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DISTORTION IN THE METRIC CHARACTERIZATION OF SUPERREFLEXIVITY IN TERMS OF THE INFINITE BINARY TREE

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Abstract. The article presents a quantitative refinement of the result of Baudier (*Archiv Math.*, **89** (2007), no. 5, 419–429): the infinite binary tree admits a bilipschitz embedding into an arbitrary non-superreflexive Banach space. According to the results of this paper, we can additionally require that, for an arbitrary $\varepsilon > 0$ and an arbitrary non-superreflexive Banach space X, there is an embedding of the infinite binary tree into X whose distortion does not exceed $4+\varepsilon$.

1. Introduction

One of the important directions in Banach space theory and metric geometry is the theory of characterizations of different isomorphic invariants of Banach spaces in purely metric terms. This direction was initiated in the papers [7] and [8]. Eventually, it became a highly developed direction with many applications, see surveys in [1, 14, 15, 20, 21], and [22, Chapter 13]. This direction is also one of the essential parts of what Bourgain [7] named the *Ribe program*.

To begin with, let us recollect the necessary definitions.

DEFINITION 1. (i) Let (A,d_A) and (Y,d_Y) be metric spaces. Given $0 \leqslant C < \infty$, a map $f:(A,d_A) \to (Y,d_Y)$ is said to be C-*Lipschitz* if, for all $u,v \in A$, the following inequality holds:

$$d_Y(f(u),f(v)) \leqslant C d_A(u,v).$$

A map f is called *Lipschitz* if there is $0 \leqslant C < \infty$ such that f is C-Lipschitz.

(ii) Let $1 \leqslant C < \infty$. A map $f: A \to Y$ is called a C-*bilipschitz embedding* if there exists $r > 0$ such that for all $u,v \in A$, the following inequalities hold:

$$r d_A(u,v) \leqslant d_Y(f(u),f(v)) \leqslant r C d_A(u,v). \tag{1}$$

A map f is a *bilipschitz embedding* if it is C-bilipschitz for some $1 \leqslant C < \infty$. The least constant C for which there exists $r > 0$ such that (1) is satisfied, is called the *distortion* of f.

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DEFINITION 2. Let $\mathscr{P}$ be a class of Banach spaces. A collection of metric spaces $\{M_\lambda\}_{\lambda\in\Lambda}$ is called a collection of *test spaces* for $\mathscr{P}$ provided that a Banach space X satisfies $X \notin \mathscr{P}$ if and only if $\{M_\lambda\}_{\lambda\in\Lambda}$ admit embeddings into X with uniformly bounded distortions.

In the sequel, the following terminology and notation related to the graph theory will be used. A *binary tree of depth* n is a finite graph in which each vertex is represented by a finite (possibly empty) sequence of 0s and 1s of length at most n. Conventionally, it is denoted by T_n. A vertex corresponding to a sequence of length n in T_n is called a *leaf*.

An *infinite binary tree* is an infinite graph in which each vertex is represented by a finite (possibly empty) sequence of 0s and 1s. Such a graph is denoted by T_∞.

For both finite and infinite binary trees, the graph structure is introduced in the following way: Two vertices are adjacent if the sequence corresponding to one of them is obtained from the sequence corresponding to the other by adding one term on the right. For example, vertices corresponding to $(1,0,1,0)$ and $(1,0,1,0,0)$ are adjacent. The vertex corresponding to the empty sequence is called a *root*.

Both finite and infinite binary trees are endowed with the shortest path distance $d(u,v)$, which is the length of the shortest path joining u and v. In this connection, recall that in a tree, there is only one path joining u and v, that is, $d(u,v)$ is the length of this path.

Throughout the paper, B_X and S_X denote the closed unit ball and the unit sphere of a Banach space X, while X^* denotes the dual space of X. We refer to [5, 6, 20] for unexplained terminology.

Recall that a Banach space is called *superreflexive* if and only if it is isomorphic to a uniformly convex space. The first metric characterization of superreflexivity was obtained by Bourgain [7].

THEOREM 1. (Bourgain's theorem) *The set* $\{T_n\}_{n=1}$ *of all finite binary trees is a collection of test spaces for superreflexivity.*

Baudier [2] proved the following result, which strengthens the part of Theorem 1 stating the embeddability of $\{T_n\}_{n=1}^{\infty}$ with uniformly bounded distortion into an arbitrary non-superreflexive Banach space.

THEOREM 2. (Baudier's theorem) *The infinite binary tree is a test space for superreflexivity.*

In his proof of the possibility to embed T_∞ into any non-superreflexive Banach space, Baudier did not attempt to find a sharp estimate for the distortion of such an embedding. The estimate of distortion derived in [2] gives the bound $\leqslant 216+\varepsilon$ for any $\varepsilon > 0$, see the bottom of page 424. The goal of the present work is to prove the result below using the approach developed in [16].

THEOREM 3. *If* X *is a non-superreflexive Banach space and* $\varepsilon > 0$*, then* T_∞ *admits an embedding into* X *with distortion not exceeding* $4+\varepsilon$.

REMARK 1. There exist non-superreflexive Banach spaces which do not admit embeddings of the infinite binary tree of distortion 1 – that is, isometric embeddings. This can be established by comparing the fact that a non-superreflexive space can be strictly convex with the observation from [19, Observation 5.1]: An unweighted graph can admit an isometric embedding into a strictly convex Banach space only if it is either a complete graph or a path.

Note that a non-superreflexive space can be strictly convex since each separable Banach space is isomorphic to a strictly convex Banach space, see [5, p. 175 and Exercise 1 on p. 186].

REMARK 2. It should be pointed out that the problem of lowering the estimate of Theorem 3 to $\leqslant 1+\varepsilon$ can be very challenging. See a related problem [16, Problem 5.1]. It is also of interest to find low-distortion embeddings for the pasting results obtained in [3], [17], and [18].

2. Proof of Theorem 3

Proof. As an initial step, notice that Bourgain's proof [7] of Theorem 1 implies that, for each non-superreflexive Banach space X, each $\varepsilon > 0$, and each $n \in \mathbb{N}$, there is an embedding of T_n into X with distortion $< 1+\varepsilon$. This may not be obvious from reading [7]; as such, we refer the reader to the presentation of Bourgain's result provided in [21, pp. 316–317], which implies that, for each non-superreflexive Banach space X, each $\varepsilon > 0$, and every $n \in \mathbb{N}$, there is a bilipschitz embedding of T_n into X with distortion $< 1+\varepsilon$.

Since the text of [21] does not explain in detail why the map can be selected to have distortion $\leqslant 1+\varepsilon$, an easy-to-follow way toward deriving this estimate is presented below. This approach employs the definition of J-convexity. The notion of J-convexity arises from the important discovery of James [11], and it has been further developed by a few significant results in [9, 12], and [23]. In our method, the fact that a Banach space is superreflexive if and only if it is J-convex will be used.

- Since we consider a non-superreflexive space X, for every $\delta \in (0,1)$, there is a sequence $\{x_i\}_{i=1}^{2^{n+1}-1}$ in B_X which satisfies the following negation of J-convexity (see a detailed explanation in [5, pp. 261–265]):

$$\left\| \sum_{i=1}^{2^{n+1}-1} \theta_i x_i \right\| > (2^{n+1}-1)-\delta,$$

for every collection $\{\theta_i\}_{i=1}^{2^{n+1}-1}$ of ± 1 with only one change of sign in the sequence.

- The previous item implies that

$$\left\| \sum_{i\in A} \theta_i x_i \right\| > |A|-\delta,$$

for every subset A in $\{1,\dots,2^{n+1}-1\}$ and each collection $\{\theta_i\}_{i=1}^{2^{n+1}-1}$ of ± 1 with only one change of sign in the sequence.

- Following Bourgain's construction as described in [21, pp. 316–317], one concludes that the inequality above implies the existence of an embedding of T_n into X with distortion $< 1/(1-\delta)$.

The proof of Theorem 3 follows the same general scheme as the proof of a quantitative version [16, Theorem 1.14] of the Baudier-Lancien theorem [4]. However, to refine the estimate for the distortion, we start with constructing spaces $\{Z_n\}$ which were not needed for [16, Theorem 1.14].

It is assumed that X is separable because each non-superreflexive Banach space contains a separable non-superreflexive subspace. This can be seen, for example, from [5, p. 265, Theorem 3] or [6, Theorem A.6]. In what follows, basic facts about ultraproducts from [20, Section 2.2] and standard notation are used.

Given two Banach spaces W and U, the space W is said to be *finitely representable* in U if, for any $\varepsilon > 0$ and any finite-dimensional subspace $F \subset W$, there exists a finite-dimensional subspace $G \subset U$ such that $d_{BM}(F,G) < 1+\varepsilon$, where d_{BM} is the Banach-Mazur distance.

Let X be a separable non-superreflexive Banach space and $\{f_i\}_{i=1}^{\infty}$ be a weak* dense subset of the sphere S_{X^*}. Let $X_k = \cap_{i=1}^{k} \ker f_i$, $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, and $Y = (\prod X_i)_{\mathscr{U}}$.

LEMMA 1. *The Banach space Y contains an isometric copy of T_∞.*

Proof. It is easy to see from the definition of superreflexivity that if X is non-superreflexive, then every subspace of X of a finite codimension is also non-superreflexive. Consequently, each X_k admits an embedding of T_k with distortion $\leqslant \left(1+\frac{1}{k}\right)$. These embeddings can be chosen in such a way that the images of the roots of T_k for all $k \in \mathbb{N}$ are zero vectors in the corresponding spaces.

Next, for each vertex $v \in T_\infty$, there is $m \in \mathbb{N}$ such that v is a vertex of T_k for $k \geqslant m$. Define the sequence $\{v_k\}_{k=1}^{\infty}$, where v_k is a zero vector when $k < m$, and is the image of v under the embedding of T_k into X_k when $k \geqslant m$. The definition of the ultraproduct implies that the mapping $v \mapsto \{v_k\}_{k=1}^{\infty}$ can be regarded as an isometric embedding of T_∞ into Y. $\square$

LEMMA 2. *If Y is the Banach space as in Lemma 1, then it is finitely representable in any of X_k, $k \in \mathbb{N}$.*

Proof. Since X_i is a subspace of X_j for $i \geqslant j$, the statement follows directly from the proof of [20, Proposition 2.31]. $\square$

At this stage, fix an isometric copy of T_∞ in Y, and assume that its root is at 0. Denote by T_n the corresponding subsets of T_∞ and by Z_n, $n \in \mathbb{N}$, the linear span of vertices of T_n in Y.

LEMMA 3. *For each* $\varepsilon > 0$ *and each subsequence* $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$, *there exists a Banach space* $\mathcal{V}$ *satisfying the following two conditions:*

1. *It has a finite-dimensional decomposition* $\oplus_{i=1}^{\infty} \mathcal{V}_i$ *with summands isometric to* Z_{n_i} *and such that each sub-sum* $\mathcal{V}_j \oplus \mathcal{V}_k$ *is isometric to the* ℓ_1*-sum* $Z_{n_j} \oplus_1 Z_{n_k}$.
2. *It admits a linear embedding into the separable non-superreflexive Banach space* X *with distortion* $< 4 + \varepsilon$.

Proof. We start with constructing in X a subspace V having a finite-dimensional decomposition $V = \oplus_{i=1}^{\infty} V_i$, which is a low-distortion image of $\oplus_{i=1}^{\infty} Z_{n_i}$.

To construct the subspace V, the method which goes back to Mazur [13, p. 4] comes in handy.

DEFINITION 3. Let $\lambda \in (0,1]$. A subspace $N \subset X^*$ is called λ*-norming over a subspace* $E \subset X$ if, for all $y \in E$, there holds:

$$\sup\{|f(y)| : f \in N, \ ||f|| \leqslant 1\} \geqslant \lambda ||y||.$$

LEMMA 4. *For any* $\lambda \in (0,1)$ *and any finite-dimensional subspace* $E \subset X$, *there exists a finite subset* $A \subset \{f_i\}_{i=1}^{\infty}$ *such that the linear span of* A *is* λ*-norming over* E.

Proof. The existence of such a subset can be established as follows. Let $\{x_k\}_{k=1}^{m}$ be a $\frac{1-\lambda}{2}$-net in the unit sphere of E. Since $\{f_i\}_{i=1}^{\infty}$ is weak* dense in S_{X^*}, it follows that, for each $k \in \{1, \dots, m\}$, one can pick $n(k)$ such that $|f_{n(k)}(x_k)| \geqslant 1 - \frac{1-\lambda}{2}$. The verification that the linear span of $A = \{f_{n(k)}\}_{k=1}^{m}$ is λ-norming is plain. $\square$

Let $\varepsilon \in (0,1)$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ be positive numbers satisfying:

$$\prod_{i=1}^{\infty} (1 - \varepsilon_i) > 1 - \varepsilon. \tag{2}$$

Lemma 2 implies that the space X contains a subspace V_1 such that there is a linear map $S_1 : V_1 \to Z_{n_1}$ satisfying

$$||y|| \leqslant ||S_1 y|| \leqslant (1+\varepsilon)||y||, \quad \forall y \in V_1.$$

Consider a finite subset $A_1 \subset \{f_i\}_{i=1}^{\infty}$ so that $N_1 = \mathrm{lin} A_1$ is $(1-\varepsilon_1)$-norming over V_1 and set

$$W_1 = (N_1)_{\top} := \{x \in X : \ \forall x^* \in N_1 \ \ x^*(x) = 0\}.$$

Observe that W_1 contains $X_{k(1)}$, where $k(1) = \max\{i : f_i \in A_1\}$. This allows us to use Lemma 2 once again. As a result, we find a subspace $V_2 \subset W_1$ and a linear map $S_2 : V_2 \to Z_{n_2}$ satisfying

$$||y|| \leqslant ||S_2 y|| \leqslant (1+\varepsilon)||y||, \quad \forall y \in V_2.$$

Consider a finite subset $A_2 \subset \{f_i\}_{i=1}^{\infty}$ so that $A_2 \supset A_1$ and also $N_2 = \mathrm{lin} A_2$ is $(1-\varepsilon_2)$-norming over the linear span of $V_1 \cup V_2$ and set $W_2 = (N_2)_\top$. Observe that W_2 contains $X_{k(2)}$, where $k(2) = \max\{i : f_i \in A_2\}$.

We continue in an obvious way. In the j-th step, we find a subspace

$$V_j \subset W_{j-1} = (N_{j-1})_\top$$

and a linear map $S_j : V_j \to Z_{n_j}$ satisfying

$$||y|| \leqslant ||S_j y|| \leqslant (1+\varepsilon)||y||, \quad \forall y \in V_j.$$

Let finite-dimensional subspace $N_j \subset X^*$ be $(1-\varepsilon_j)$-norming over the linear span of $V_1 \cup \ldots \cup V_j$. It is clear that, for $u \in V_1 \cup \ldots \cup V_j$ and $v \in (N_j)_\top$, the following inequality holds:

$$||u+v|| \geqslant (1-\varepsilon_j)||u||. \tag{3}$$

It is easy to see that subspaces $\{V_i\}_{i=1}^{\infty}$ form a finite-dimensional decomposition of the closed linear span V of $\bigcup_{i=1}^{\infty} V_i$. When writing a sum of the form $\sum_{i=1}^{\infty} y_i$, it is understood by tacit agreement that $y_i \in V_i$. Let us introduce the following norm on V:

$$\left\|\sum_{i=1}^{\infty} y_i\right\|_a = \max\left\{\left\|\sum_{i=1}^{\infty} y_i\right\|_X, \quad \max\{||S_j y_j|| + ||S_k y_k|| : \quad j,k \in \mathbb{N}\}\right\} \tag{4}$$

and verify that $||\cdot||_a$ is $\dfrac{4(1+\varepsilon)}{1-\varepsilon}$-equivalent to $||\cdot||_X$. In fact, it is clear that

$$\left\|\sum_{i=1}^{\infty} y_i\right\|_X \leqslant \left\|\sum_{i=1}^{\infty} y_i\right\|_a.$$

On the other hand, inequality (3) yields:

$$(1-\varepsilon_k)\left\|\sum_{i=1}^{k} y_i\right\|_X \leqslant \left\|\sum_{i=1}^{\infty} y_i\right\|_X$$

and

$$(1-\varepsilon_{k-1})\left\|\sum_{i=1}^{k-1} y_i\right\|_X \leqslant \left\|\sum_{i=1}^{\infty} y_i\right\|_X.$$

By the triangle inequality,

$$||y_k||_X \leqslant \left(\frac{1}{1-\varepsilon_k} + \frac{1}{1-\varepsilon_{k-1}}\right)\left\|\sum_{i=1}^{\infty} y_i\right\|_X.$$

The above-stated equivalence of $\|\cdot\|_a$ and $\|\cdot\|_X$ now follows from $||S_k y_k|| \leqslant (1+\varepsilon)||y_k||_X$ and (2).

Denote V with the norm $\|\cdot\|_a$ by $\mathscr{V}$ and V_i with the norm $\|\cdot\|_a$ by $\mathscr{V}_i$. It is clear that $\mathscr{V}_j \oplus \mathscr{V}_k$ with the norm $\|\cdot\|_a$ is isometric to $Z_{n_j} \oplus_1 Z_{n_k}$. This proves Lemma 3. $\square$

To complete the proof of Theorem 3, it remains to prove that, for each $\varepsilon > 0$, there exists an embedding of T_∞ into $\mathscr{V}$ with distortion $< 1+\varepsilon$.

Recall that we have identified T_∞ with its isometric image M in Y, so that the image of the root is 0.

Let $\{R_i\}_{i=1}^\infty$ be an increasing sequence of positive real numbers. Additional conditions on $\{R_i\}_{i=1}^\infty$ will be imposed later. Consider finite subsets M_i of M defined as:

$$M_i = \{x \in M : \ ||x|| \leqslant R_i\}.$$

Notice that each M_i can be identified with the binary tree T_{n_i}, where $n_i = \lfloor R_i \rfloor$ and $\lfloor \cdot \rfloor$ denotes the integer part of a real number. Taking into account the way in which spaces Z_i were introduced, one can conclude that each M_i admits an isometric embedding E_i into Z_{n_i}. Embedding E_i may be also regarded as an embedding of M_i into the summand of $\mathscr{V}$ that is equal to Z_{n_i}.

The rest of the proof goes along the path laid in [16] with the application of logarithmic spirals, which being quasi-geodesics in $\mathbb{R}^2$ and far from geodesics (see [10, p. 4]) can, after proper modifications, become expedient to construct embeddings with distortion $\leqslant (1+\varepsilon)$.

REMARK 3. The idea behind the proof of Theorem 3 is to find a low-distortion pasting technique for natural isometric embeddings of balls in M with increasing radii into $\mathscr{V}$. This is exactly what is achieved by the forthcoming formulae (8), (9), and (10), which can be regarded as "flows" of ℓ_1-versions of logarithmic spirals.

To execute this idea, let us select an increasing sequence $\{R_i\}_{i=1}^\infty$ of positive real numbers in such a way that

$$R_1 = 1, \tag{5}$$

$$\varepsilon \ln(R_{2i}/R_{2i-1}) = \frac{\pi}{2}, \tag{6}$$

$$\frac{R_{2i+1}}{R_{2i}} \geqslant \frac{1}{\varepsilon}. \tag{7}$$

Further, let c_{2i-1} and s_{2i-1}, $i \in \mathbb{N}$ be real-valued functions on M defined as:

$$c_{2i-1}(x) = \begin{cases} \cos^2(\varepsilon \ln(R_{2i-1}/R_{2i-1})) = 1 & \text{when } ||x|| \leqslant R_{2i-1}, \\ \cos^2(\varepsilon \ln(||x||/R_{2i-1})) & \text{when } R_{2i-1} \leqslant ||x|| \leqslant R_{2i}, \\ \cos^2(\varepsilon \ln(R_{2i}/R_{2i-1})) = 0 & \text{when } ||x|| \geqslant R_{2i}, \end{cases} \tag{8}$$

$$s_{2i-1}(x) = \begin{cases} \sin^2(\varepsilon \ln(R_{2i-1}/R_{2i-1})) = 0 & \text{when } ||x|| \leqslant R_{2i-1}, \\ \sin^2(\varepsilon \ln(||x||/R_{2i-1})) & \text{when } R_{2i-1} \leqslant ||x|| \leqslant R_{2i}, \\ \sin^2(\varepsilon \ln(R_{2i}/R_{2i-1})) = 1 & \text{when } ||x|| \geqslant R_{2i}. \end{cases} \tag{9}$$

Clearly, the equalities in the last lines of formulae (8) and (9) follow from (6). Now, let us introduce the map $D : M \to \oplus_{i=1}^\infty Z_{n_{2i}}$ (considered as a subspace of $\mathscr{V}$), as a casewise function of the form:

$$D(x) = \begin{cases} c_1(x)E_2(x) + s_1(x)E_4(x) & \text{when } x \in M_3, \\ c_3(x)E_4(x) + s_3(x)E_6(x) & \text{when } x \in M_5 \setminus M_3, \\ \dots & \dots \\ c_{2i-1}(x)E_{2i}(x) + s_{2i-1}(x)E_{2i+2}(x) & \text{when } x \in M_{2i+1} \setminus M_{2i-1}, \\ \dots & \dots, \end{cases} \tag{10}$$

where, by default, it is accepted that a product of 0 and an undefined quantity is 0. For example, $E_2(x)$ is not defined for $x \in M_3 \setminus M_2$, but $c_1(x) = 0$ for $x \in M_3 \setminus M_2$, therefore the first term in the first line is regarded as 0 for such x. Obviously, $c_{2i-1}(x) + s_{2i-1}(x) = 1$ for all i and x. Therefore, taking into account (10), equality $E_n 0 = 0$, and the fact that $\mathscr{V}_j \oplus \mathscr{V}_k$ is isometric to the ℓ_1-sum $Z_{n_j} \oplus_1 Z_{n_k}$, one obtains:

$$\forall x \in M \quad ||x|| = ||D(x)||. \tag{11}$$

What is yet to be supplied is an estimate of the form:

$$\forall x, y \in M \quad (1 - \psi(\varepsilon))||x - y|| \leqslant ||D(x) - D(y)|| < (1 + \xi(\varepsilon))||x - y||, \tag{12}$$

where functions ψ and ξ take positive values and comply with

$$\lim_{\varepsilon \downarrow 0} \psi(\varepsilon) = \lim_{\varepsilon \downarrow 0} \xi(\varepsilon) = 0.$$

Evidently, it suffices to look into the case $||y|| \leqslant ||x||$ only. The simpler case $||y|| \leqslant \varepsilon ||x||$ is of no difficulty. Indeed, $||y|| \leqslant \varepsilon ||x||$ implies

$$(1 - \varepsilon)||x|| \leqslant ||x|| - ||y|| \leqslant ||x - y|| \leqslant ||x|| + ||y|| \leqslant (1 + \varepsilon)||x|| \tag{13}$$

and

$$\begin{aligned} (1 - \varepsilon)||x|| \leqslant ||x|| - ||y|| &= ||D(x)|| - ||D(y)|| \\ &\leqslant ||D(x) - D(y)|| \leqslant ||D(x)|| + ||D(y)|| \\ &= ||x|| + ||y|| \leqslant (1 + \varepsilon)||x||. \end{aligned} \tag{14}$$

Combining (13) and (14) yields:

$$\frac{1 - \varepsilon}{1 + \varepsilon} ||x - y|| \leqslant ||D(x) - D(y)|| \leqslant \frac{1 + \varepsilon}{1 - \varepsilon} ||x - y||, \tag{15}$$

which is exactly an estimate of the desired type (12).

Set $R_0 = 0$. By condition (7) and inequality (15), one may focus only on the case where

$$R_{2i-2} \leqslant ||y|| \leqslant ||x|| \leqslant R_{2i+1}, \quad i = 1, 2, \dots. \tag{16}$$

We begin with the case $R_{2i-1} \leqslant ||y|| \leqslant ||x|| \leqslant R_{2i}$.

In what follows, it is convenient – for the sake of transparency in the calculations – to write c for c_{2i-1}, s for s_{2i-1} as well as E for E_{2i}, and F for E_{2i+2}. Having stated so, we write:

$$\begin{aligned} ||D(x)-D(y)|| &= ||c(x)E(x)-c(y)E(y)||+||s(x)F(x)-s(y)F(y)|| \\ &= ||c(x)(E(x)-E(y))+(c(x)-c(y))E(y)|| \\ &\quad +||s(x)(F(x)-F(y))+(s(x)-s(y))F(y)||. \end{aligned} \tag{17}$$

We shall dwell longer upon each of the summands in (17). In the first instance, the Mean Value Theorem implies that, for some number $\tau \in (||y||,||x||)$, the inequality below is true:

$$\begin{aligned} &c(x)-c(y) = \cos^2(\varepsilon \ln(||x||/R_{2i-1})) - \cos^2(\varepsilon \ln(||y||/R_{2i-1})) \\ &= 2\cos(\varepsilon \ln(\tau/R_{2i-1}))\cdot(-\sin(\varepsilon \ln(\tau/R_{2i-1})))\cdot \varepsilon \frac{1}{\tau}(||x||-||y||). \end{aligned} \tag{18}$$

Consequently,

$$||(c(x)-c(y))E(y)|| \leqslant 2\varepsilon \frac{1}{\tau}(||x||-||y||)\cdot ||y|| \leqslant 2\varepsilon ||x-y||. \tag{19}$$

In the same way, it can be established that

$$||(s(x)-s(y))F(y)|| \leqslant 2\varepsilon ||x-y||. \tag{20}$$

Collecting inequalities (17), (19), and (20), one arrives at:

$$\begin{aligned} &(\max\{c(x)-2\varepsilon,0\}+\max\{s(x)-2\varepsilon,0\})||x-y|| \\ &\qquad \leqslant ||D(x)-D(y)|| \leqslant ((c(x)+2\varepsilon)+(s(x)+2\varepsilon))||x-y||. \end{aligned} \tag{21}$$

In addition, since $c(x), s(x) \geqslant 0$ and $c(x)+s(x)=1$, the next equalities are valid:

$$\lim_{\varepsilon \downarrow 0}(\max\{c(x)-2\varepsilon,0\}+\max\{s(x)-2\varepsilon,0\}) = 1$$

and

$$\lim_{\varepsilon \downarrow 0}((c(x)+2\varepsilon)+(s(x)+2\varepsilon)) = 1,$$

whence the targeted estimate (12) is justified by inequality (21).

The remaining subcases of the case $R_{2i-2} \leqslant ||y|| \leqslant ||x|| \leqslant R_{2i+1}$ can be considered using similar formulae as in the case above, and the observation, that for $\|x\| \in [R_{2i-2}, R_{2i-1}]$, $i \geqslant 2$, the formulae

$$D(x) = c_{2i-1}(x)E_{2i}(x)+s_{2i-1}(x)E_{2i+2}(x) = E_{2i}(x) \tag{22}$$

and

$$D(x) = c_{2i-3}(x)E_{2i-2}(x)+s_{2i-3}(x)E_{2i}(x) = E_{2i}(x)$$

give the same result.

To illustrate this statement, consider the case where $||y|| \in [R_{2i-2}, R_{2i-1}]$ and $||x|| \in [R_{2i-1}, R_{2i}]$. By the observation just made, in this case we may use (22) with $c_{2i-1}(y) = \cos^2(\varepsilon \ln(R_{2i-1}/R_{2i-1}))$, which – with the help of the reasoning as in (18) and in the left inequality of (19) – reveals that, for some number $\tau \in (R_{2i-1}, ||x||)$, there holds:

$$||(c(x)-c(y))E(y)|| \leqslant 2 \cdot \varepsilon \frac{1}{\tau}(||x|| - R_{2i-1}) \cdot ||y||.$$

Hence, inequality

$$||(c(x)-c(y))E(y)|| \leqslant 2\varepsilon ||x-y||$$

is true in this case, too. In the same manner, estimate (20) can be established. All of the other subcases of

$$R_{2i-2} \leqslant ||y|| \leqslant ||x|| \leqslant R_{2i+1}$$

can be analyzed by means of similar arguments. The idea is that in the case when either x or y is in the range where functions c and s have values 0 and 1, we may use the same estimates as in the case $R_{2i-1} \leqslant ||y|| \leqslant ||x|| \leqslant R_{2i}$ replacing $\|x\|$ or $\|y\|$ by the corresponding R_j in some places. $\square$

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