type inequalities and iterated Hardy-type inequalities.

1. Introduction

Our principle goal in this paper is to obtain two-sided estimates of the best constant c in the inequality

$$\left(\int_0^\infty \left(\int_0^t f(s)^{p_2} v_2(s)^{p_2} ds \right)^{\frac{q_2}{p_2}} u_2(t)^{q_2} dt \right)^{\frac{1}{q_2}}$$
$$\leqslant c \left(\int_0^\infty \left(\int_0^t f(s)^{p_1} v_1(s)^{p_1} ds \right)^{\frac{q_1}{p_1}} u_1(t)^{q_1} dt \right)^{\frac{1}{q_1}}, \tag{1}$$

where $0 < p_1, p_2, q_1, q_2 < \infty$ and u_1, u_2, v_1, v_2 are non-negative measurable functions.

Let X and Y be quasi normed vector spaces. If $X \subset Y$ and the identity operator is continuous from X to Y, that is, there exists a positive constant c such that $\|I(z)\|_Y \leqslant c\|z\|_X$ for all $z \in X$, we say that X is embedded into Y and write $X \hookrightarrow Y$. We denote by $\mathscr{M}$, the set of all measurable functions on $(0, \infty)$. We also define $\mathscr{M}^+ = \{f \in \mathscr{M} : f \geqslant 0\}$. The family of all weights, that is, measurable, positive and finite a.e. on $(0, \infty)$, is given by $\mathscr{W}$.

We denote by $\mathrm{Ces}_{p,q}(u, v)$, the weighted Cesàro function spaces and $\mathrm{Cop}_{p,q}(u, v)$, the weighted Copson function spaces, the collection of all functions on $\mathscr{M}$ such that

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} = \left(\int_0^\infty \left(\int_0^t |f(s)|^p v(s)^p ds \right)^{\frac{q}{p}} u(t)^q dt \right)^{\frac{1}{q}} < \infty,$$

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and

$$\|f\|_{\mathrm{Cop}_{p,q}(u,v)} = \left(\int_0^\infty \left(\int_t^\infty |f(s)|^p v(s)^p ds \right)^{\frac{q}{p}} u(t)^q dt \right)^{\frac{1}{q}} < \infty,$$

respectively, where $p, q \in (0, \infty)$, $u \in \mathcal{M}^+$ and $v \in \mathcal{W}$. Then with this denotation, we can formulate the main aim of this paper as the characterization of the embeddings between weighted Cesàro function spaces, that is,

$$\mathrm{Ces}_{p_1, q_1}(u_1, v_1) \hookrightarrow \mathrm{Ces}_{p_2, q_2}(u_2, v_2). \tag{2}$$

The classical Cesàro function spaces $\mathrm{Ces}_{1,p}(x^{-1}, 1)$ have been defined by Shiue in [28] and it was shown in [21] that these spaces are Banach spaces when $p > 1$.

In [11], it was shown that $\mathrm{Ces}_{1,p}(x^{-1}, 1)$ and $\mathrm{Cop}_{1,p}(1, x^{-1})$ coincide when $1 < p < \infty$ and the dual of the $\mathrm{Ces}_{1,p}(x^{-1}, 1)$ function spaces is given with a simpler description than in [29] as a remark.

During the past decade, these spaces have not been studied to a high degree but recently Astashkin and Maligranda began to examine the properties of classical Cesàro and Copson spaces in various aspects ([1, 2, 3, 4, 5, 6, 7, 8]), for the detailed information see the survey paper [9]. In [3], they gave the proof of the characterization of dual spaces of classical Cesàro function spaces. Later, in [22] authors computed the dual norm of the spaces $Ces_{1,p}(w, 1)$ generated by an arbitrary positive weight w, where $1 < p < \infty$. In [10], factorizations of spaces $\mathrm{Ces}_{1,p}(1, x^{-1}, v)$ and $\mathrm{Cop}_{1,p}(x^{-1}, v)$ are presented.

Let X and Y be (quasi-) Banach spaces of measurable functions on $(0, \infty)$. Denote by $\mathsf{M}(X, Y)$, the space of all multipliers, that is,

$$\mathsf{M}(X, Y) := \{ f : f \cdot g \in Y \quad \text{for all} \quad g \in X \}.$$

The Köthe dual X' of X is defined as the space $\mathsf{M}(X, L_1)$ of multipliers into L_1.

The space of all multipliers from X into Y is a quasi normed space with the quantity

$$\|f\|_{\mathsf{M}(X,Y)} := \sup_{g \neq 0} \frac{\|fg\|_Y}{\|g\|_X}.$$

Now, define a weighted space $Y_f = \{ g : f \cdot g \in Y, \quad f \in \mathcal{W} \}$. Then

$$\|f\|_{\mathsf{M}(X,Y)} = \sup_{g \neq 0} \frac{\|g\|_{Y_f}}{\|g\|_X} = \|\mathrm{I}\|_{X \to Y_f}.$$

Therefore, characterization of (2) will be enough to characterize the pointwise multipliers between weighted Cesàro function spaces. We should mention that the characterization of the multipliers between Cesàro and Copson spaces is stated to be difficult in [20] and note that the weighted Cesàro and Copson spaces are related to the spaces C and D defined in [20] as follows:

$$\mathrm{Ces}_{p,q}(u, v) = C(p, q, u)_v \quad \text{and} \quad \mathrm{Cop}_{p,q}(u, v) = D(p, q, u)_v.$$

This motivation started the study of the embeddings between weighted Cesáro and Copson function spaces. The approach to these problems begins with the well-known duality principle in weighted Lebesgue spaces. Recall that $p \in (1, \infty)$, $f \in \mathcal{M}^+$ and v is a weight on $(0, \infty)$, then

$$\left(\int_0^\infty f(t)^p v(t) dt \right)^{\frac{1}{p}} = \sup_{h \in \mathcal{M}^+} \frac{\int_0^\infty f(t) h(t) dt}{\left(\int_0^\infty h(t)^{p'} v(t)^{1-p'} dt \right)^{\frac{1}{p'}}}. \tag{3}$$

Combination of duality principle with estimates of the best constants in the weighted Hardy and reverse Hardy inequalities reduces the characterization of (2) to the characterizations of iterated Hardy-type inequalities.

With this technique, in [16], the embeddings between weighted Copson and Cesàro function spaces have been characterized under the restriction $p_2 \leqslant q_2$ arises from the duality. Also, using these results pointwise multipliers between weighted Cesàro and Copson function spaces are given in [17]. We should mention that recently in [25], multipliers between $\mathrm{Ces}_{1,p}(x^{-1}, 1)$ and $\mathrm{Cop}_{1,q}(1, x^{-1})$ are given when $1 < q \leqslant p \leqslant \infty$. At the time when [16] is written iterated Hardy-type inequalities of forms from [13] and [19] made it possible to characterize the embeddings between weighted Cesàro and Copson spaces.

However, duality argument reduces inequality (1) to the inequalities which contains iterated Copson operators. The solution of these problems were not known until recently, but lately different characterizations have been given for these inequalities, see [23, 14, 15, 26, 24]. Therefore, now we are able to continue this study. We will use characterizations from [23] and [24].

Note that, when $p_2 = q_2$ or $q_1 = p_1$, (1) has been characterized in [16]. Unfortunately in this paper we will solve (1) under the restriction $p_2 < q_2$ arising from duality, we will deal with the case when $q_2 < p_2$ in the future paper with a different approach. On the other hand we always assume that $p_2 \leqslant p_1$, since otherwise inequality (1) holds only for trivial functions (see Lemma 1).

The paper is organized as follows. In the next section we formulate main results of this paper. In the third section we present the necessary back-ground material. Finally, in the last section we prove the the main results of this paper.

2. Main results

Now, we will present the main results of the paper.

THEOREM 1. *Let* $0 < q_1 \leqslant p_2 < \min\{p_1, q_2\}$. *Assume that* $v_1, v_2 \in \mathcal{W}$ *and* $u_1, u_2 \in \mathcal{M}^+$ *such that* $\int_t^\infty u_i^{q_i} < \infty$, $i = 1, 2$ *for all* $t \in (0, \infty)$.

(i) *If* $p_1 \leqslant q_2 < \infty$, *then inequality* (1) *holds for all* $f \in \mathcal{M}^+$ *if and only if* $A_1 < \infty$, *where*

$$A_1 := \sup_{x \in (0, \infty)} \left(\int_x^\infty u_1^{q_1} \right)^{-\frac{1}{q_1}} \sup_{t \in (x, \infty)} \left(\int_x^t v_1^{-\frac{p_1 p_2}{p_1 - p_2}} v_2^{\frac{p_1 p_2}{p_1 - p_2}} \right)^{\frac{p_1 - p_2}{p_1 p_2}} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{1}{q_2}}.$$

Moreover, the best constant in (1) *satisfies* $c \approx A_1$.

(ii) *If* $q_2 < p_1 < \infty$, *then inequality* (1) *holds for all* $f \in \mathcal{M}^+$ *if and only if* $A_2 < \infty$, *where*

$$A_2 := \sup_{x \in (0,\infty)} \left(\int_x^\infty u_1^{q_1} \right)^{-\frac{1}{q_1}} \left(\int_x^\infty \left(\int_x^t v_1^{-\frac{p_1 p_2}{p_1 - p_2}} v_2^{\frac{p_1 p_2}{p_1 - p_2}} \right)^{\frac{q_2(p_1 - p_2)}{p_2(p_1 - q_2)}} \right.$$
$$\left. \times \left(\int_t^\infty u_2^{q_2} \right)^{\frac{q_2}{p_1 - q_2}} u_2(t)^{q_2} dt \right)^{\frac{p_1 - q_2}{p_1 q_2}}.$$

Moreover, the best constant in (1) *satisfies* $c \approx A_2$.

THEOREM 2. *Let* $0 < q_1 \leqslant p_1 = p_2 < q_2 < \infty$. *Assume that* $v_1, v_2 \in \mathcal{W}$ *and* $u_1, u_2 \in \mathcal{M}^+$ *such that* $\int_t^\infty u_i^{q_i} < \infty$, $i = 1, 2$ *for all* $t \in (0, \infty)$. *Then inequality* (1) *holds for all* $f \in \mathcal{M}^+$ *if and only if* $A_3 < \infty$, *where*

$$A_3 := \sup_{t \in (0,\infty)} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{1}{q_2}} \operatorname*{ess\,sup}_{s \in (0,t)} v_1(s)^{-1} v_2(s) \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{1}{q_1}}.$$

Moreover, the best constant in (12) *satisfies* $c \approx A_3$.

THEOREM 3. *Let* $0 < p_2 < \min\{p_1, q_1, q_2\}$. *Assume that* $u_1, u_2, v_1, v_2 \in \mathcal{W}$ *such that* $\int_t^\infty u_i^{q_i} < \infty$, $i = 1, 2$ *for all* $t \in (0, \infty)$. *Suppose that*

$$0 < \int_0^t \left(\int_s^t v_1^{-\frac{p_1 p_2}{p_1 - p_2}} v_2^{\frac{p_1 p_2}{p_1 - p_2}} \right)^{\frac{q_1(p_1 - p_2)}{p_1(q_1 - p_2)}} \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_2}} u_1(s)^{q_1} ds < \infty$$

holds for all $t \in (0, \infty)$.

(i) *If* $\max\{p_1, q_1\} \leqslant q_2 < \infty$, *then inequality* (1) *holds for all* $f \in \mathcal{M}^+$ *if and only if* $A_4 < \infty$ *and* $A_5 < \infty$, *where*

$$A_4 := \left(\int_0^\infty u_1^{q_1} \right)^{-\frac{1}{q_1}} \sup_{t \in (0,\infty)} \left(\int_0^t v_1^{-\frac{p_1 p_2}{p_1 - p_2}} v_2^{\frac{p_1 p_2}{p_1 - p_2}} \right)^{\frac{p_1 - p_2}{p_1 p_2}} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{1}{q_2}} \tag{4}$$

and

$$A_5 := \sup_{t \in (0,\infty)} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{1}{q_2}} \left(\int_0^t \left(\int_s^t v_1^{-\frac{p_1 p_2}{p_1 - p_2}} v_2^{\frac{p_1 p_2}{p_1 - p_2}} \right)^{\frac{q_1(p_1 - p_2)}{p_1(q_1 - p_2)}} \right.$$
$$\left. \times \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1 - p_2}} u_1(s)^{q_1} ds \right)^{\frac{q_1 - p_2}{q_1 p_2}}. \tag{5}$$

Moreover, the best constant in (1) *satisfies* $c \approx A_4 + A_5$.

(ii) *If $p_1 \leqslant q_2 < q_1 < \infty$, then inequality* (1) *holds for all $f \in \mathcal{M}^+$ if and only if $A_4 < \infty$, $A_6 < \infty$ and $A_7 < \infty$, where A_4 is defined in* (4),

$$
A_6 := \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(s)^{q_1} ds \right)^{\frac{q_1(q_2-p_2)}{p_2(q_1-q_2)}} \left(\int_t^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(t)^{q_1} \right.
$$

$$
\left. \times \sup_{z \in (t,\infty)} \left(\int_t^z v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}} \right)^{\frac{q_1 q_2 (p_1-p_2)}{p_1 p_2 (q_1-q_2)}} \left(\int_z^\infty u_2^{q_2} \right)^{\frac{q_1}{q_1-q_2}} dt \right)^{\frac{q_1-q_2}{q_1 q_2}}
$$

and

$$
A_7 := \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(s)^{q_1} \left(\int_s^t v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}} \right)^{\frac{q_1(p_1-p_2)}{p_1(q_1-p_2)}} ds \right)^{\frac{q_1(q_2-p_2)}{p_2(q_1-q_2)}} \right.
$$

$$
\times \sup_{z \in (t,\infty)} \left(\int_t^z v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}} \right)^{\frac{q_1(p_1-p_2)}{p_1(q_1-q_2)}} \left(\int_z^\infty u_2^{q_2} \right)^{\frac{q_1}{q_1-q_2}}
$$

$$
\left. \times \left(\int_t^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(t)^{q_1} dt \right)^{\frac{q_1-q_2}{q_1 q_2}} . \tag{6}
$$

Moreover, the best constant in (1) *satisfies $c \approx A_4 + A_6 + A_7$.*

(iii) *If $q_1 \leqslant q_2 < p_1 < \infty$, then inequality* (1) *holds for all $f \in \mathcal{M}^+$ if and only if $A_5 < \infty$, $A_8 < \infty$ and $A_9 < \infty$, where A_5 is defined in* (5),

$$
A_8 := \left(\int_0^\infty u_1^{q_1} \right)^{-\frac{1}{q_1}} \left(\int_0^\infty \left(\int_0^t v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}} \right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{q_2}{p_1-q_2}} u_2(t)^{q_2} dt \right)^{\frac{p_1-q_2}{p_1 q_2}} \tag{7}
$$

and

$$
A_9 := \sup_{t \in (0,\infty)} \left(\int_t^\infty \left(\int_t^s v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}} \right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \left(\int_s^\infty u_2^{q_2} \right)^{\frac{q_2}{p_1-q_2}} u_2(s)^{q_2} ds \right)^{\frac{p_1-q_2}{p_1 q_2}}
$$

$$
\times \left(\int_0^t \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(s)^{q_1} ds \right)^{\frac{q_1-p_2}{q_1 p_2}} .
$$

Moreover, the best constant in (1) *satisfies $c \approx A_5 + A_8 + A_9$.*

(iv) *If $p_1 < \infty$, $q_1 < \infty$ and $q_2 < \min\{p_1, q_1\}$, then inequality* (1) *holds for all $f \in \mathcal{M}^+$ if and only if $A_7 < \infty$, $A_8 < \infty$ and $A_{10} < \infty$, where A_7 and A_8 are defined in* (6) *and* (7), *respectively, and*

$$
A_{10} := \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(s)^{q_1} ds \right)^{\frac{q_1(q_2-p_2)}{p_2(q_1-q_2)}} \left(\int_t^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_2}} u_1(t)^{q_1} \right.
$$

$$
\left. \times \left(\int_t^\infty \left(\int_t^s v_1^{-\frac{p_1 p_2}{p_1-p_2}} v_2^{\frac{p_1 p_2}{p_1-p_2}} \right)^{\frac{q_2(p_1-p_2)}{p_2(p_1-q_2)}} \left(\int_s^\infty u_2^{q_2} \right)^{\frac{q_2}{p_1-q_2}} u_2(s)^{q_2} ds \right)^{\frac{q_1(p_1-q_2)}{p_1(q_1-q_2)}} dt \right)^{\frac{q_1-q_2}{q_1 q_2}}.
$$

Moreover, the best constant in (1) *satisfies $c \approx A_7 + A_8 + A_{10}$.*

THEOREM 4. *Let $0 < p_1 = p_2 < \min\{q_1, q_2\}$. Assume that $v_1, v_2 \in \mathcal{W}$ such that $v_1^{-1} v_2$ is continuous and $u_1, u_2 \in \mathcal{W}$ such that $\int_t^\infty u_i^{q_i} < \infty$, $i = 1, 2$ for all $t \in (0, \infty)$. Suppose that*

- $0 < \int_0^t v_1^{-\frac{p_1 q_1}{q_1-p_1}} v_2^{\frac{p_1 q_1}{q_1-p_1}} < \infty$,

- $0 < \int_0^t \left(\int_x^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_1}} u_1(x)^{q_1} dx < \infty$,

- $0 < \int_0^t u_2^{-\frac{p_1 q_2}{q_2-p_1}} < \infty$

hold for all $t \in (0, \infty)$.

(i) *If $q_1 \leqslant q_2 < \infty$, then inequality* (1) *holds for all $f \in \mathcal{M}^+$ if and only if $A_{11} < \infty$ and $A_{12} < \infty$, where*

$$
A_{11} := \left(\int_0^\infty u_1^{q_1} \right)^{-\frac{1}{q_1}} \sup_{t \in (0, \infty)} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{1}{q_2}} \operatorname*{ess\,sup}_{s \in (0, t)} v_1(s)^{-1} v_2(s) \tag{8}
$$

and

$$
A_{12} := \sup_{t \in (0, \infty)} \left(\int_t^\infty u_2^{q_2} \right)^{\frac{1}{q_2}} \left(\int_0^t \left(\int_x^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_1}} u_1(x)^{q_1} \sup_{z \in (x, t)} v_1(z)^{-\frac{q_1 p_1}{q_1-p_1}} v_2(z)^{\frac{q_1 p_1}{q_1-p_1}} dx \right)^{\frac{q_1-p_1}{q_1 p_1}}.
$$

Moreover, the best constant in (1) *satisfies $c \approx A_{11} + A_{12}$.*

(ii) *If $q_2 < q_1 < \infty$, then inequality* (1) *holds for all $f \in \mathcal{M}^+$ if and only if $A_{11} < \infty$, $A_{13} < \infty$ and $A_{14} < \infty$, where A_{11} is defined in* (8),

$$
A_{13} := \left(\int_0^\infty \left(\int_0^t \left(\int_x^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_1}} u_1(x)^{q_1} dx \right)^{\frac{q_1(q_2-p_1)}{p_1(q_1-q_2)}} \left(\int_t^\infty u_1^{q_1} \right)^{-\frac{q_1}{q_1-p_1}} u_1(t)^{q_1}
$$

$$\times \sup_{z\in(t,\infty)} v_1(z)^{\frac{q_1q_2}{q_1-q_2}} v_2(z)^{\frac{q_1q_2}{q_1-q_2}} \left(\int_z^\infty u_2^{q_2}\right)^{\frac{q_1}{q_1-q_2}} dt\right)^{\frac{q_1-q_2}{q_1q_2}}$$

and

$$A_{14} := \left(\int_0^\infty \left(\int_0^t \left(\int_x^\infty u_1^{q_1}\right)^{-\frac{q_1}{q_1-p_1}} u_1(x)^{q_1} \sup_{z\in(x,t)} v_1(z)^{-\frac{q_1p_1}{q_1-p_1}} v_2(z)^{-\frac{q_1p_1}{q_1-p_1}} dx\right)^{\frac{q_1(q_2-p_1)}{p_1(q_1-q_2)}}\right.$$

$$\left.\times \sup_{z\in(t,\infty)} v_1(z)^{-\frac{q_1p_1}{q_1-p_1}} v_2(z)^{-\frac{q_1p_1}{q_1-p_1}} \left(\int_z^\infty u_2^{q_2}\right)^{\frac{q_1}{q_1-q_2}} \left(\int_t^\infty u_1^{q_1}\right)^{-\frac{q_1}{q_1-p_1}} u_1(t)^{q_1} dt\right)^{\frac{q_1-q_2}{q_1q_2}}.$$

Moreover, the best constant in (1) *satisfies* $c \approx A_{11} + A_{13} + A_{14}$.

It should be noted that, using (2) one can obtain the characterization of the embeddings between weighted Copson function spaces. Indeed, using change of variables $x = 1/t$, it is easy to see that the embedding

$$\mathrm{Cop}_{p_1,q_1}(u_1,v_1) \hookrightarrow \mathrm{Cop}_{p_2,q_2}(u_2,v_2)$$

is equivalent to the embedding

$$\mathrm{Ces}_{p_1,q_1}(\tilde{u}_1,\tilde{v}_1) \hookrightarrow \mathrm{Ces}_{p_2,q_2}(\tilde{u}_2,\tilde{v}_2),$$

where $\tilde{u}_i(t) = t^{-2/q_i} u_i(1/t)$ and $\tilde{v}_i(t) = t^{-2/p_i} v_i(1/t)$, $i = 1,2$, $t > 0$. We will not formulate the results here.

3. Notations and background material

We adopt the following usual conventions. Throughout the paper we put $0/0 = 0$, $0 \cdot (\pm\infty) = 0$ and $1/(\pm\infty) = 0$. For $p \in (1,\infty)$, we define $p' = \frac{p}{p-1}$. We always denote by c and C a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript or superscript such as c_1 does not change in different occurrences. By $a \lesssim b$, ($b \gtrsim a$) we mean that $a \leqslant \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. Since the expressions on our main results are too long, to make the formulas plain we sometimes omit the differential element dx.

Now, we will present some background information we need to prove our main results. Let us begin with the characterization of the modified versions of the well-known Hardy-type inequalities (see, for instance, [27], Section 1.)

THEOREM 5. *Assume that $1 < p < \infty$, $0 < q < \infty$ and $v, w \in \mathcal{M}^+$. Let*

$$H = \sup_{f \in \mathcal{M}^+} \frac{\left(\int_0^\infty \left(\int_t^\infty f(s)ds \right)^q w(t)dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t)dt \right)^{\frac{1}{p}}}.$$

(i) *If $p \leqslant q$, then $H \approx H_1$, where*

$$H_1 = \sup_{t \in (0,\infty)} \left(\int_0^t w(s)ds \right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}ds \right)^{\frac{1}{p'}}.$$

(ii) *If $q < p$, then $H \approx H_2$, where*

$$H_2 = \left(\int_0^\infty \left(\int_0^t w(s)ds \right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}ds \right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}dt \right)^{\frac{p-q}{pq}}.$$

THEOREM 6. *Assume that $1 < p < \infty$ and $v, w \in \mathcal{M}^+$. Let*

$$H = \sup_{f \in \mathcal{M}^+} \frac{\operatorname*{ess\,sup}_{t \in (0,\infty)} \left(\int_t^\infty f(s)ds \right) w(t)}{\left(\int_0^\infty f(t)^p v(t)dt \right)^{\frac{1}{p}}}.$$

Then $H \approx H_3$, where

$$H_3 = \sup_{t \in (0,\infty)} \left(\operatorname*{ess\,sup}_{s \in (0,t)} w(s) \right) \left(\int_t^\infty v(s)^{1-p'}ds \right)^{\frac{1}{p'}}.$$

Let us now recall the characterizations of the following reverse Hardy-type inequalities.

THEOREM 7. [12, Theorem 5.1] *Assume that $0 < q \leqslant p \leqslant 1$. Suppose that $v, w \in \mathcal{M}^+$ such that w satisfies $\int_t^\infty w < \infty$ for all $t \in (0,\infty)$. Let*

$$R = \sup_{f \in \mathcal{M}^+} \frac{\left(\int_0^\infty f(t)^p v(t)dt \right)^{\frac{1}{p}}}{\left(\int_0^\infty \left(\int_0^t f(s)ds \right)^q w(t)dt \right)^{\frac{1}{q}}}. \tag{9}$$

(i) *If $p < 1$, then $R \approx R_1$, where*

$$R_1 = \sup_{t \in (0,\infty)} \left(\int_t^\infty w(s)ds \right)^{-\frac{1}{q}} \left(\int_t^\infty v(s)^{\frac{1}{1-p}}ds \right)^{\frac{1-p}{p}}.$$

(ii) If $p = 1$, then $R \approx R_2$, where

$$R_2 = \sup_{t \in (0,\infty)} \left(\int_t^\infty w(s)ds \right)^{-\frac{1}{q}} \left(\operatorname*{ess\,sup}_{s \in (t,\infty)} v(s) \right).$$

THEOREM 8. [12, Theorem 5.4] *Assume that $0 < p \leqslant 1$ and $p < q < \infty$. Suppose that $v, w \in \mathcal{M}^+$ such that w satisfies $\int_t^\infty w < \infty$ for all $t \in (0,\infty)$ and $w \neq 0$ a.e. on $(0,\infty)$. Let R be defined by* (9).
(i) If $p < 1$, then $R \approx R_3$, where

$$R_3 = \left(\int_0^\infty \left(\int_t^\infty v^{\frac{1}{1-p}} \right)^{\frac{q(1-p)}{q-p}} \left(\int_t^\infty w \right)^{-\frac{q}{q-p}} w(t)dt \right)^{\frac{q-p}{qp}} + \left(\int_0^\infty v^{\frac{1}{1-p}} \right)^{\frac{1-p}{p}} \left(\int_0^\infty w \right)^{-\frac{1}{q}}.$$

(ii) If $p = 1$, then $R \approx R_4$, where

$$R_4 = \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{s \in (t,\infty)} v(s)^{\frac{q}{q-1}} \right) \left(\int_t^\infty w \right)^{-\frac{q}{q-1}} w(t)dt \right)^{\frac{q-1}{q}} + \operatorname*{ess\,sup}_{s \in (0,\infty)} v(s) \left(\int_0^\infty w \right)^{-\frac{1}{q}}.$$

THEOREM 9. [24, Theorem 1.1] *Let $1 < p < \infty$ and $0 < q, m < \infty$ and define $r := \frac{pq}{p-q}$. Assume that $u, v, w \in \mathcal{M}^+$ such that*

$$0 < \left(\int_0^t \left(\int_s^t u \right)^{\frac{q}{m}} w(s)ds \right)^{\frac{1}{q}} < \infty$$

for all $t \in (0,\infty)$. Let

$$I = \sup_{f \in \mathcal{M}^+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty f \right)^m u(s)ds \right)^{\frac{q}{m}} w(t)dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t)dt \right)^{\frac{1}{p}}}.$$

(i) *If $p \leqslant \min\{m,q\}$, then $I \approx I_1$, where*

$$I_1 := \sup_{t \in (0,\infty)} \left(\int_0^t w(s) \left(\int_s^t u \right)^{\frac{q}{m}} ds \right)^{\frac{1}{q}} \left(\int_t^\infty v^{1-p'} \right)^{\frac{1}{p'}}. \tag{10}$$

(ii) *If $q < p \leqslant m$, then $I \approx I_2 + I_3$, where*

$$I_2 := \left(\int_0^\infty \left(\int_0^t w \right)^{\frac{r}{p}} w(t) \sup_{z \in (t,\infty)} \left(\int_t^z u \right)^{\frac{r}{m}} \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} dt \right)^{\frac{1}{r}},$$

and

$$I_3 := \left(\int_0^\infty \sup_{z \in (t,\infty)} \left(\int_t^z u \right)^{\frac{q}{m}} \left(\int_z^\infty v^{1-p'} \right)^{\frac{r}{p'}} \left(\int_0^t w(s) \left(\int_s^t u \right)^{\frac{q}{m}} ds \right)^{\frac{r}{p}} w(t) dt \right)^{\frac{1}{r}}. \tag{11}$$

(iii) *If* $m < p \leqslant q$, *then* $I \approx I_1 + I_4$, *where* I_1 *is defined in* (10) *and*

$$I_4 := \sup_{t \in (0,\infty)} \left(\int_0^t w \right)^{\frac{1}{q}} \left(\int_t^\infty \left(\int_t^s u \right)^{\frac{p}{p-m}} \left(\int_s^\infty v^{1-p'} \right)^{\frac{p(m-1)}{p-m}} v(s)^{1-p'} ds \right)^{\frac{p-m}{pm}}.$$

(iv) *If* $\max\{m,q\} < p$ *then* $I \approx I_3 + I_5$, *where* I_3 *is defined in* (11) *and*

$$I_5 := \left(\int_0^\infty \left(\int_0^t w \right)^{\frac{r}{p}} w(t) \left(\int_t^\infty \left(\int_t^s u \right)^{\frac{p}{p-m}} \left(\int_s^\infty v^{1-p'} \right)^{\frac{p(m-1)}{p-m}} v(s)^{1-p'} ds \right)^{\frac{q(p-m)}{m(p-q)}} dt \right)^{\frac{1}{r}}.$$

THEOREM 10. [23, Theorem 6] *Let* $1 < p < \infty$ *and* $0 < q < \infty$ *and set* $r := \frac{pq}{p-q}$. *Assume that* $u, v, w \in \mathcal{M}^+$ *such that* u *is continuous and*

$$0 < \int_0^t u < \infty, \quad 0 < \int_0^t v < \infty, \quad 0 < \int_0^t w < \infty$$

hold for all $t \in (0,\infty)$. *Let*

$$I = \sup_{f \in \mathcal{M}^+} \frac{\left(\int_0^\infty \left(\sup_{s \in (t,\infty)} u(s) \int_s^\infty f \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t) dt \right)^{\frac{1}{p}}}.$$

(i) *If* $p \leqslant q$ *then* $I \approx I_6$, *where*

$$I_6 := \sup_{t \in (0,\infty)} \left(\int_0^t w(s) \sup_{z \in (s,t)} u(z)^q ds \right)^{\frac{1}{q}} \left(\int_t^\infty v^{1-p'} \right)^{\frac{1}{p'}}.$$

(ii) *If* $q < p$, *then* $I \approx I_7 + I_8$, *where*

$$I_7 := \left(\int_0^\infty \left(\int_0^t w \right)^{\frac{r}{p}} w(t) \sup_{s \in (t,\infty)} u(s)^r \left(\int_s^\infty v^{1-p'} \right)^{\frac{r}{p'}} dt \right)^{\frac{1}{r}},$$

and

$$I_8 := \left(\int_0^\infty \left(\int_0^t w(s) \sup_{z \in (s,t)} u(z)^q ds \right)^{\frac{r}{p}} w(t) \sup_{z \in (t,\infty)} u(z)^q \left(\int_z^\infty v^{1-p'} ds \right)^{\frac{r}{p'}} dt \right)^{\frac{1}{r}}.$$

4. Proofs of the main results

REMARK 1. Note that for $0 < p,q < \infty$, $v \in \mathcal{W}$ and $u \in \mathcal{M}^+$, if $\int_t^\infty u^q = \infty$ for all $t > 0$, then $\mathrm{Ces}_{p,q}(u,v)$ consists of only functions equivalent to 0. Therefore, we will always assume that $\int_t^\infty u^q < \infty$, $t > 0$.

Following lemma explains the assumption $p_2 \leqslant p_1$, when characterizing our main inequality.

LEMMA 1. *Let $0 < p_1,p_2,q_1,q_2 < \infty$. Assume that $v_1,v_2 \in \mathcal{W}$ and $u_1,u_2 \in \mathcal{M}^+$ such that $0 < \int_t^\infty u_i^{q_i} < \infty$, $i = 1,2$ for all $t \in (0,\infty)$. If $p_1 < p_2$, then inequality* (1) *holds only for trivial functions.*

Proof. Suppose that there exists $c > 0$ such that (1) holds for all $f \in \mathcal{M}^+$.

Let $0 < \tau_1 < \tau_2 < \infty$ and assume that $h \in \mathcal{M}^+$ such that $\mathrm{supp}\, h \subset [\tau_1,\tau_2]$. Testing inequality (1) with h, one can see that

$$\left(\int_0^\infty \left(\int_0^t h^{p_2} v_2^{p_2} \right)^{\frac{q_2}{p_2}} u_2(t)^{q_2} dt \right)^{\frac{1}{q_2}} \geqslant \left(\int_{\tau_2}^\infty \left(\int_0^t h^{p_2} v_2^{p_2} \right)^{\frac{q_2}{p_2}} u_2(t)^{q_2} dt \right)^{\frac{1}{q_2}}$$

$$= \left(\int_{\tau_1}^{\tau_2} h^{p_2} v_2^{p_2} \right)^{\frac{1}{p_2}} \left(\int_{\tau_2}^\infty u_2^{q_2} \right)^{\frac{1}{q_2}}$$

and

$$\left(\int_0^\infty \left(\int_0^t h^{p_1} v_1^{p_1} \right)^{\frac{q_1}{p_1}} u_1(t)^{q_1} dt \right)^{\frac{1}{q_1}} = \left(\int_{\tau_1}^\infty \left(\int_{\tau_1}^t h^{p_1} v_1^{p_1} \right)^{\frac{q_1}{p_1}} u_1(t)^{q_1} dt \right)^{\frac{1}{q_1}}$$

$$\leqslant \left(\int_{\tau_1}^{\tau_2} h^{p_1} v_1^{p_1} \right)^{\frac{1}{p_1}} \left(\int_{\tau_1}^\infty u_1^{q_1} \right)^{\frac{1}{q_1}}$$

hold. Since $0 < \int_t^\infty u_i^{q_i} < \infty$, $i = 1,2$, the validity of inequality (1) implies that when $p_1 < p_2$, $L_{p_1}(v_1) \hookrightarrow L_{p_2}(v_2)$ holds, which is a contradiction.

In order to shorten the formulas and simplify the notation, in the proofs we will use the following inequality:

$$\left(\int_0^\infty \left(\int_0^t f(s)^p v(s) ds \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty \left(\int_0^t f(s) ds \right)^\theta w(t) dt \right)^{\frac{1}{\theta}}. \tag{12}$$

It is clear that we can obtain the characterization of inequality (1) with $c \approx C^{\frac{1}{p_1}}$, where c and C are the best constants of inequalities (1) and (12), respectively, by taking parameters and weights as follows:

$$p = \frac{p_2}{p_1}, \; q = \frac{q_2}{p_1}, \; \theta = \frac{q_1}{p_1}; \; v = v_1^{-p_2} v_2^{p_2}, \; u = u_2^{q_2}, \; w = u_1^{q_1}. \tag{13}$$

Proof of Theorem 1. It is clear that since in our case $q/p > 1$, using (3), the best constant of inequality (12) satisfies

$$C = \sup_{f \in \mathcal{M}^+} \frac{1}{\left(\int_0^\infty \left(\int_0^t f \right)^\theta w(t) dt \right)^{\frac{1}{\theta}}} \sup_{h \in \mathcal{M}^+} \frac{\left(\int_0^\infty h(t) \int_0^t f(s)^p v(s) ds dt \right)^{\frac{1}{p}}}{\left(\int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{qp}}}.$$

Interchanging suprema and applying Fubini, we get that

$$C = \sup_{h \in \mathcal{M}^+} \frac{1}{\left(\int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{qp}}} \sup_{f \in \mathcal{M}^+} \frac{\left(\int_0^\infty f(t)^p v(t) \int_t^\infty h(s) ds dt \right)^{\frac{1}{p}}}{\left(\int_0^\infty \left(\int_0^t f \right)^\theta w(t) dt \right)^{\frac{1}{\theta}}}.$$

Since $\theta \leqslant p < 1$, we have by applying [Theorem 7, (i)] that

$$C \approx \sup_{h \in \mathcal{M}^+} \frac{\sup\limits_{x \in (0,\infty)} \left(\int_x^\infty v(s)^{\frac{1}{1-p}} \left(\int_s^\infty h \right)^{\frac{1}{1-p}} ds \right)^{\frac{1-p}{p}} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}}}{\left(\int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{qp}}}.$$

Interchanging suprema yields that

$$C \approx \sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \sup_{h \in \mathcal{M}^+} \frac{\left(\int_0^\infty \left(\int_s^\infty h \right)^{\frac{1}{1-p}} v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{1-p}{p}}}{\left(\int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{qp}}}.$$

Then, it remains to apply Theorem 5. To this end, we need to split into two cases.

(i) If $1 \leqslant q$, in this case $\frac{1}{1-p} \geqslant \frac{q}{q-p}$, then applying [Theorem 5, (i)], we obtain that

$$C \approx \sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \sup_{t \in (0,\infty)} \left(\int_0^t v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{1-p}{p}} \left(\int_t^\infty u \right)^{\frac{1}{q}}$$

$$= \sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \max \left\{ \sup_{t \in (0,x)} \left(\int_0^t v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{1-p}{p}} \left(\int_t^\infty u \right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{t \in (x,\infty)} \left(\int_0^t v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{1-p}{p}} \left(\int_t^\infty u \right)^{\frac{1}{q}} \right\}$$

$$= \sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \sup_{t \in (x,\infty)} \left(\int_x^t v^{\frac{1}{1-p}} \right)^{\frac{1-p}{p}} \left(\int_t^\infty u \right)^{\frac{1}{q}}.$$

(ii) If $q < 1$, in this case $\frac{1}{1-p} < \frac{q}{q-p}$, then applying [Theorem 5, (ii)], we arrive at

$$C \approx \sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \left(\int_0^\infty \left(\int_0^t v^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_t^\infty u \right)^{\frac{q}{1-q}} u(t) dt \right)^{\frac{1-q}{q}}$$

$$= \sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \left(\int_x^\infty \left(\int_x^t v^{\frac{1}{1-p}} ds \right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_t^\infty u \right)^{\frac{q}{1-q}} u(t) dt \right)^{\frac{1-q}{q}}.$$

Since the best constant of inequality (1) satisfies $c \approx C^{\frac{1}{p_1}}$, applying (13) the result follows.

Proof of Theorem 2. Since, in this case $\theta \leqslant p = 1$, as in the previous proof duality approach combined with [Theorem 7, (ii)] yields that,

$$C \approx \sup_{h \in \mathcal{M}^+} \frac{\sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} \operatorname{ess\,sup}_{s \in (x,\infty)} v(s) \int_s^\infty h}{\left(\int_0^\infty h(t)^{\frac{q}{q-1}} u(t)^{-\frac{1}{q-1}} dt \right)^{\frac{q-1}{q}}}.$$

Recall that if F is a non-negative, non-decreasing measurable function on $(0,\infty)$, then

$$\operatorname{ess\,sup}_{t \in (0,\infty)} F(t)G(t) = \operatorname{ess\,sup}_{t \in (0,\infty)} F(t) \operatorname{ess\,sup}_{\tau \in (t,\infty)} G(\tau), \tag{14}$$

holds (see, for instance, page 85 in [18]). On using (14), we obtain that

$$C \approx \sup_{h \in \mathcal{M}^+} \frac{\sup_{x \in (0,\infty)} \left(\int_x^\infty w \right)^{-\frac{1}{\theta}} v(x) \int_x^\infty h}{\left(\int_0^\infty h(t)^{\frac{q}{q-1}} u(t)^{-\frac{1}{q-1}} dt \right)^{\frac{q-1}{q}}}.$$

Finally, applying Theorem 6, we arrive at

$$C \approx \sup_{t \in (0,\infty)} \left(\int_t^\infty u \right)^{\frac{1}{q}} \operatorname{ess\,sup}_{s \in (0,t)} v(s) \left(\int_s^\infty w \right)^{-\frac{1}{\theta}}.$$

Hence, applying (13) the proof is complete.

Proof of Theorem 3. Since in this case $p < 1$ and $p < \theta$, similar to the previous proofs, duality argument combined with [Theorem 8, (i)] gives that

$$
C \approx \left(\int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{h \in \mathcal{M}^+} \frac{\left(\int_0^\infty \left(\int_s^\infty h \right)^{\frac{1}{1-p}} v(s)^{\frac{1}{1-p}} ds \right)^{\frac{1-p}{p}}}{\left(\int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{qp}}}
$$

$$
+ \sup_{h \in \mathcal{M}^+} \frac{\left(\int_0^\infty \left(\int_x^\infty \left(\int_t^\infty h \right)^{\frac{1}{1-p}} v(t)^{\frac{1}{1-p}} dt \right)^{\frac{\theta(1-p)}{\theta-p}} \left(\int_x^\infty w \right)^{-\frac{\theta}{\theta-p}} w(x) dx \right)^{\frac{\theta-p}{\theta p}}}{\left(\int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{qp}}}
$$

$$=: C_1 + C_2.$$

Let us first consider C_1. We need to consider the cases $q < 1$ and $1 \leqslant q$ seperately. Hence, we begin with the condition $p < 1 \leqslant q$. Using [Theorem 5, (i)], we obtain that

$$
C_1 \approx \left(\int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{t \in (0,\infty)} \left(\int_0^t v^{\frac{1}{1-p}} \right)^{\frac{1-p}{p}} \left(\int_t^\infty u \right)^{\frac{1}{q}} =: A_4^*.
$$

On the other hand, if $p < q < 1$, using [Theorem 5, (ii)], we get that

$$
C_1 \approx \left(\int_0^\infty w \right)^{-\frac{1}{\theta}} \left(\int_0^\infty \left(\int_0^t v^{\frac{1}{1-p}} \right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_t^\infty u \right)^{\frac{q}{1-q}} u(t) dt \right)^{\frac{1-q}{q}} =: A_8^*.
$$

Let us now eveluate C_2. We will apply Theorem 9 with parameters

$$
m = \frac{1}{1-p}, \quad q = \frac{\theta}{\theta-p}, \quad p = \frac{q}{q-p}.
$$

Thus, we need to consider the conditions on parameters in four cases.

(i) If $p < \min\{1, q, \theta\}$ and $\max\{1, \theta\} \leqslant q$, then applying [Theorem 9, (i)], we get that $C_2 \approx I_1^{\frac{1}{p}}$, where

$$
I_1^{\frac{1}{p}} = \sup_{t \in (0,\infty)} \left(\int_0^t \left(\int_s^\infty w \right)^{-\frac{\theta}{\theta-p}} w(s) \left(\int_s^t v^{\frac{1}{1-p}} \right)^{\frac{\theta(1-p)}{\theta-p}} ds \right)^{\frac{\theta-p}{\theta p}} \left(\int_t^\infty u \right)^{\frac{1}{q}} = A_5^*. \quad (15)
$$

Then, since $1 \leqslant q$ in this case, we have that $C_1 \approx A_4^*$. Therefore $C = C_1 + C_2 \approx A_4^* + A_5^*$. Finally, applying (13) we arrive at $c \approx A_4 + A_5$.

(ii) If $p < \min\{1, q, \theta\}$ and $1 \leqslant q < \theta$, then applying [Theorem 9, (ii)], we get that $C_2 \approx I_2^{\frac{1}{p}} + I_3^{\frac{1}{p}}$, where

$$I_2^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty w \right)^{-\frac{\theta}{\theta-p}} w(s)ds \right)^{\frac{\theta(q-p)}{p(\theta-q)}} \left(\int_t^\infty w \right)^{-\frac{\theta}{\theta-p}} w(t) \right.$$

$$\left. \times \sup_{z \in (t,\infty)} \left(\int_t^z v^{\frac{1}{1-p}} \right)^{\frac{\theta q(1-p)}{p(\theta-q)}} \left(\int_z^\infty u \right)^{\frac{\theta}{\theta-q}} dt \right)^{\frac{\theta-q}{\theta q}} =: A_6^*.$$

and

$$I_3^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty w \right)^{-\frac{\theta}{\theta-p}} w(s) \left(\int_s^t v^{\frac{1}{1-p}} \right)^{\frac{\theta(1-p)}{\theta-p}} ds \right)^{\frac{\theta(q-p)}{p(\theta-q)}} \left(\int_t^\infty w \right)^{-\frac{\theta}{\theta-p}} w(t) \right.$$

$$\left. \times \sup_{z \in (t,\infty)} \left(\int_t^z v^{\frac{1}{1-p}} \right)^{\frac{\theta(1-p)}{\theta-p}} \left(\int_z^\infty u \right)^{\frac{\theta}{\theta-q}} dt \right)^{\frac{\theta-q}{\theta q}} =: A_7^*. \tag{16}$$

Since, $1 < q$ in this case, we have that $C_1 \approx A_4^*$. Therefore $C = C_1 + C_2 \approx A_4^* + A_6^* + A_7^*$. Applying (13), we obtain that $c \approx A_4 + A_6 + A_7$.

(iii) If $p < \min\{1, q, \theta\}$ and $\theta \leqslant q < 1$, then applying [Theorem 9, (iii)], we get that $C_2 \approx I_1^{\frac{1}{p}} + I_4^{\frac{1}{p}}$, where I_1 is given in (15) and

$$I_4^{\frac{1}{p}} := \sup_{t \in (0,\infty)} \left(\int_0^t \left(\int_s^\infty w \right)^{-\frac{\theta}{\theta-p}} w(s)ds \right)^{\frac{\theta-p}{\theta p}}$$

$$\times \left(\int_t^\infty \left(\int_t^s v^{\frac{1}{1-p}} \right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_s^\infty u \right)^{\frac{q}{1-q}} u(s)ds \right)^{\frac{1-q}{q}}$$

$$=: A_9^*.$$

Since $q < 1$, we have that $C_1 \approx A_8^*$. Thus, $C = C_1 + C_2 \approx A_8^* + A_5^* + A_9^*$. Using (13), the result follows.

(iv) If $p < \min\{1, q, \theta\}$ and $q < \min\{1, \theta\}$, then applying [Theorem 9, (iv)], we get that $C_2 \approx I_3^{\frac{1}{p}} + I_5^{\frac{1}{p}}$, where I_3 is given in (16) and

$$I_5^{\frac{1}{p}} := \left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty w \right)^{-\frac{\theta}{\theta-p}} w(s)ds \right)^{\frac{\theta(q-p)}{p(\theta-q)}} \left(\int_t^\infty w \right)^{-\frac{\theta}{\theta-p}} w(t) \right.$$

$$\times \left(\int_t^\infty \left(\int_t^s v^{\frac{1}{1-p}} \right)^{\frac{q(1-p)}{p(1-q)}} \left(\int_s^\infty u \right)^{\frac{q}{1-q}} u(s)ds \right)^{\frac{\theta(1-q)}{\theta-q}} dt \right)^{\frac{\theta-q}{\theta q}} =: A_{10}^*.$$

Since, $q < 1$, again we have that $C_1 \approx A_8^*$, which yields that $C = C_1 + C_2 \approx A_8^* + A_7^* + A_{10}^*$. Applying (13), the proof is complete.

Proof of Theorem 4. In this case $p = 1$ and $p < \theta$, therefore, we have, by using duality and applying [Theorem 8, (ii)], that

$$C \approx \left(\int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{h \in \mathscr{M}^+} \frac{\operatorname*{ess\,sup}_{x \in (0,\infty)} v(x) \int_x^\infty h}{\left(\int_0^\infty h(t)^{\frac{q}{q-1}} u(t)^{-\frac{1}{q-1}} dt \right)^{\frac{q-1}{q}}}$$

$$+ \sup_{h \in \mathscr{M}^+} \frac{\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{s \in (x,\infty)} v(s) \int_s^\infty h \right)^{\frac{\theta}{\theta-1}} \left(\int_x^\infty w \right)^{-\frac{\theta}{\theta-1}} w(x)dx \right)^{\frac{\theta-1}{\theta}}}{\left(\int_0^\infty h(t)^{\frac{q}{q-1}} u(t)^{-\frac{1}{q-1}} dt \right)^{\frac{q-1}{q}}}$$

$$=: C_3 + C_4.$$

Since $\frac{q}{q-1} > 1$, applying Theorem 6, we have that

$$C_3 \approx \left(\int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{t \in (0,\infty)} \left(\operatorname*{ess\,sup}_{s \in (0,t)} v(s) \right) \left(\int_t^\infty u \right)^{\frac{1}{q}} =: A_{11}^*.$$

On the other hand, in order to calculate C_4, we will apply Theorem 10 with parameters

$$q = \frac{\theta}{\theta - 1} \quad \text{and} \quad p = \frac{q}{q-1}.$$

We need to apply this theorem to the cases $\theta \leqslant q$ and $q < \theta$ seperately.

(i) If $\theta \leqslant q$, then applying [Theorem 10, (i)], we have that $C_4 \approx I_6$, where

$$I_6 = \sup_{t \in (0,\infty)} \left(\int_0^t \left(\int_x^\infty w \right)^{-\frac{\theta}{\theta-1}} w(x) \sup_{z \in (x,t)} v(z)^{\frac{\theta}{\theta-1}} dx \right)^{\frac{\theta-1}{\theta}} \left(\int_t^\infty u \right)^{\frac{1}{q}} =: A_{12}^*.$$

Therefore, $C = C_3 + C_4 \approx A_{11}^* + A_{12}^*$. Applying (13), we arrive at $c \approx A_{11} + A_{12}$.

(ii) If $q < \theta$, then applying [Theorem 10, (ii)], we have that $C_4 \approx I_7 + I_8$, where

$$I_7 = \left(\int_0^\infty \left(\int_0^t \left(\int_x^\infty w \right)^{-\frac{\theta}{\theta-1}} w(x)dx \right)^{\frac{\theta(q-1)}{\theta-q}} \left(\int_t^\infty w \right)^{-\frac{\theta}{\theta-1}} w(t)$$

$$\times \sup_{z\in(t,\infty)} v(z)^{\frac{\theta q}{\theta-q}} \left(\int_z^\infty u\right)^{\frac{\theta}{\theta-q}} dt\right)^{\frac{\theta-q}{\theta q}} =: A_{13}^*,$$

and

$$I_8 = \left(\int_0^\infty \left(\int_0^t \left(\int_x^\infty w\right)^{-\frac{\theta}{\theta-1}} w(x) \sup_{z\in(x,t)} v(z)^{\frac{\theta}{\theta-1}} dx\right)^{\frac{\theta(q-1)}{\theta-q}} \left(\int_t^\infty w\right)^{-\frac{\theta}{\theta-1}} w(t)\right.$$

$$\left.\times \sup_{z\in(t,\infty)} v(z)^{\frac{\theta}{\theta-1}} \left(\int_z^\infty u\right)^{\frac{\theta}{\theta-q}} dt\right)^{\frac{\theta-q}{\theta q}} =: A_{14}^*.$$

Then, we arrive at $C = C_3 + C_4 \approx A_{11}^* + A_{13}^* + A_{14}^*$. Using (13), we have that $c \approx A_{11} + A_{13} + A_{14}$.

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