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WEIGHTED HELLINGER DISTANCE AND IN–BETWEENNESS PROPERTY

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Abstract. In this paper we introduce the weighted Hellinger distance for matrices which is an interpolating between the Euclidean distance and the Hellinger distance. We show the equivalence of the weighted Hellinger distance and the Alpha Procrustes distance. As a consequence, we prove that the matrix power mean $\mu_p(t,A,B) = (tA^p + (1-t)B^p)^{1/p}$ satisfies in-betweenness property in the weighted Hellinger and Alpha Procrustes distances.

1. Introduction

Let M_n be the algebra of $n \times n$ matrices over $\mathbb{C}$ and $\mathscr{D}_n$ denote the cone of positive definite elements in M_n. Denote by I the identity matrix of M_n. For a real-valued function f and a Hermitian matrix $A \in M_n$, the matrix $f(A)$ is understood by means of the functional calculus. The space of density matrices or quantum states is as

$$\mathscr{D}_n^1 = \{\rho \in \mathscr{D}_n : Tr\rho = 1\}.$$

In recent years, many researchers have paid attention to different distance functions on the set $\mathscr{D}_n$ of positive definite matrices. Along with the traditional Riemannian metric $d_R(A,B) = \left(\sum_{i=1}^n \log^2 \lambda_i(A^{-1}B)\right)^{1/2}$ (where $\lambda_i(A^{-1}B)$ are eigenvalues of the matrix $A^{-1/2}BA^{-1/2}$), there are other important functions. Two of them are the Bures-Wasserstein distance, which are adapted from the theory of optimal transport [2]

$$d_b(A,B) = \left(Tr(A+B) - 2Tr((A^{1/2}BA^{1/2})^{1/2})\right)^{1/2},$$

and the Hellinger metric or Bhattacharya metric in quantum information [13]

$$d_h(A,B) = \left(Tr(A+B) - 2Tr(A^{1/2}B^{1/2})\right)^{1/2}.$$

Notice that the metric d_h is the same as the Euclidean distance between $A^{1/2}$ and $B^{1/2}$, i.e., $\|A^{1/2} - B^{1/2}\|_F$.

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Recently, Ha [11] introduced the Alpha Procrustes distance as follows: For $\alpha > 0$ and for positive semidefinite matrices A and B,

$$d_{b,\alpha} = \frac{1}{\alpha} d_b(A^{2\alpha}, B^{2\alpha}).$$

He showed that the Alpha Procrustes distances are the Riemannian distances corresponding to a family of Riemannian metrics on the manifold of positive definite matrices, which encompass both the Log-Euclidean and Wasserstein Riemannian metrics. Since the Alpha Procrustes distances are defined based on the Bures-Wasserstein distance, we also call them the *weighted Bures-Wasserstein distances*. In that flow, one can define the *weighted Hellinger metric* for two positive semidefinite matrices as

$$d_{h,\alpha}(A,B) = \frac{1}{\alpha} d_h(A^{2\alpha}, B^{2\alpha}). \tag{1}$$

It turns out that $d_{h,\alpha}(A,B)$ is an interpolating metric between the Log-Euclidean and the Hellinger metrics (Proposition 1).

In 2016, Audenaert introduced the in-betweenness property of matrix means [1]. We say that a matrix mean σ satisfies the *in-betweenness property* with respect to the metric d if for any pair of positive definite operators A and B,

$$d(A, A\sigma B) \leqslant d(A,B).$$

In [10] the first and the third authors, together with their coauthors, introduced and studied the in-sphere property of matrix means. Dinh, Franco and Dumitru also published several papers [7]-[9] on geometric properties of the matrix power mean $\mu_p(t;A,B) := (tA^p + (1-t)B^p)^{1/p}$ with respect to different distance functions. They also considered the case of the matrix power mean in the sense of Kubo-Ando [12] which is defined as

$$P_p(t,A,B) = A^{1/2}\left(tI + (1-t)(A^{-1/2}BA^{-1/2})^p\right)^{1/p} A^{1/2}.$$

In this paper, we focus our study on the monotonicity and in-betweenness properties of the matrix power means with respect to the weighted Bures-Wasserstein and weighted Hellinger distances.

The paper is organized as follows: We start the next section by showing that the limit of the weighted Hellinger distance as α tends to 0 is the Log-Euclidean distance. We also show that the weighted Bures-Wasserstein and weighted Hellinger distances are equivalent (Proposition 2). As a consequence of the equivalence, using the operator convexity and concavity of the power functions, we show that the matrix power mean satisfies the in-betweenness property with respect to $d_{h,\alpha}$ (Theorem 3) and $d_{b,\alpha}$ (Theorem 4). We also show that among symmetric means, the arithmetic mean is the only one that satisfies the in-betweenness property in the weighted Bures-Wasserstein and weighted Hellinger distances. Finally, we prove an inequality for the weighted quantum fidelity involving the matrix power mean.

2. Main results

PROPOSITION 1. For positive semidefinite matrices A and B,

$$\lim_{\alpha\to 0} d_{h,\alpha}^2(A,B) = ||\log(A)-\log(B)||_F^2.$$

Proof. We rewrite the expression of $d_{h,\alpha}(A,B)$ as

$$\begin{aligned} d_{h,\alpha}^2(A,B) &= \frac{1}{\alpha^2} d_h^2(A^{2\alpha}B^{2\alpha}) \\ &= \frac{1}{\alpha^2}\Big[Tr\Big(A^{2\alpha}+B^{2\alpha}-2A^{\alpha}B^{\alpha}\Big)\Big] \\ &= \frac{||A^{\alpha}-I||_F^2}{\alpha^2}+\frac{||B^{\alpha}-I||_F^2}{\alpha^2}-\frac{2}{\alpha^2}Tr\Big(A^{\alpha}B^{\alpha}-A^{\alpha}-B^{\alpha}+I\Big). \end{aligned}$$

We have

$$\lim_{\alpha\to 0}\frac{||A^{\alpha}-I||_F^2}{\alpha^2} = ||\log A||_F^2, \quad \lim_{\alpha\to 0}\frac{||B^{\alpha}-I||_F^2}{\alpha^2} = ||\log B||_F^2.$$

Since

$$A^{\alpha} = \exp(\alpha\log A) = I+\alpha\log A+\frac{\alpha^2}{2!}(\log A)^2+\ldots,$$

$$B^{\alpha} = \exp(\alpha\log B) = I+\alpha\log B+\frac{\alpha^2}{2!}(\log B)^2+\ldots,$$

we have

$$A^{\alpha}B^{\alpha} = I+\alpha(\log A+\log B)+\frac{\alpha^2}{2}\Big((\log A)^2+(\log B)^2+2\log A.\log B\Big)+\ldots.$$

Therefore,

$$A^{\alpha}B^{\alpha}-A^{\alpha}-B^{\alpha}+I = \alpha^2\log A\cdot\log B+\ldots.$$

Consequently,

$$\begin{aligned} d_{h,\alpha}^2(A,B) &= \frac{||A^{\alpha}-I||_F^2}{\alpha^2}+\frac{||B^{\alpha}-I||_F^2}{\alpha^2}-2Tr(\log A.\log B) \\ &= \frac{||A^{\alpha}-I||_F^2}{\alpha^2}+\frac{||B^{\alpha}-I||_F^2}{\alpha^2}-2\Big\langle \log A,\log B\Big\rangle_F. \end{aligned}$$

Tending α to zero, we obtain

$$d_{h,\alpha}^2(A,B) = ||\log A||_F^2+||\log B||_B^2-2\Big\langle \log A,\log B\Big\rangle_F = ||\log A-\log B||_F^2.$$

This completes the proof.

It is interesting to note that the weighted Bures-Wasserstein and weighted Hellinger distances are equivalent.

PROPOSITION 2. Let $A, B \in \mathscr{D}_n$. Then

$$d_{b,\alpha}(A,B) \leqslant d_{h,\alpha}(A,B) \leqslant \sqrt{2} d_{b,\alpha}(A,B).$$

Proof. According the Araki-Lieb-Thirring inequality [11], we have

$$Tr(A^{1/2}BA^{1/2})^r \geqslant Tr(A^r B^r), \quad |r| \leqslant 1.$$

Replacing A with $A^{2\alpha}$, B with $B^{2\alpha}$ and r with $\frac{1}{2}$ we obtain the following

$$Tr(A^\alpha B^{2\alpha} A^\alpha)^{1/2} \geqslant Tr(A^\alpha B^\alpha).$$

Thus,

$$\frac{1}{\alpha^2} Tr\Big(A^{2\alpha} + B^{2\alpha} - 2(A^\alpha B^{2\alpha} A^\alpha)^{1/2}\Big) \leqslant \frac{1}{\alpha^2} Tr\Big(A^{2\alpha} + B^{2\alpha} - 2A^\alpha B^\alpha\Big).$$

In other words,

$$d_{b,\alpha}(A,B) \leqslant d_{h,\alpha}(A,B).$$

With $\rho, \sigma \in \mathscr{D}_n^1$, we have

$$d_h^2(\rho,\sigma) = 2 - 2Tr(\rho^{1/2}\sigma^{1/2}) \leqslant 4 - 4Tr((\rho^{1/2}\sigma\rho^{1/2})^{1/2}) = 2d_b^2(\rho,\sigma),$$

or

$$2Tr((\rho^{1/2}\sigma\rho^{1/2})^{1/2}) \leqslant 1 + Tr(\rho^{1/2}\sigma^{1/2}).$$

In the above inequality replace ρ with $\dfrac{A^{2\alpha}}{Tr(A^{2\alpha})}$ and σ with $\dfrac{B^{2\alpha}}{Tr(B^{2\alpha})}$ we have

$$\begin{aligned} 2Tr\Big[(A^\alpha B^{2\alpha} A^\alpha)^{1/2}\Big] &\leqslant Tr(A^{2\alpha})^{1/2} Tr(B^{2\alpha})^{1/2} + Tr(A^\alpha B^\alpha) \\ &\leqslant \frac{1}{2} Tr(A^{2\alpha} + B^{2\alpha}) + Tr(A^\alpha B^\alpha). \end{aligned}$$

It follows that

$$4Tr[(A^\alpha B^{2\alpha} A^\alpha)^{1/2}] \leqslant Tr(A^{2\alpha} + B^{2\alpha}) + 2Tr(A^\alpha B^\alpha).$$

The above inequality is equivalent to

$$2[Tr(A^{2\alpha} + B^{2\alpha} - 2Tr(A^\alpha B^{2\alpha} A^\alpha)^{1/2}] \geqslant Tr(A^{2\alpha} + B^{2\alpha} - 2A^\alpha B^\alpha),$$

or,

$$d_{h,\alpha}^2(A,B) \leqslant 2d_{b,\alpha}^2(A,B).$$

Consequently,

$$d_{h,\alpha}(A,B) \leqslant \sqrt{2} d_{b,\alpha}(A,B).$$

Now we are ready to show that the matrix power means $\mu_p(t;A,B)$ satisfy the in-betweenness property in $d_{h,\alpha}$ and $d_{b,\alpha}$.

THEOREM 3. *Let* $A,B \in \mathscr{D}_n$, $p/2 \leqslant \alpha \leqslant p$ *and* $0 \leqslant t \leqslant 1$. *Then*

$$d_{h,\alpha}(A,\mu_p(t;A,B)) \leqslant d_{h,\alpha}(A,B).$$

Proof. We have

$$d_{h,\alpha}^2(A,\mu_p(t;A,B)) = \frac{1}{\alpha^2}Tr\Big(A^{2\alpha}+\mu_p^{2\alpha}-2A^{\alpha}\mu_p^{\alpha}(t;A,B)\Big),$$

and

$$d_{h,\alpha}^2(A,B) = \frac{1}{\alpha^2}Tr\Big(A^{2\alpha}+B^{2\alpha}-2A^{\alpha}B^{\alpha}\Big).$$

Therefore, the above result follows if

$$Tr\Big(\mu_p^{2\alpha}(t;A,B)-2A^{\alpha}\mu_p^{\alpha}(t;A,B)\Big) \leqslant Tr\Big(B^{2\alpha}-2A^{\alpha}B^{\alpha}\Big).$$

By the operator convexity of the map $x \mapsto x^{2\alpha/p}$, when $\frac{p}{2} \leqslant \alpha \leqslant p$,

$$\mu_p^{2\alpha}(t;A,B) = \Big(tA^p+(1-t)B^p\Big)^{2\alpha/p} \leqslant tA^{2\alpha}+(1-t)B^{2\alpha}.$$

Thus, the desired result follows if

$$Tr\Big[t\Big(A^{2\alpha}-B^{2\alpha}\Big)-2A^{\alpha}\mu_p^{\alpha}(t;A,B)\Big] \leqslant -2Tr(A^{\alpha}B^{\alpha}).$$

By the operator concavity of the map $x \mapsto x^{\alpha/p}$, when $\frac{p}{2} \leqslant \alpha \leqslant p$,

$$\mu_p^{\alpha}(t;A,B) = \Big(tA^p+(1-t)B^p\Big)^{\alpha/p} \geqslant tA^{\alpha}+(1-t)B^{\alpha}.$$

Therefore, the distance monotonicity follows if

$$Tr\Big[t(A^{2\alpha}-B^{2\alpha})-2A^{\alpha}\Big(tA^{\alpha}+(1-t)B^{\alpha}\Big)\Big] \leqslant -2Tr(A^{\alpha}B^{\alpha}),$$

or

$$tTr\Big(A^{2\alpha}+B^{2\alpha}-2A^{\alpha}B^{\alpha}\Big) \geqslant 0,$$

which is from AM-GM inequality.

THEOREM 4. *Let* $A,B \in \mathscr{D}_n$, $p/2 \leqslant \alpha \leqslant p$ *and* $1/2 \leqslant t \leqslant 1$. *Then,*

$$d_{b,\alpha}(A,\mu_p(t;A,B)) \leqslant d_{b,\alpha}(A,B).$$

Proof. Firstly, we show that for any positive semidefinite matrices A and B, for $p/2 \leqslant \alpha \leqslant p$ and $1/2 \leqslant t \leqslant 1$,

$$d_{b,\alpha}(A,\mu_p(t;A,B)) \leqslant d_{h,\alpha}(A,\mu_p(t;A,B)) \leqslant \sqrt{1-t}\,d_{h,\alpha}(A,B).$$

By the Araki-Lieb-Thirring inequality, we have

$$Tr\Big(A^{\alpha}B^{2\alpha}A^{\alpha}\Big)^{1/2} \geqslant Tr\Big(A^{\alpha}B^{\alpha}\Big).$$

Therefore,

$$\begin{aligned} d_{b,\alpha}^2(A,\mu_p(t;A,B)) &= \frac{1}{\alpha^2}d_b(A^{2\alpha},\mu_p^{2\alpha}(t;A,B)) \\ &= \frac{1}{\alpha^2}Tr\Big(A^{2\alpha}+\mu_p^{2\alpha}(t;A,B)-2(A^{\alpha}\mu_p^{2\alpha}(t;A,B)A^{\alpha})^{1/2}\Big) \\ &\leqslant \frac{1}{\alpha^2}Tr\Big(A^{2\alpha}+\mu_p^{2\alpha}(t;A,B)-2A^{\alpha}\mu_p^{\alpha}(t;A,B)\Big). \end{aligned}$$

By the operator convexity of the function $x\mapsto x^{2\alpha/p}$ and the operator concavity of the function $x\mapsto x^{\alpha/p}$, we obtain

$$\begin{aligned} d_{b,\alpha}^2(A,\mu_p(t;A,B)) &\leqslant \frac{1}{\alpha^2}Tr\Big[A^{2\alpha}+tA^{2\alpha}+(1-t)B^{2\alpha}-2A^{\alpha}\Big(tA^{\alpha}+(1-t)B^{\alpha}\Big)\Big] \\ &= \frac{1-t}{\alpha^2}Tr\Big(A^{2\alpha}+B^{2\alpha}-2A^{\alpha}B^{\alpha}\Big) \\ &= (1-t)d_{h,\alpha}^2(A,B). \end{aligned}$$

From here, applying the square root function to both sides with $t\in[1/2,1]$, we have

$$d_{b,\alpha}(A,\mu_p(t;A,B)) \leqslant \sqrt{1-t}\,d_{h,\alpha}(A,B) \leqslant \frac{1}{\sqrt{2}}d_{h,\alpha}(A,B) \leqslant d_{b,\alpha}(A,B).$$

This completes the proof.

In [9, Theorem 2] the authors proved that the matrix Kubo-Ando power mean $P_p(t,A,B)$ satisfies the in-betweenness property which follows from the fact that the function $g(t)=Tr(A^{1/2}P_p(t;A,B)^{1/2})$ is concave. Note that $P_t(A,B)\neq P_t(B,A)$, i.e., P_t is not symmetric. However, for the symmetric means we may have the following result whose proof is adapted from [6].

THEOREM 5. *Let σ be a symmetric mean and assume that one of the following inequalities holds for any pair of positive definite matrices A and B:*

$$d_{h,\alpha}(A,A\sigma B) \leqslant d_{h,\alpha}(A,B) \tag{2}$$

or

$$d_{b,\alpha}(A,A\sigma B) \leqslant d_{b,\alpha}(A,B). \tag{3}$$

Then σ is the arithmetic mean.

Proof. By [12, Theorem 4.4], the symmetric operator mean σ is represented as follows:

$$A\sigma B = \frac{\delta}{2}(A+B)+\int_{(0,\infty)}\frac{\lambda+1}{\lambda}\{(\lambda A):B+A:(\lambda B)\}d\mu(\lambda), \tag{4}$$

where $A,B \geqslant 0$, $\lambda \geqslant 0$ and μ is a positive measure on $(0,\infty)$ with $\delta+\mu((0,\infty))=1$, and the parallel sum $A:B$ is given by $A:B=(A^{-1}+B^{-1})^{-1}$, where A and B are invertible.

For two orthogonal projections P,Q acting on a Hilbert space H, let us denote by $P\wedge Q$ their infimum which is the orthogonal projection on the subspace $P(H)\cap Q(H)$. If $P\wedge Q=0$, then by [12, Theorem 3.7],

$$(\lambda P):Q=P:(\lambda Q)=\frac{\lambda}{\lambda+1}P\wedge Q.$$

Consequently, from (4) we get

$$P\sigma Q=\frac{\delta}{2}(P+Q).$$

Let us consider the following orthogonal projections

$$P=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad Q_\theta=\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}.$$

Notice that $Q_\theta\to P$ as $\theta\to 0$ and $Q_\theta\wedge P=0$. From the projections above, it is easy to see that the inequality (2) becomes

$$d_{h,\alpha}(P,\delta(P+Q_\theta)/2)\leqslant d_{h,\alpha}(P,Q_\theta).$$

Since this is true for all $\theta>0$, we can take a limit as $\theta\to 0^+$ to obtain

$$d_{h,\alpha}(P,\delta P)\leqslant d_{h,\alpha}(P,P)$$

whose equality occurs if and only if $\delta=1$. This shows that $\mu=0$ and σ is the arithmetic mean.

The statement for $d_{h,\alpha}$ can be proved similarly.

To finish the paper, in relation to the matrix power mean, we prove an inequality which is called a *parameterized version of quantum fidelity* which was introduced by Bhatia, Jain, and Lim [14].

Let $A,B\in\mathscr{D}_n$, a parameterized version of fidelity defined as

$$F_\alpha(A,B)=Tr\left(A^{\frac{1-\alpha}{2\alpha}}BA^{\frac{1-\alpha}{2\alpha}}\right)^\alpha,\alpha\in(0,\infty),$$

where $A,B\in\mathscr{D}_n$.

PROPOSITION 6. Let $A,B\in\mathscr{D}_n^1$, $p\geqslant 1$ and $0\leqslant t\leqslant 1, 0<\alpha<1$. Then

$$F_\alpha(A,\mu_p(t;A,B))\geqslant F_\alpha(A,B)$$

and

$$F_\alpha(A,P_p(t;A,B))\geqslant F_\alpha(A,B).$$

Proof. Let $p=1$. Notice that the function x^{α} $(0<\alpha<1)$ is operator concave, and that $0\leqslant F_{\alpha}(A,B)\leqslant Tr\Big(tA+(1-t)B\Big)=t+1-t=1$ [14, Theorem 11]. We have

$$\begin{aligned}
F_{\alpha}(A,\mu_1(t;A,B)) &= Tr\Big(A^{\frac{1-\alpha}{2\alpha}}\Big(tA+(1-t)B\Big)A^{\frac{1-\alpha}{2\alpha}}\Big)^{\alpha} \\
&= Tr\Big(tA^{\frac{1}{\alpha}}+(1-t)A^{\frac{1-\alpha}{2\alpha}}BA^{\frac{1-\alpha}{2\alpha}}\Big)^{\alpha} \\
&\geqslant Tr\Big(tA+(1-t)\Big(A^{\frac{1-\alpha}{2\alpha}}BA^{\frac{1-\alpha}{2\alpha}}\Big)^{\alpha}\Big) \\
&= t+(1-t)F_{\alpha}(A,B) \\
&\geqslant F_{\alpha}(A,B).
\end{aligned}$$

Now, let us consider the case where $p>1$. In this case, the function $x\mapsto x^{1/p}$ is operator concave, hence

$$\mu_p(t;A,B)=\Big(tA^p+(t-1)B^p\Big)^{1/p}\geqslant tA+(1-t)B=\mu_1(t;A,B).$$

This implies

$$F_{\alpha}(A,\mu_p(t;A,B))\geqslant F_{\alpha}(A,\mu_1(t;A,B)),$$

from which the result for $\mu_p(t;A,B)$ follows.
The proof for $P_p(t;A,B)$ is similar to $P_1(t;A,B)=\mu_1(t;A,B)$ and

$$\Big(tI+(1-t)(A^{-1/2}BA^{-1/2})^p\Big)^{1/p}\geqslant tI+(1-t)(A^{-1/2}BA^{-1/2}),$$

which implies

$$F_{\alpha}(A,P_p(t;A,B))\geqslant F_{\alpha}(A,P_1(t;A,B)).$$

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