DISCRETE HARDY'S TYPE INEQUALITIES AND STRUCTURE OF DISCRETE CLASS OF WEIGHTS SATISFY REVERSE HÖLDER'S INEQUALITY

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Abstract. In this paper, we will prove a new discrete weighted Hardy's type inequality with different powers. Next, we will apply this inequality to prove that the forward and backward propagation properties (*self-improving properties*) for the general discrete class $\mathscr{B}^{p,q}$ of weights that satisfy the reverse Hölder inequality hold. As special cases, we will deduce the *self-improving properties* of discrete Gehring and Muckenhoupt weights. An example is considered for illustrations.

1. Introduction

The classical Muckenhoupt class of weights has been introduced by Muckenhoupt [17] in connection with the boundedness of the Hardy and Littlewood maximal operator in the space $L_w^p(\mathbb{R}_+)$ with weight w. A year later, a different class of weights satisfying reverse Hölder's inequality has been introduced by Gehring [7, 8] in connection with the integrability properties of the gradient of quasiconformal mappings. For further studies of these two classes, we refer the reader to [6, 10, 17, 19, 20, 21, 28] and the references cited therein.

During the past few years there has been renewed interest in the area of discrete harmonic analysis and then it becomes an active field of research [12]. For example, the study of regularity and boundedness of discrete operator on $\ell^p(\mathbb{Z}_+)$ analogues for $L^p(\mathbb{R}_+)$-regularity and boundedness has been considered by some authors, see for example [3, 11, 14] and the references they are cited. This began with an observation of Riesz in his work on the Hilbert transform in 1928 and later in the work of Calderón and Zygmund on singular integrals in 1952. Whereas some results from Euclidean harmonic analysis admit an obvious variant in the discrete setting, some others do not. It is well known that passage from integral operators to their discrete analogues is not trivial (see, e.g., [5]) and each of these two settings requires its own techniques. The discrete weighted theory came of age with the paper [9] of Hunt, Muckenhoupt and Wheeden, showing that for non-negative weight v, the discrete Hilbert operator is bounded on

$\ell_v^p(\mathbb{Z}_+)$ if and only if v belongs to the discrete Muckenhoupt class of weights. We confine ourselves in this paper in studying the structure of a general class of discrete weights which contains the discrete Muckenhoupt and discrete Gehring classes as special cases.

In the following, we introduce some notation and basic results in discrete harmonic analysis that will be used in the rest of this paper. Throughout, we assume that $1 < p < \infty$. A discrete weight on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is a sequence $v = \{v(n)\}_{n=1}^\infty$ of nonnegative real numbers. Given a weight v, we denote by $\ell_v^p(\mathbb{Z}_+)$ the Banach space of all real-valued sequences $x = \{x(n)\}_{n=1}^\infty$ such that

$$\|x\|_{\ell_v^p(\mathbb{Z}_+)} := \left(\sum_{n=1}^\infty |x(n)|^p v(n) \right)^{1/p} < \infty.$$

A discrete weight v is said to belong to the discrete Muckenhoupt class $\mathscr{A}^1(A)$ on $\mathbb{I} \subset \mathbb{Z}_+$ for $p > 1$ and $A > 1$, if the inequality

$$\frac{1}{|J|} \sum_{k \in J} v(k) \leqslant Av(k), \text{ for all } k \in J, \tag{1}$$

holds for every subinterval $J \subset \mathbb{I}$ and $|J|$ is the cardinality of the set J. In [1] Ariño and Muckenhoupt proved that if v is nonincreasing and satisfies (1) then the space $d(v^{-q^*/q}, q^*)$ is the dual space of the discrete classical Lorentz space

$$d(v, q) = \left\{ x : \|x\|_{v,q} = \left(\sum_{n=1}^\infty |x^*(n)|^q v(n) \right)^{1/q} < \infty \right\},$$

where $x^*(n)$ is the nonincreasing rearrangement of $|x(n)|$ and q^* is the conjugate of q. A discrete weight v is said to belong to the discrete Muckenhoupt class $\mathscr{A}^2(A)$ on the interval $\mathbb{I} \subseteq \mathbb{Z}_+$ for $p > 1$ and $A > 1$, if the inequality

$$\sum_{k \in J} v(k) \sum_{k \in J} v^{-1}(k) \leqslant A |J|^2, \tag{2}$$

holds for every subinterval $J \subset \mathbb{I}$. In [18] Pavlov gave a full description of all complete interpolating sequences on the real line by using the integral from of (2). In particular, he proved that the sequence λ_n of real numbers is a complete interpolating sequence if and only if the function $w = |F(x + iy)|^2$, $x \in \mathbb{R}$, satisfies the Muckenhoupt condition

$$\int_J w(x)dx \int_J w^{-1}(x)dx \leqslant A |J|^2, \tag{3}$$

for some constant $A > 0$, some $y \neq 0$ for all intervals $J \subset \mathbb{R}$ of finite length $|J|$, where

$$F(z) = \lim_{R \to \infty} \prod_{|\lambda_n| < R} \left(1 - \frac{z}{\lambda_n} \right).$$

Lyubarskii and Seip [13] shown that the condition (3) can be replaced by a discrete version (2) and proved that sequence λ_n of real numbers is a complete interpolating sequence if and only if there is a relatively dense subsequence λ_{n_k} such that the numbers $d(k) = \left| F'(\lambda_{n_k}) \right|^2$ satisfies the discrete Muckenhoupt condition (2) for some constant $A > 0$ and all finite sets J of consecutive integers containing $|J|$ elements. Checking the Muckenhoupt condition (3) for a function F given by an infinite product (covering in the Cauchy principle value sense) is particularly quite hard. Condition (2) is already easier to verify since it involves only countably many sets J instead of all finite intervals.

A discrete nonnegative sequence v is said to belong to the discrete Muckenhoupt class $\mathscr{A}^p(A)$ on the interval $\mathbb{I} \subseteq \mathbb{Z}_+$ for $p > 1$ and $A > 1$ if the inequality

$$\left(\frac{1}{|J|} \sum_{k \in J} v(k) \right) \left(\frac{1}{|J|} \sum_{k \in J} v^{\frac{-1}{p-1}}(k) \right)^{p-1} \leqslant A, \tag{4}$$

holds for every subinterval interval $J \subset \mathbb{I}$. For a given exponent $p > 1$, we define the $\mathscr{A}^p$-norm of the discrete weight v by the following quantity

$$\mathscr{A}^p(v) := \sup_{J \subset \mathbb{I}} \left(\frac{1}{|J|} \sum_{k \in J} v(k) \right) \left(\frac{1}{|J|} \sum_{k \in J} v^{\frac{-1}{p-1}}(k) \right)^{p-1}, \tag{5}$$

where the supremum is taken over all intervals $J \subset \mathbb{I}$. When we fix a constant $A > 1$ the couple of real numbers (p, A) defines the $\mathscr{A}^p$ discrete Muckenhoupt class $\mathscr{A}^p(A)$:

$$v \in \mathscr{A}^p(A) \iff \mathscr{A}^p(v) \leqslant A,$$

and we will refer to A as the $\mathscr{A}^p$-constant of the class. Note that by Hölder's inequality $\mathscr{A}^p(v) \geqslant 1$ for all $1 < p < \infty$ and that the following inclusion is true:

$$\text{if } 1 < p \leqslant q < \infty, \text{ then } \mathscr{A}^p \subset \mathscr{A}^q \text{ and } \mathscr{A}^q(v) \leqslant \mathscr{A}^p(v).$$

A discrete nonnegative weight v is said to belong to the discrete Gehring class $\mathscr{G}^q(\mathscr{K})$ for a given exponent $q > 1$ and a constant $\mathscr{K} > 1$, (or satisfies the reverse Hölder inequality) on the interval $\mathbb{I} \subset \mathbb{Z}_+$ if for every subinterval $J \subseteq \mathbb{I}$, we have

$$\left(\frac{1}{|J|} \sum_{k \in J} v^q(k) \right)^{\frac{1}{q}} \leqslant \mathscr{K} \frac{1}{|J|} \sum_{k \in J} v(k).$$

For a given exponent $q > 1$, we define the $\mathscr{G}^q$-norm of v as

$$\mathscr{G}^q(v) := \sup_{J \subset I} \left[\left(\frac{1}{|J|} \sum_{k \in J} v(k) \right)^{-1} \left(\frac{1}{|J|} \sum_{k \in J} v^q(k) \right)^{\frac{1}{q}} \right]^{\frac{q}{q-1}},$$

where the supremum is taken over all intervals $J \subset \mathbb{I}$ and represents the best constant for which the $\mathscr{G}^q$-condition holds true independently on the interval $J \subseteq \mathbb{I}$. We say that v is a discrete Gehring weight if its $\mathscr{G}^q$-norm is finite, i.e.,

$$v \in \mathscr{G}^q \iff \mathscr{G}^q(v) < \infty.$$

When we fix a constant $\mathscr{K} > 1$ the couple of real numbers $(q, \mathscr{K})$ defines the $\mathscr{G}^p$-discrete Gehring class $\mathscr{G}^p(\mathscr{K})$:

$$v \in \mathscr{G}^q(\mathscr{K}) \iff \mathscr{G}^q(v) \leqslant \mathscr{K},$$

and we will refer to $\mathscr{K}$ as the $\mathscr{G}^q$-constant of the class. Note that by Hölder's inequality $\mathscr{G}^q(v) \geqslant 1$ for all $1 < q < \infty$ and that the following inclusion is true:

$$\text{if } 1 < p \leqslant q < \infty, \text{ then } \mathscr{G}^q \subset \mathscr{G}^p \text{ and } 1 \leqslant \mathscr{G}^p(v) \leqslant \mathscr{G}^q(v).$$

It is evident from the condition (4) that v and $v^{-1/(p-1)}$ must be summable and consequently almost every where finite. In particular, $v > 0$ almost everywhere. The following simple facts follow directly from the definition of the condition (4) and the properly applicable Hölder's inequality: The weight $v \in \mathscr{A}^p$ $(1 < p < \infty)$ if and only if $v^{-\frac{1}{p-1}} \in \mathscr{A}^{p'}$ where p' is the conjugate of p, and if $v \in \mathscr{A}^p$ $(1 < p < \infty)$ then $v^\alpha \in \mathscr{A}^p$ for any $0 < \alpha < 1$.

In [4] Böttcher and Seybold proved that if $v \in \mathscr{A}^p$ satisfies (4) then there is an $v \in \mathscr{A}^{p-\varepsilon}$ which is called the *self-improving* property for discrete weights. This result also has been stated in the book of Strömberg and Torchinsky [29] for the class Muckenhoupt weights as a result in an abstract context but nothing has been said regarding the exact value of ε.

In [25] the authors established the discrete versions of the Korenovskii result [10] and proved that if $q > 1$, $A > 1$ and v is a nondecreasing weight belonging to $\mathscr{A}^q(A)$, then $v \in \mathscr{A}^p(A_1)$ for $p \in (p_0, q]$ where $p_0 > 1$ is the unique solution of the equation

$$(Ap_0)^{\frac{1}{q-1}} \left(\frac{q - p_0}{q - 1} \right) = 1. \tag{6}$$

This result shows that if $v \in \mathscr{A}^q(A)$ then there exist an $\varepsilon > 0$ and a constant $A_1 = A_1(p, A)$ such that $v \in A^{q-\varepsilon}(A_1)$, (self-improving property) and thus

$$A^q(A) \subset A^{q-\varepsilon}(A_1), \tag{7}$$

where $\varepsilon = q - p$ for $p \in (p_0, q]$ where $p_0 > 1$ is determined from the solution of the equation (6). In [4] Böttcher and Seybold proved that if v satisfies the Muckenhoupt condition (4), then there exists a constant $\delta > 0$ and $\mathscr{K}_1 < \infty$ depending only on p and v such that the reverse of the inequality Hölder inequality

$$\frac{1}{|J|} \sum_{k \in J} v^{p(1+\varepsilon)}(k) \leqslant \mathscr{K}_1 \left(\frac{1}{|J|} \sum_{k \in J} v^p(k) \right)^{1+\varepsilon}, \tag{8}$$

holds (*a transition property*) for all $\varepsilon \in [0, \delta]$ and all J of the form $|J| = 2^r$ with $r \in \mathbb{N}$, the set of natural numbers. In [22] the authors proved that if v is a nonincreasing sequence and satisfies (1) for $A > 1$, then for $p \in [1, C/(C-1))$ the inequality

$$\frac{1}{|J|} \sum_{k \in J} v^p(k) \leqslant A_1 \left(\frac{1}{|J|} \sum_{k \in J} v(k) \right)^p, \text{ for } J \subset \mathbb{I}, \qquad (9)$$

holds for every subinterval interval $J \subset \mathbb{I}$. This result proves that any Muckenhoupt $\mathscr{A}^1$ weight belongs to some Gehring classes of weights (*a transition property*). In [24] the authors proved the discrete versions of the results of D'Apuzzo and Sbordone [6] for the general case of nondecreasing functions and proved that if $q > 1$ and $\mathscr{K}_q > 1$ and v is a nonincreasing sequence belonging to $\mathscr{G}^q(\mathscr{K}_q)$, then $v \in \mathscr{G}^p(\mathscr{K}_p)$ for $p \in [q, q^*)$ where q^* is determined from the equation

$$\left(\frac{x}{x-1} \right)^{-1} \left(\frac{x}{x-q} \right)^{\frac{1}{x}} = \mathscr{K}_q. \qquad (10)$$

For more details of properties of discrete Muckenhoupt and Gehring weights, we refer the reader to the papers [23, 26, 27].

The natural question now is: *Is it possible to prove the self-improving properties of general class of discrete weights which as special cases contain the properties of the discrete Muckenhoupt and Gehring weights?*

Our aim in this paper, is to give an affirmative answer to the above question for the generalized discrete class of weights that satisfy a generalized reverse Hölder inequality

$$\left(\frac{1}{|J|} \sum_{n \in J} v^q(n) \right)^{1/q} \leqslant \mathscr{C} \left(\frac{1}{|J|} \sum_{n \in J} v^p(n) \right)^{1/p}, \text{ for all } J \subset \mathbb{I}. \qquad (11)$$

For $\mathscr{C} \geqslant 1$ and $q > p > 1$, we denote by $\mathscr{B}^{p,q}(\mathscr{C})$ the class of all nonnegative weights v that satisfy the inequality (11). By recalling the classical Hölder inequality it is clear that the definition of $\mathscr{B}^{p,q}$ is well posed only for $\mathscr{C} \geqslant 1$, where the equality prevails in case of constant sequences. The smallest constant, independent on the interval J, satisfying the inequality (11) is called the $\mathscr{B}^{p,q}$-norm of the weight v and will be denoted by $\mathscr{B}^{p,q}(v)$ and given by

$$\mathscr{B}^{p,q}(v) := \sup_{J \subset \mathbb{I}} \left(\frac{1}{|J|} \sum_{n \in J} v^p(n) \right)^{-\frac{1}{p}} \left(\frac{1}{|J|} \sum_{n \in J} v^q(n) \right)^{\frac{1}{q}}. \qquad (12)$$

We say that v is a $\mathscr{B}^{p,q}$-weight if its $\mathscr{B}^{p,q}$-norm is finite, i.e.,

$$v \in \mathscr{B}^{p,q} \iff \mathscr{B}^{p,q}(v) < +\infty.$$

When we fix a constant $\mathscr{C} > 1$ the triple of real numbers $(p, q, \mathscr{C})$ defines the $\mathscr{B}^{p,q}$ *discrete class*:

$$v \in \mathscr{B}^{p,q}(\mathscr{C}) \iff \mathscr{B}^{p,q}(v) \leqslant \mathscr{C},$$

and we will refer to $\mathscr{C}$ as the $\mathscr{B}^{p,q}-$constant of the class. Moreover

$$v \in \mathscr{B}^{p,q}(\mathscr{C}) \Leftrightarrow v^p \in \mathscr{B}^{1,q/p}(\mathscr{C}^p) \Leftrightarrow v^q \in \mathscr{B}^{p/q,1}(\mathscr{C}^q),$$

and the following properties hold:

$$\mathscr{B}^{p,q}(\mathscr{C}) \subset \mathscr{B}^{p,r}(\mathscr{C}), \quad \text{for } p < r \leqslant q,$$

$$\mathscr{B}^{p,q}(\mathscr{C}) \subset \mathscr{B}^{r,q}(\mathscr{C}), \quad \text{for } p \leqslant r < q.$$

It is immediate to observe that the classes $\mathscr{A}^p$ and $\mathscr{G}^q$ are special cases of the $\mathscr{B}^{p,q}(\mathscr{C})$ of discrete weights as follows:

(1). $\mathscr{A}^p(\mathscr{C}) = \mathscr{B}^{\frac{1}{1-p},1}(\mathscr{C}) \Leftrightarrow \mathscr{A}^{\frac{p-1}{p}}(\mathscr{C}) = \mathscr{B}^{1,p}(\mathscr{C}),$
(2). $\mathscr{G}^q(\mathscr{C}) = \mathscr{B}^{1,q}(\mathscr{C}).$

The goal of this paper is to prove that the forward ($\mathscr{B}^{p,q} \subset \mathscr{B}^{p,q+\varepsilon}$) and the backward ($\mathscr{B}^{p,q} \subset \mathscr{B}^{p-\varepsilon,q}$) propagation properties for the general class $\mathscr{B}^{p,q}$ of all nonnegative weights v satisfying the reverse Hölder inequality (11) hold.

The paper is organized as follows: In the next section, we state and prove the essential inequalities that will be needed in the proofs of the main results. These basic inequalities include the weighted inequality of Hardy's type. In Section 3, we state and prove the main results for self-improving properties of the general class $\mathscr{B}^{p,q}(\mathscr{C})$ and determine the exact values of constants and exponents in the respective inequalities. Next, we obtain the self-improving properties of $\mathscr{A}^p$ and $\mathscr{G}^q$ classes as special cases. An example is considered to illustrate the sharpness of the results.

2. Basic lemmas

In this section, we state and prove the basic lemmas that will be used in the proofs of the main results in the next section. Throughout the rest of the paper, we will assume that the weights are nonnegative sequences defined on $\mathbb{I} \subset \mathbb{Z}_+$ and use the conventions $0 \cdot \infty = 0$ and $0/0 = 0$ and $\sum_{k=m}^{b} y(k) = 0$, whenever $m > b$, $\Delta\left(\sum_{s=a}^{k-1} y(s)\right) = y(k)$ and $\sum_{s=a}^{k-1} \Delta y(s) = y(k) - y(a)$.

LEMMA 1. [2, Lemma 3] *If $p \geqslant 1$, then for all $n \in \mathbb{Z}_+$*

$$\sum_{k=1}^{N} a(k) \left(\sum_{s=1}^{k} a(s)\right)^{p-1} \leqslant \left(\sum_{k=1}^{N} a(k)\right)^{p} \leqslant p \sum_{k=1}^{N} a(k) \left(\sum_{s=1}^{k} a(s)\right)^{p-1}. \quad (13)$$

The inequalities reverse direction if $0 < p < 1$ and $a(1) > 0$. The constants $(1$ and $p)$ are best possible.

We begin with the following lemma is the discrete version of the Hardy-Littlewood inequality.

LEMMA 2. *Suppose that v is a nonincreasing sequence with $v(1) > 0$. Then for $r \geqslant 1$ the following inequality*

$$\left(\sum_{k=1}^{N} v^r(k) \right)^{1/r} \leqslant \left[\sum_{k=1}^{N} (k)^{(1/r)-1} v(k) \right], \tag{14}$$

holds.

Proof. Since v^r is nonincreasing on $\mathbb{Z}_+$, we have that

$$k v^r(k) \leqslant \sum_{s=1}^{k} v^r(s), \text{ for all } k \in \mathbb{Z}_+. \tag{15}$$

Using the fact that the function $\psi(u) = u^{(1/r)-1}$ is nonincreasing on $(0, \infty)$, we have from (15) that

$$\left(\sum_{s=1}^{k} v^r(s) \right)^{(1/r)-1} v^r(k) \leqslant (k v^r(k))^{(1/r)-1} v^r(k) = (k)^{(1/r)-1} v(k). \tag{16}$$

By using the power rule (13) with $p = 1/r \leqslant 1$ and $a(k) = v^r(k)$ with $a(1) > 0$, we have that

$$\left(\sum_{k=1}^{N} v^r(k) \right)^{p} \leqslant \sum_{k=1}^{N} v^r(k) \left(\sum_{s=1}^{k} v^r(s) \right)^{p-1}. \tag{17}$$

Summing the inequality (16) over k from 1 to N and using (17), we get that

$$\left(\sum_{k=1}^{N} v^r(k) \right)^{1/r} \leqslant \left[\sum_{k=1}^{N} (k)^{(1/r)-1} v(k) \right], \tag{18}$$

which is the desired inequality (14). The proof is complete. $\square$

By replacing v by v^r and setting $r = s/r \geqslant 1$ in (14), where r and s are positive numbers such that $s \geqslant r$, we get the following result.

COROLLARY 1. *Let $v : \mathbb{Z}_+ \to \mathbb{R}^+$ be a nonincreasing sequence and there exist positive numbers r and s such that $r \leqslant s$. Then*

$$\left(\sum_{k=1}^{N} v^s(k) \right)^{r/s} \leqslant \left(\sum_{k=1}^{N} k^{(r/s)-1} v^r(k) \right). \tag{19}$$

In Corollary 1, if the sequence v is replaced by $1/g$, where g is a positive and nondecreasing sequence, we get the following result.

Corollary 2. *Let $g : \mathbb{Z}_+ \to \mathbb{R}^+$ be a positive and nondecreasing sequence. If r and s are positive real numbers such that $r \leqslant s$, then*

$$\left(\sum_{k=1}^{N} g^{-s}(k) \right)^{\frac{r}{s}} \leqslant \left(\sum_{k=1}^{N} k^{\frac{r}{s}-1} g^{-r}(k) \right). \tag{20}$$

Lemma 3. *Assume that φ, ψ are nonnegative sequences, then*

$$\sum_{k=1}^{N} \varphi(k) \left(\sum_{s=k}^{N} \psi(s) \right) = \sum_{k=1}^{N} \psi(k) \left(\sum_{s=1}^{k} \varphi(k) \right). \tag{21}$$

Proof. Assume that $\Psi(k) = \sum_{s=k}^{N} \psi(s)$. Applying the summation by parts

$$\sum_{k=1}^{N} \Delta u(n) v(n+1) = u(k)v(k)|_{k=1}^{N+1} - \sum_{k=1}^{N} u(n)\Delta v(n), \tag{22}$$

on the term $\sum_{k=1}^{N} \varphi(k)\Psi(k)$ with $u(k) = \Psi(k)$ and $\Delta v(k) = \varphi(k)$, we see that

$$\sum_{k=1}^{N} \varphi(k) \left(\sum_{s=k}^{N} \psi(s) \right) = \Psi(k)v(k)|_{1}^{N+1} - \sum_{k=1}^{N} \Delta\Psi(k)v(k+1),$$

where $v(k) = \sum_{s=1}^{k-1} \varphi(s)$. Using $v(a) = \Psi(N+1) = 0$, we obtain that

$$\sum_{k=1}^{N} \varphi(k) \left(\sum_{s=k}^{N} \psi(s) \right) = \sum_{k=1}^{N} (-\Delta\Psi(k))v(k+1)$$

$$= \sum_{k=1}^{N} \psi(k) \left(\sum_{s=1}^{k} \varphi(s) \right).$$

The proof is complete. $\square$

For any weight $g : \mathbb{I} \to \mathbb{R}^+$, we define the Hardy operator $\mathscr{M}g : \mathbb{I} \to \mathbb{R}^+$ by

$$\mathscr{M}g(n) = \frac{1}{n} \sum_{k=1}^{n} g(k), \quad \text{for all } n \in \mathbb{I}. \tag{23}$$

From the definition of $\mathscr{M}g$, we see that if g is nonincreasing, then

$$\mathscr{M}g(n) = \frac{1}{n} \sum_{s=1}^{n} g(s) \geqslant \frac{1}{n} \sum_{s=1}^{n} g(n) = g(n).$$

By using the above inequality one can see that $\Delta(\mathscr{M}g(n)) < 0$. These two facts, give us the following properties of $\mathscr{M}g$.

Lemma 4. *If v is nonincreasing then so is $\mathscr{M}g$ and $\mathscr{M}g \geqslant g$.*

REMARK 1. As a consequence of Lemma 4, we notice that if g is nonnegative and nonincreasing, then $\mathscr{M} g^q \geqslant g^q$. We also notice that if g is nonnegative and nonincreasing, then so is $\mathscr{M} g^q$ for $q > 1$.

LEMMA 5. *Let $g : \mathbb{Z}_+ \to \mathbb{R}^+$ be a nonnegative sequence and $\mathscr{M} g$ defined be as in (23). If $\lambda < 0$, then*

$$\sum_{k=1}^{N} k^\lambda \mathscr{M} g(k) \geqslant \frac{1}{\lambda} \left[N^\lambda \sum_{k=1}^{N} g(k) - \sum_{k=1}^{N} k^\lambda g(k) \right]. \tag{24}$$

Proof. From the definition of $\mathscr{M} g$, we see that

$$\sum_{k=1}^{N} k^\lambda \mathscr{M} g(k) = \sum_{k=1}^{N} \frac{k^\lambda}{k} \sum_{s=1}^{k} g(s).$$

By applying Lemma 3, we get that

$$\sum_{k=1}^{N} k^\lambda \mathscr{M} g(k) \geqslant \sum_{k=1}^{N} g(k) \left(\sum_{s=k}^{N-1} k^{\lambda-1} \right). \tag{25}$$

By employing the inequality

$$\gamma x^{\gamma-1}(x-y) \geqslant x^\gamma - y^\gamma \geqslant \gamma y^{\gamma-1}(x-y), \tag{26}$$

for $x \geqslant y > 0$, for $\gamma = \lambda < 0$, we have that

$$\lambda s^{\lambda-1} \leqslant \Delta s^\lambda \leqslant \lambda(s+1)^{\lambda-1}.$$

This implies that (note that $\lambda < 0$)

$$\sum_{s=k}^{N-1} s^{\lambda-1} \geqslant \frac{1}{\lambda} \sum_{s=k}^{N-1} \Delta s^\lambda = \frac{1}{\lambda} \left[N^\lambda - k^\lambda \right].$$

This and (25) give us that

$$\sum_{k=1}^{N} k^\lambda \mathscr{M} g(k) \geqslant \frac{1}{\lambda} \left[N^\lambda \sum_{k=1}^{N} g(k) - \sum_{k=1}^{N} k^\lambda g(k) \right],$$

which is the desired inequality (24). The proof is complete. $\square$

THEOREM 1. *Let u be a nonnegative sequence. If $0 < p \leqslant q$ and $q \geqslant 1$, then*

$$\left[\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} u(k) \right)^q n^{-p-1} \right]^{1/q} \leqslant \left(\frac{q}{p} \right) \left[\sum_{n=1}^{\infty} (nu(n))^q n^{-p-1} \right]^{1/q}. \tag{27}$$

Proof. Since $\sum_{k=1}^{n} \Delta (k-1)^{p/q} = n^{p/q}$, we have that

$$\left(\sum_{k=1}^{n} u(k) \right)^{q} = \left(n^{p/q} \sum_{k=1}^{n} \frac{u(k)}{\Delta (k-1)^{p/q}} \frac{\Delta (k-1)^{p/q}}{n^{p/q}} \right)^{q}$$

$$= n^{p} \left(\sum_{k=1}^{n} \frac{u(k)}{\Delta (k-1)^{p/q}} \frac{\Delta (k-1)^{p/q}}{\sum_{k=1}^{n} \Delta (k-1)^{p/q}} \right)^{q}. \tag{28}$$

Now, by applying Jensen's inequality

$$\varphi \left(\frac{\sum \omega_k u_k}{\sum \omega_k} \right) \leqslant \frac{\sum \omega_k \varphi (u_k)}{\sum \omega_k}, \tag{29}$$

with the convex function $\varphi(x) = |x|^{q}$, for $q > 1$,

$$u_k = \frac{u(k)}{\Delta (k-1)^{p/q}}, \text{ and } \omega_k = \Delta (k-1)^{p/q},$$

we have that

$$\left(\sum_{k=1}^{n} u(k) \right)^{q} \leqslant n^{p} \sum_{k=1}^{n} \frac{\left(\frac{u(k)}{\Delta (k-1)^{p/q}} \right)^{q} \Delta (k-1)^{p/q}}{\sum_{k=1}^{n} \Delta (k-1)^{p/q}}$$

$$= n^{p} \sum_{k=1}^{n} u^{q}(k) \frac{\left(\Delta (k-1)^{p/q} \right)^{1-q}}{n^{p/q}}$$

$$= n^{p-p/q} \sum_{k=1}^{n} u^{q}(k) \left(\Delta (k-1)^{p/q} \right)^{1-q}. \tag{30}$$

By applying the elementary inequality

$$\gamma x^{\gamma-1}(x-y) \leqslant x^{\gamma} - y^{\gamma} \leqslant \gamma y^{\gamma-1}(x-y), \tag{31}$$

for $x \geqslant y > 0$, $0 \leqslant \gamma \leqslant 1$, with $0 \leqslant p/q < 1$, we obtain

$$\frac{p}{q} k^{p/q-1} \leqslant \Delta (k-1)^{p/q} \leqslant \frac{p}{q} (k-1)^{p/q-1}. \tag{32}$$

Then (30) becomes

$$\left(\sum_{k=1}^{n} u(k) \right)^{q} \leqslant \left(\frac{q}{p} \right)^{q-1} n^{p-p/q} \sum_{k=1}^{n} k^{q-p-1+p/q} u^{q}(k). \tag{33}$$

By using inequality (33), we have

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} u(k) \right)^{q} n^{-p-1} \leqslant \left(\frac{q}{p} \right)^{q-1} \sum_{n=1}^{\infty} n^{-p/q-1} \left(\sum_{k=1}^{n} u^{q}(k) k^{q-p-1+p/q} \right)$$

$$\leqslant \left(\frac{q}{p} \right)^{q-1} \sum_{n=1}^{\infty} n^{-p/q-1} \left(\sum_{k=1}^{n+1} u^{q}(k) k^{q-p-1+p/q} \right). \tag{34}$$

The elementary inequality (26) with $\gamma = -p/q < 0$ implies that

$$-\frac{p}{q}(n-1)^{-p/q-1} \leqslant \Delta(n-1)^{-p/q} \leqslant -\frac{p}{q}n^{-p/q-1}. \tag{35}$$

Also, by employing (35) and (21) on (34), we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}u(k)\right)^{q}n^{-p-1}$$

$$\leqslant \left(\frac{q}{p}\right)^{q-1}\sum_{n=1}^{\infty}\left(-\frac{q}{p}\Delta(n-1)^{-p/q}\right)\left(\sum_{k=1}^{n+1}k^{q-p-1+p/q}u^{q}(k)\right)$$

$$= -\left(\frac{q}{p}\right)^{q}\sum_{n=1}^{\infty}u^{q}(n)n^{q-p+p/q-1}\left(\sum_{k=n+1}^{\infty}\Delta(k-1)^{-p/q}\right)$$

$$= -\left(\frac{q}{p}\right)^{q}\sum_{n=1}^{\infty}u^{q}(n)n^{q-p+p/q-1}\left(-n^{-p/q}\right) \leqslant \left(\frac{q}{p}\right)^{q}\sum_{n=1}^{\infty}[nu(n)]^{q}n^{-p-1},$$

which is the desired inequality (27). The proof is complete. $\quad\square$

Now, by putting $u = g$ and $q = \beta$ and $p = \beta - \alpha$ with $0 < \alpha < 1$, we have from Theorem 1 the following new inequality of Hardy's type which plays a crucial rule in the proof of the main results.

THEOREM 2. *Let g be a nonnegative sequence. Suppose that $\beta > 1$ and $0 < \alpha < 1$ or $\beta < 0$, $\alpha < 0$ and $\mathscr{M}g$ be defined as in (23). Then*

$$\sum_{k=1}^{N}\frac{1}{k^{1-\alpha}}(\mathscr{M}g(k))^{\beta} \leqslant \left(\frac{\beta}{\beta-\alpha}\right)^{\beta}\sum_{k=1}^{N}\frac{1}{k^{1-\alpha}}g^{\beta}(k). \tag{36}$$

By replacing g by g^{p}, $\alpha = q/s < 1$ and $\beta = q/p > 1$ in Theorem 2 we get the following result.

LEMMA 6. *Let g be a nonincreasing sequence. If p, q and s are positive numbers such that $s > q > 0$ and $q > p$, then*

$$\sum_{n=1}^{N}n^{\frac{q}{s}-1}\left(\frac{1}{n}\sum_{k=1}^{n}g^{p}(k)\right)^{\frac{q}{p}} \leqslant \left(\frac{s}{s-p}\right)^{\frac{q}{p}}\sum_{n=1}^{N}(n)^{\frac{q}{s}-1}g^{q}(n). \tag{37}$$

In Theorem 2 if we set $\alpha = q/r$ and $\beta = q/p$ where $q > 0$ and $r < p < 0$, and replacing g with g^{p} where g is a nondecreasing sequence, we get the following result.

LEMMA 7. *Let v be a nondecreasing sequence. If $q > 0$ and p, $r < 0$, then*

$$\sum_{n=1}^{N}(n)^{\frac{q}{r}-1}\left(\frac{1}{n}\sum_{k=1}^{n}g^{p}(k)\right)^{q/p} \leqslant \left(\frac{r}{r-p}\right)^{\frac{q}{p}}\sum_{n=1}^{N}(n)^{\frac{q}{r}-1}g^{p}(n). \tag{38}$$

3. Main results

In this section, we prove the main results. First, let us recall and present some properties of the $\mathscr{B}^{p,q}(\mathscr{C})$-characteristic equation

$$\left(\frac{x}{x-q}\right)^{\frac{1}{q}} = \mathscr{C}\left(\frac{x}{x-p}\right)^{\frac{1}{p}}. \tag{39}$$

This equation (see [19] and [20])) can be written in terms of ω-function in the equivalent form $\omega(p,q,x) = \mathscr{C}$, where

$$\omega(p,q,x) = \left(\frac{x}{x-p}\right)^{-\frac{1}{p}}\left(\frac{x}{x-q}\right)^{\frac{1}{q}}.$$

Actually, by observing that the ω function is strictly increasing for x in $(-\infty,0)$ and strictly decreasing in $(0,+\infty)$, we see that the equation $\omega(p,q,x) = \mathscr{C}$ admits only one negative root $v_- = v_-(p,q,\mathscr{C})$ and one positive root $v_+ = v_+(p,q,\mathscr{C})$.

THEOREM 3. *Let $\mathscr{C} > 1$ and p, $q \in \mathbb{R} - \{0\}$ real numbers such that $p < q$ and $v \in \mathscr{B}^{p,q}(\mathscr{C})$. Furthermore assume that v_- and v_+ are the two solutions of the equation*

$$\omega(p,q,x) = \mathscr{C}. \tag{40}$$

(i). If $p > 0$ then there exists $s \in [q,q^*)$ where $q^* = v_+(p,q,\mathscr{C})$ such that

$$\frac{1}{N^{\frac{q}{s}}}\sum_{n=1}^{N}(n)^{\frac{q}{s}-1}v^q(n) \leqslant \frac{1}{\varphi^{q,p}(s)}\frac{1}{N}\sum_{n=1}^{N}v^q(n), \tag{41}$$

where

$$\varphi^{q,p}(s) = 1 - \left[\mathscr{C}\left(\frac{s-q}{s}\right)^{\frac{1}{q}}\left(\frac{s}{s-p}\right)^{\frac{1}{p}}\right]^q > 0.$$

(ii). If $p < 0$ then there exists $r \in (p^*,\ p]$ where $p^* = v_-(p,q,\mathscr{C})$ such that

$$\frac{1}{N^{\frac{p}{r}}}\sum_{n=1}^{N}(n)^{\frac{p}{r}-1}v^p(n) \leqslant \frac{1}{\psi^{p,q}(r)}\frac{1}{N}\sum_{n=1}^{N}v^p(n), \tag{42}$$

where

$$\psi^{q,p}(r) = 1 - \left[\frac{1}{\mathscr{C}}\left(\frac{r-p}{r}\right)^{\frac{1}{p}}\left(\frac{r}{r-q}\right)^{\frac{1}{q}}\right]^p > 0.$$

Proof. First, we prove the Statement (i) and assume that v is a nonincreasing sequence defined on $\mathbb{Z}_+$. By using the assumption $v \in \mathscr{B}^{p,q}(\mathscr{C})$, we have from (11) that

$$(V_q(n))^{1/q} \leqslant \mathscr{C}(V_p(n))^{1/p}, \tag{43}$$

where

$$V_p(n) := \frac{1}{n} \sum_{k=1}^{n} v^p(k), \text{ and } V_q(n) := \frac{1}{n} \sum_{k=1}^{n} v^q(k).$$

From the inequality (43), we have that

$$\sum_{n=1}^{N} (n)^{(q/s)-1} V_q(n) \leqslant \mathscr{C}^q \sum_{n=1}^{N} (n)^{(q/s)-1} (V_p(n))^{q/p}. \tag{44}$$

By setting $\alpha = q/s$ and $\beta = q/p$, we have that $\beta > 1$ and $0 < \alpha < 1$ if $q > 0$ and $\alpha < 0$, $\beta < 0$ if $q < 0$. So the inequality (37) of Lemma 6 works (by replacing g by v^p, $\alpha = q/s < 1$ and $\beta = q/p > 1$), and we get that

$$\sum_{n=1}^{N} (n)^{(q/s)-1} (V_p(n))^{q/p} \leqslant \left(\frac{s}{s-p}\right)^{q/p} \sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n). \tag{45}$$

Now by combining (44) and (45), we see that

$$\sum_{n=1}^{N} (n)^{(q/s)-1} V_q(n) \leqslant \mathscr{C}^q \left(\frac{s}{s-p}\right)^{q/p} \sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n). \tag{46}$$

By applying Lemma 5 on the left-hand side of (46) with $\lambda = (q/s) - 1 < 0$ and $g = v^q$, we see that

$$\sum_{n=1}^{N} (n)^{(q/s)-1} V_q(n)$$

$$\geqslant \frac{s}{(q-s)} \left[N^{\frac{q}{s}-1} \sum_{n=1}^{N} v^q(n) - \sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n) \right]$$

$$= \frac{s}{(s-q)} \left[\sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n) - N^{\frac{q}{s}-1} \sum_{n=1}^{N} v^q(n) \right]. \tag{47}$$

From (46) and (47), we have that

$$\left[\sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n) - (N)^{\frac{q}{s}-1} \sum_{n=1}^{N} v^q(n) \right]$$

$$\leqslant \mathscr{C}^q \left(\frac{s-q}{s}\right) \left(\frac{s}{s-p}\right)^{q/p} \sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n)$$

$$\leqslant \mathscr{C}^q \left(\frac{s-q}{s}\right) \left(\frac{s}{s-p}\right)^{q/p} \sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n),$$

which leads to

$$\frac{\left[1 - \mathscr{C}^q \left(\frac{s-q}{s}\right) \left(\frac{s}{s-p}\right)^{q/p}\right]}{N^{\frac{q}{s}}} \sum_{n=1}^{N} (n)^{(q/s)-1} v^q(n) \leqslant \frac{1}{N} \sum_{n=1}^{N} v^q(n). \tag{48}$$

Let us now introduce the auxiliary function

$$\varphi^{q,p}(s) = 1 - \mathscr{C}^q \left(\frac{s-q}{s} \right) \left(\frac{s}{s-p} \right)^{\frac{q}{p}},$$

which can be written in terms of ω-function as

$$\varphi^{q,p}(s) = 1 - \left[\mathscr{C} \left(\frac{s-q}{s} \right)^{\frac{1}{q}} \left(\frac{s}{s-p} \right)^{\frac{1}{p}} \right]^q = 1 - \left[\frac{\mathscr{C}}{\omega(p,q,s)} \right]^q.$$

Clearly $\varphi^{q,p}(q) = 1$ and as $\omega(p,q,s)$ strictly decreases for positive values of s, the same does $\varphi^{q,p}(s)$, which will be zero for a certain value $q^* > q$ given by the unique positive root v_+ to the equation $\omega(p,q,s) = \mathscr{C}$. So $\varphi^{q,p}(q^*) = 0$ and

$$\varphi^{q,p}(s) > 0 \Leftrightarrow \frac{\mathscr{C}}{\omega(p,q,s)} < 1 \Leftrightarrow \left(\frac{s-p}{s} \right)^{\frac{1}{p}} > \mathscr{C} \left(\frac{s-q}{s} \right)^{\frac{1}{q}}.$$

So we have that $\varphi^{q,p}(s) > 0$ in $[q,q^*)$ and from (48), we get that

$$\frac{1}{N^{\frac{q}{s}}} \sum_{n=1}^{N} (n)^{\frac{q}{s}-1} v^q(n) \leqslant \frac{1}{\varphi^{q,p}(s)} \left(\frac{1}{N} \sum_{n=1}^{N} v^q(n) \right),$$

which completes the proof of the statement (i). Now, we consider the Statement (ii) and assume that $v \in \mathscr{B}^{p,q}(\mathscr{C})$ be a nondecreasing sequence defined on $\mathbb{I}$. Following similar steps as in the proof of the Statement (i) by using the inequality (38) in Lemma 7 with $g = v$, we obtain that

$$\left[1 - \mathscr{C}^{-p} \left(\frac{r-p}{r} \right) \left(\frac{r}{r-q} \right)^{\frac{p}{q}} \right] \sum_{n=1}^{N} (n)^{\frac{p}{r}-1} v^p \leqslant (N)^{\frac{p}{r}-1} \sum_{n=1}^{N} v^p(n). \qquad (49)$$

By setting

$$\psi^{p,q}(r) = \left[1 - \mathscr{C}^{-p} \left(\frac{r-p}{r} \right) \left(\frac{r}{r-q} \right)^{\frac{p}{q}} \right],$$

we get that

$$\psi^{p,q}(r) = 1 - \left[\frac{1}{\mathscr{C}} \left(\frac{r-p}{r} \right)^{\frac{1}{p}} \left(\frac{r}{r-q} \right)^{\frac{1}{q}} \right]^p = 1 - \left[\frac{\omega(p,q,r)}{\mathscr{C}_1} \right]^p.$$

It is clear that $\psi^{p,q}(r) = 1$ for $r = p$ and as $\omega(p,q,r)$ strictly increases for negative values of r, the same does $\psi^{p,q}(r)$, which will be zero for a certain value $p^* < p$ given by the unique negative solution v_- to the equation $\omega(p,q,x) = \mathscr{C}$. So $\psi^{q,q}(p^*) = 0$, and

$$\psi^{p,q}(r) > 0 \iff \frac{\omega(p,q,r)}{\mathscr{C}} < 1 \iff \left(\frac{r-p}{r} \right)^{\frac{1}{p}} > \mathscr{C} \left(\frac{r-q}{r} \right)^{\frac{1}{q}}.$$

Hence, as $\psi^{p,q}(r) > 0$ in $(p^*, p]$, from (49) we can obtain

$$\frac{1}{N^{\frac{p}{r}}} \sum_{n=1}^{N} (n)^{\frac{p}{r}-1} v^p(n) \leqslant \frac{1}{\psi^{p,q}(r)N} \sum_{n=1}^{N} v^p(n),$$

which completes the proof of the Statement (*ii*). The proof is complete. □

Now, we prove the *self-improving* properties of the discrete class $\mathscr{B}^{p,q}(\mathscr{C})$.

THEOREM 4. *Let* $\mathscr{C} > 1$ *and* $p,\ q \in \mathbb{R} - \{0\}$ *real numbers such that* $p < q$. *If* $v_- \leqslant p$ *and* $v_+ \geqslant q$ *are the two solutions of the equation* (40), *then we have the following:*

(1). *If* $pq > 0$ *and* $v \in \mathscr{B}^{p,q}(\mathscr{C})$ *be a nonincreasing sequence, then* $v \in \mathscr{B}^{p,s}(C_s)$ *for any* $s \in \mathbb{R}^+$ *such that* $q \leqslant s < q^*$ *where* $q^* = v_+(p,q,\mathscr{C})$ *and*

$$C_s = \mathscr{C} \left[\frac{1}{\varphi^{q,p}(s)} \right]^{\frac{1}{q}}. \tag{50}$$

(2). *If* $pq < 0$ *and* $v \in \mathscr{B}^{p,q}(\mathscr{C})$ *be a nondecreasing sequence, then* $v \in \mathscr{B}^{r,q}(D_r)$ *for any* $r \in \mathbb{R}^-$ *such that* $p^* < r \leqslant p$ *where* $p^* = v_-(p,q,\mathscr{C})$ *and*

$$D_r = \mathscr{C} \left[\frac{1}{\psi^{p,q}(r)} \right]^{-\frac{1}{p}}. \tag{51}$$

Proof. First we prove (1). Let us suppose that $p > 0$ and $q > 0$. Since v be a nonincreasing sequence in $\mathscr{B}^{p,q}(\mathscr{C})$, then the sequence v satisfies the conditions of Lemma 1 and then from the inequality (19) by replacing r with q and noting that $q/s < 1$, we get that

$$\left(\sum_{n=1}^{N} v^s(n) \right)^{q/s} \leqslant \left(\sum_{n=1}^{N} (n)^{\frac{q}{s}-1} v^q(n) \right).$$

This implies that

$$\frac{1}{N^{\frac{q}{s}}} \left(\sum_{n=1}^{N} v^s(n) \right)^{q/s} \leqslant \frac{1}{N^{\frac{q}{s}}} \left(\sum_{n=1}^{N} (n)^{\frac{q}{s}-1} v^q(n) \right). \tag{52}$$

By combining (41) and (52), we get that

$$\frac{1}{N^{\frac{q}{s}}} \left(\sum_{n=1}^{N} v^s(n) \right)^{q/s} \leqslant \frac{1}{\varphi^{q,p}(s)N} \sum_{n=1}^{N} v^q(n),$$

and this implies that

$$\left(\frac{1}{N} \sum_{n=1}^{N} v^s(n) \right)^{1/s} \leqslant \left(\frac{1}{\varphi^{q,p}(s)} \right)^{1/q} \left(\frac{1}{N} \sum_{n=1}^{N} v^q(n) \right)^{1/q}.$$

Finally, by setting $K_s = [1/\varphi^{q,p}(s)]^{\frac{1}{q}}$, and applying the $\mathscr{B}^{p,q}(\mathscr{C})$ condition on the last inequality, we obtain that

$$\left(\frac{1}{N} \sum_{n=1}^{N} v^s(n) \right)^{1/s} \leqslant \mathscr{C} K_s \left(\frac{1}{N} \sum_{n=1}^{N} v^p(n) \right)^{\frac{1}{p}}. \tag{53}$$

This implies that $v \in \mathscr{B}^{p,s}(\mathscr{C} K_s)$ for all values of s such that $\varphi^{q,p}(s) > 0$ or, equivalently, for all $s \in [q, q^*)$ with $q^* = v_+(p, q, \mathscr{C})$ unique positive solution to the equation $\varphi^{q,p}(x) = 0$, i.e.

$$\omega(p, q, q^*) = \mathscr{C}. \tag{54}$$

This proves the equation (40) and then proves Statement (1).

    Now, we prove the Statement (2). Let us suppose that $p < 0$ and $q > 0$ and $v \in \mathscr{B}^{p,q}(\mathscr{C})$ be a nonnegative and nondecreasing sequence. Since $p < 0$, we see that v^p is nonincreasing and then Lemma 1 works for the sequence v^p and for $r/p < 1$, we have that

$$\left(\sum_{n=1}^{N} v^r(n) \right)^{\frac{p}{r}} \leqslant \sum_{n=1}^{N} (n)^{\frac{p}{r}-1} v^p(n). \tag{55}$$

Then from (42) and (55), we get that

$$\frac{1}{N^{\frac{p}{r}-1}} \left(\sum_{n=1}^{N} v^r(n) \right)^{\frac{p}{r}} \leqslant \sum_{n=1}^{N} v^p(n).$$

Taking into account that $p < 0$, we obtain

$$\left[\frac{1}{\psi^{p,q}(r)} \right]^{\frac{1}{p}} \left(\frac{1}{N} \sum_{n=1}^{N} v^p(n) \right)^{\frac{1}{p}} \leqslant \left(\frac{1}{N} \sum_{n=1}^{N} v^r(n) \right)^{\frac{1}{r}}$$

By setting

$$K_r = \left[\frac{1}{\psi^{p,q}(r)} \right]^{\frac{1}{p}},$$

and applying the $\mathscr{B}^{p,q}(\mathscr{C})$ condition on the last inequality we conclude that

$$\left(\frac{1}{N} \sum_{n=1}^{N} v^q(n) \right)^{\frac{1}{q}} \leqslant \frac{\mathscr{C}}{K_r} \left(\frac{1}{N} \sum_{n=1}^{N} v^r(n) \right)^{\frac{1}{r}}. \tag{56}$$

This implies that $v \in \mathscr{B}^{r,q}(\mathscr{C}/K_r)$ for all values of r such that $\psi^{p,q}(r) > 0$, or equivalently, for all $r \in (p^*, p]$ where $p^* = v_-(p, q, \mathscr{C})$ is the unique negative solution of the equation $\psi^{p,q}(x) = 0$, i.e., $\omega(p, q, p^*) = \mathscr{C}$. This proves the Statement (2). The proof is complete. $\quad\square$

REMARK 2. It is immediate to see that for $p = 1$, Theorem 4 gives an improvement of a result due to Saker and Krnić [24] for Gehring class $\mathscr{G}^q(\mathscr{C}) = \mathscr{B}^{1,q}(\mathscr{C})$. In fact Theorem 4 proves that if $v \in \mathscr{G}^q(\mathscr{C})$ then then $v \in \mathscr{G}^s(C_s)$, where s is determined from the equation

$$\left(\frac{x}{x-1}\right)^{-1} \left(\frac{x}{x-q}\right)^{\frac{1}{q}} = \mathscr{C},$$

with a constant given in (50).

REMARK 3. Also for $q = 1$ and $p = 1/(1-p)$, we see that Theorem 4 gives a sharp self-improving property due to Saker, O'Regan and Agarwal [25] for the Muckenhoupt class $\mathscr{A}^p(\mathscr{C}) = \mathscr{B}^{1,\frac{1}{1-p}}(\mathscr{C})$. In this case, we see that $\omega(p,q,x) = \mathscr{C}$ becomes

$$\left(\frac{x}{x-1}\right) \left(\frac{(p-1)x}{(p-1)x+1}\right)^{p-1} = \mathscr{C},$$

and then by applying the transform $r \to 1/(1-x)$, we see that r is determined from the equation

$$\frac{p-r}{p-1} (\mathscr{C}r)^{\frac{1}{p-1}} = 1. \tag{57}$$

Then Theorem 4 gives a result for Muckenhoupt class and proves that if $v \in \mathscr{A}^p(\mathscr{C})$ then $v \in \mathscr{A}^r(D_r)$ where r is determined from the equation (57) with a constant given in (51).

In the following lemma, we will prove that the results proved in Theorem 4 are optimal.

LEMMA 8. (1). *Let v_+ be the unique positive solution of the equation* (39). *Then the sequence $v(n) = n^{-1/v_+}$ satisfies $\mathscr{B}^{p,q}(v) < \infty$ but $\mathscr{B}^{p,v_+}(v) = \infty$ if $0 < p < q$.*
(2). *Let v_- be the unique negative solution of the equation* (39). *Then the sequence $v(n) = n^{-1/v_-}$ satisfies $\mathscr{B}^{p,q}(v) < \infty$ but $\mathscr{B}^{v_-,q}(v) = \infty$ if $p < 0 < q$.*
(3). *If $0 < \alpha < \beta$, then $(\mathscr{B}^{p,q}(n^\alpha)) < (\mathscr{B}^{p,q}(n^\beta))$.*

Proof. Consider the interval $J = [a,N) \subset \mathbb{Z}_+$. From the definition of the norm of $\mathscr{B}^{p,q}(v)$, we see that

$$\mathscr{B}^{p,q}(v) := \sup_{J \subset \mathbb{Z}_+} \frac{\left(\frac{1}{|J|}\sum_{n \in J} v^q(n)\right)^{\frac{1}{q}}}{\left(\frac{1}{|J|}\sum_{n \in J} v^p(n)\right)^{\frac{1}{p}}} \leqslant \sup_{a < N \leqslant \infty} \frac{\left(\frac{1}{N-a}\sum_{n=a}^{N-1} v^q(n)\right)^{\frac{1}{q}}}{\left(\frac{1}{N-a}\sum_{n=a}^{N-1} v^p(n)\right)^{\frac{1}{p}}}.$$

(1). We begin to prove the assertion in the first case when $1 < p < q$. Let us consider the sequence $v(n) = (n)^\alpha$ such that $-1/q < \alpha < 0$. Since $0 < p < q$, we see

also that $-1/p < \alpha < 0$. We start by the summation

$$A_p = \left(\frac{1}{N-a} \sum_{k=a}^{N-1} (n)^{p\alpha} \right)^{\frac{1}{p}}$$

By noting that $-1/p < \alpha < 0$, we see $0 < 1 + p\alpha < 1$, and then by employing the inequality (31) with $\gamma = 1 + p\alpha < 1$, we see that

$$\Delta(n)^{1+p\alpha} \leqslant (n+1)^{1+p\alpha} - n^{1+p\alpha} \leqslant (1+p\alpha)(n)^{p\alpha},$$

and so we obtain that

$$
\begin{aligned}
A_p &= \left(\frac{1}{N-a} \sum_{k=a}^{N-1} (n)^{p\alpha} \right)^{\frac{1}{p}} \geqslant \frac{1}{(1+p\alpha)^{1/p}} \left(\frac{1}{N-a} \sum_{k=a}^{N-1} \Delta(n)^{1+p\alpha} \right)^{\frac{1}{p}} \\
&= \frac{1}{(1+p\alpha)^{1/p}} \frac{1}{(N-a)^{1/p}} (N^{1+p\alpha} - a^{1+p\alpha})^{1/p}.
\end{aligned}
\tag{58}
$$

So, we have that

$$A_p \geqslant \frac{1}{(1+p\alpha)^{1/p}} \frac{1}{(N-a)^{\frac{1}{p}}} (N^{1+p\alpha} - a^{1+p\alpha})^{1/p}. \tag{59}$$

Also, one can prove that

$$B_q = \left(\frac{1}{N-a} \sum_{k=a}^{N-1} (n)^{q\alpha} \right)^{\frac{1}{q}} \leqslant \frac{1}{(1+q\alpha)^{1/q}} \frac{1}{(N-a)^{1/q}} (N^{1+q\alpha} - a^{1+q\alpha})^{1/q}. \tag{60}$$

By combining(59) and (60), we have that

$$\frac{B_q}{A_p} \leqslant \frac{(1+p\alpha)^{1/p}}{(1+q\alpha)^{1/q}} \frac{(N-a)^{1-1/q}}{(N-a)^{1-1/p}} \frac{(N^{1+q\alpha} - a^{1+q\alpha})^{1/q}(N^{1+\alpha} - a^{1+\alpha})^{-1}}{(N^{1+p\alpha} - a^{1+p\alpha})^{1/p}(N^{1+\alpha} - a^{1+\alpha})^{-1}}.$$

Denote $t = N/a > 1$, we see that

$$
\begin{aligned}
&(N^{1+p\alpha} - a^{1+p\alpha})^{1/p} (N-a)^{1-\frac{1}{p}} (N^{1+\alpha} - a^{1+\alpha})^{-1} \\
&= (t-1)^{1-\frac{1}{p}} (t^{1+p\alpha} - 1)^{1/p}(t^{1+\alpha} - 1)^{-1}.
\end{aligned}
$$

We define

$$\zeta(t, p, \alpha) = (t-1)^{1-\frac{1}{p}} (t^{1+p\alpha} - 1)^{1/p}(t^{1+\alpha} - 1)^{-1},$$

for $t > 1$, $p > 1$ and $\alpha > -1/p$. Now, by using Lemma 2.2 in [16], we see that

$$\sup_{t>1} \zeta(t, p, \alpha) = 1.$$

Also one can get that

$$\sup_{t>1} \zeta(t,q,\alpha) = \sup_{t>1}(N^{1+q\alpha} - a^{1+q\alpha})^{1/p} (N-a)^{1-\frac{1}{q}} (N^{1+\alpha} - a^{1+\alpha})^{-1}$$

$$= \sup_{t>1} (t-1)^{1-\frac{1}{q}} (t^{1+q\alpha} - 1)^{1/q} (t^{1+\alpha} - 1)^{-1} = 1.$$

This gives us that

$$(\mathscr{B}^{p,q}(n^\alpha)) = \Psi(p,q,\alpha) = \frac{(1+p\alpha)^{1/p}}{(1+q\alpha)^{1/q}}.$$

This means that $v(n) = n^\alpha$ belongs to the class $\mathscr{B}^{p,q}(\mathscr{C})$ for $-1/q < \alpha < 0$ with the constant in the right hand side $\mathscr{C} = \Psi(p,q,\alpha)$. From Theorem 4 $v(n) = n^\alpha$ also belongs to $\mathscr{B}^{p,s}(\mathscr{C})$ class for $q \leqslant s < v_+$ and clearly $\alpha = -1/v_+$ is the upper bound of those values for which the which $\mathscr{B}^{p,q}$-norm is finite but $v(n) = n^{-1/v_+}$ does not belong to $\mathscr{B}^{p,v_+}(\mathscr{C})$.

(2). Now, we prove the assertion in this case when $p < 0 < q$. Let us consider the sequence $v(n) = n^\alpha$ such that $0 < \alpha < -1/p$. Since $p < 0 < q$, we see that $\alpha > -1/q$. We start by the summation

$$A_p = \left(\frac{1}{N-a} \sum_{k=a}^{N-1} (n)^{p\alpha} \right)^{\frac{1}{p}},$$

where $-1 < p\alpha < 0$ and then $0 < p\alpha + 1 < 1$. So as in the first case, we have that

$$A_p^* \geqslant \frac{1}{(1+p\alpha)^{1/p}} \frac{1}{(N-a)^{1/p}} (N^{1+p\alpha} - a^{1+p\alpha})^{1/p}. \tag{61}$$

Also, since $\alpha > 0$ and $q > 0$, we see that $q\alpha + 1 > 1$, and then we get that

$$n^{q\alpha} \leqslant \frac{1}{(q\alpha+1)} \Delta(n)^{q\alpha+1},$$

and then

$$B_q^* = \left(\frac{1}{N-a} \sum_{k=a}^{N-1} (n)^{q\alpha} \right)^{1/q} \leqslant \frac{1}{(q\alpha+1)^{1/q}} \left(\frac{1}{N-a} \sum_{k=a}^{N-1} \Delta(n)^{q\alpha+1} \right)^{1/q}$$

$$= \frac{1}{(q\alpha+1)^{1/q}} (N^{1+q\alpha} - a^{1+q\alpha})^{1/q}. \tag{62}$$

So as in the proof of the first case, by combining (61) and (62), we have also that

$$(\mathscr{B}^{p,q}(n^\alpha)) = \frac{(1+p\alpha)^{1/p}}{(1+q\alpha)^{1/q}} = \Psi(p,q,\alpha).$$

Arguing in the same way as in the first case, we deduce that $v(n) = n^\alpha$ belongs to the class $\mathscr{B}^{p,q}(\mathscr{C})$ for $0 < \alpha < -1/p$ with the constant in the right hand side $\mathscr{C} = \Psi(p,q,\alpha)$.

From Theorem 4 $v(n) = n^\alpha$ also belongs to $\mathscr{B}^{s,q}(\mathscr{C})$ class for $v_- \leqslant s < p$ and clearly $\alpha = -1/v_-$ is the lower bound of those values for which the which $\mathscr{B}^{p,q}$-norm is finite but $v(n) = n^{-1/v_-}$ does not belong to $\mathscr{B}^{v_-,q}(\mathscr{C})$. This proves the second case.

(3). By defining the function $F(x)$ by

$$F(x) = \frac{(1+px)^{1/p}}{(1+qx)^{1/q}},$$

we see that

$$F'(x) = \frac{d}{dx} \frac{(1+px)^{1/p}}{(1+qx)^{1/q}} = \frac{(px+1)^{\frac{1}{p}}}{(qx+1)^{\frac{1}{q}}} \left[\frac{1}{(px+1)} - \frac{1}{(qx+1)} \right]$$

$$= \frac{(px+1)^{\frac{1}{p}}}{(qx+1)^{\frac{1}{q}}} \left[\frac{(q-p)x}{(px+1)(qx+1)} \right] > 0, \text{ for } x > 0.$$

So that F is an increasing function for $x > 0$ and $q > p > 0$, and then we have that $F(\alpha) < F(\beta)$ if $0 < \alpha < \beta$. This completes the proof of (3). The proof is complete. $\square$

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