AVERAGE SAMPLING AND RECONSTRUCTION IN SHIFT–INVARIANT SUBSPACES OF MIXED LEBESGUE SPACE $L^{p,q}(\mathbb{R}^{d+1})$

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Abstract. In this paper, we mainly study the average sampling and reconstruction for signals in a shift-invariant subspace of mixed Lebesgue space $L^{p,q}(\mathbb{R}^{d+1})$ with the generator belonging to a mixed Wiener amalgam space. First, the sampling stability for two kinds of average sampling functionals are considered. Second, two kinds of iterative approximation projection reconstruction algorithms with exponential convergence are utilized for recovering the corresponding signals. Finally, error estimations are also considered under three different conditions.

1. Introduction

Mixed Lebesgue space arose from considering some functions which depended on independent quantities with different properties. It was first described in [3] and was further studied in [4, 5, 6, 7]. Compared with the classical Lebesgue space which makes it compulsory for all variables to have the same properties, the mixed Lebesgue space which could realize the separate integrability of each variable has a greater advantage in modelling and measuring some time-varying signals, especially for the time-spatial signals. And this advantage also provides some flexibility for the study of time-based partial differential equations [8].

The mixed Lebesgue space $L^{p,q}(\mathbb{R}^{d+1})$ consists of all measurable functions $f = f(x,y)$ defined on $\mathbb{R} \times \mathbb{R}^d$ such that

$$\|f\|_{L^{p,q}} = \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |f(x,y)|^q dy \right)^{p/q} dx \right]^{1/p} < \infty, \ \ 1 \leqslant p,q < \infty. \tag{1}$$

The corresponding sequence space is defined by

$$\ell^{p,q}(\mathbb{Z}^{d+1}) = \left\{ c : \|c\|_{\ell^{p,q}}^p = \sum_{k_1 \in \mathbb{Z}} \left(\sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|^q \right)^{p/q} < \infty \right\}, \ \ 1 \leqslant p,q < \infty. \tag{2}$$

The case for $p = \infty$ or $q = \infty$ obeys the usual adjustment. Obviously, $L^{p,p}(\mathbb{R}^{d+1}) = L^p(\mathbb{R}^{d+1})$, $\ell^{p,p}(\mathbb{Z}^{d+1}) = \ell^p(\mathbb{Z}^{d+1})$.

As a fundamental problem in signal processing, the sampling problem in mixed Lebesgue space has already attracted a lot of researchers' attention and many marvelous sampling findings of the signals in various subspaces of mixed Lebesgue space have already been presented such as the bandlimited space [13] and the shift-invariant space [8]. However, as we have learned from the previous sampling results of the signals in the classical Lebesgue space [1, 2, 11, 12], we need to overcome a number of shortcomings about the existing results in order to meet the realistic requirements. For example, the well-known Shannon sampling theorem once told us that signals f belonging to the band-limited subspaces of L^2 could be recovered from the uniform samples $\{f(n\delta)\}_{n\in\mathbb{Z}}$ if the sampling gap δ was sufficient small [10, 14]. Nevertheless, the existence of the non-bandlimited signals and the fact that some sampling data will be lost during the transmission which finally leads to the obtained sampling set is nonuniform, make it impractical in reality. Taking these two factors into account, the researchers prefer to study the nonuniform sampling problems for signals in more general subspaces of Lebesgue space such that the corresponding sampling results could be suitable for more signals in practice. Inspired by this thought, the investigation of the nonuniform ideal sampling problem in shift-invariant subspaces of mixed Lebesgue space has already been considered in [8] under the assumption that the generator is subject to a strong constraint condition.

In this paper, we mainly study the sampling and reconstruction in shift-invariant subspace

$$V_{p,q}(\varphi) = \left\{ \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1,y-k_2) : \{c(k_1,k_2)\}_{k_1\in\mathbb{Z},k_2\in\mathbb{Z}^d} \in \ell^{p,q}(\mathbb{Z}^{d+1}) \right\}$$
(3)

with the generator φ satisfying

(A1) The generator φ belongs to a mixed Wiener amalgam space $W(L^{1,1})(\mathbb{R}^{d+1})$, whose general forms are defined as

$$\|f\|_{W(L^{p,q})} := \left(\sum_{n\in\mathbb{Z}} \sup_{x\in[0,1]} \left[\sum_{l\in\mathbb{Z}^d} \sup_{y\in[0,1]^d} |f(x+n,y+l)|^q \right]^{p/q} \right)^{1/p} < \infty;$$

(A2) $\lim_{\delta\to 0} \|\omega_\delta(\varphi)\|_{W(L^{1,1})} = 0$, where the modulus of continuity is defined by

$$\omega_\delta(\varphi)(x,y) = \sup_{|s|\leqslant\delta,|t|\leqslant\delta} |\varphi(x+s,y+t) - \varphi(x,y)|;$$

(A3) $\sum_{k\in\mathbb{Z}^{d+1}} |\widehat{\varphi}(\xi+2k\pi)|^2 > 0, \ \xi\in\mathbb{R}^{d+1};$

(A4) There exits $\delta_0 > 0$, such that $\omega_\delta(\varphi)\in W(L^{1,1})$, for any $\delta\leqslant\delta_0$.

In fact, if the continuous function φ is in the classical Wiener amalgam space $W(L^1)(\mathbb{R}^{d+1})$, which means that

$$\|\varphi\|_{W(L^1)} = \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}^d} \sup_{(x,y)\in[0,1]^{d+1}} |\varphi(x+k_1,y+k_2)| < \infty,$$

then the assumption $(A2)$ holds due to

$$\lim_{\delta \to 0} \|\omega_\delta(\varphi)\|_{W(L^{1,1})} \leqslant \lim_{\delta \to 0} \|\omega_\delta(\varphi)\|_{W(L^1)} = 0.$$

In this paper, the sampling set $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j, k \in \mathscr{J}\} \subset \mathbb{R}^{d+1}$ is assumed to be relatively-separated for both variables, that is,

$$B_{\Gamma,x}(\delta_1) := \sup_{x \in \mathbb{R}} \sum_{j \in \mathscr{J}} \chi_{B(x_j, \delta_1)}(x) < \infty$$

and

$$B_{\Gamma,y}(\delta_2) := \sup_{y \in \mathbb{R}^d} \sum_{k \in \mathscr{J}} \chi_{B(y_k, \delta_2)}(y) < \infty$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. Furthermore, $\delta_1 > 0$ and $\delta_2 > 0$ are said to be gaps of Γ if

$$A_{\Gamma,x}(\delta_1) := \inf_{x \in \mathbb{R}} \sum_{j \in \mathscr{J}} \chi_{B(x_j, \delta_1)}(x) \geqslant 1$$

and

$$A_{\Gamma,y}(\delta_2) := \inf_{y \in \mathbb{R}^d} \sum_{k \in \mathscr{J}} \chi_{B(y_k, \delta_2)}(y) \geqslant 1.$$

Here, $\mathscr{J}$ is a countable index set, $B(x, \delta)$ and $B(y, \delta)$ are balls in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

Moreover, due to the limitation of the sampling devices, the obtained sampling value in reality is not the exact value of signal at each sampling point but is the local average value near the corresponding sampling location. Thus, for a given relatively-separated sampling set Γ, we will consider two kinds of average sampling schemes for obtaining the corresponding sampling values. The first average sampling scheme is

$$\langle f, \psi_{j,k} \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(x, y) \psi_{j,k}(x, y) dx dy, \ j, k \in \mathscr{J},$$

where the average sampling functionals $\{\psi_{j,k} : j, k \in \mathscr{J}\}$ satisfy

(i) $\int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi_{j,k}(x, y) dx dy = 1$ for all $j, k \in \mathscr{J}$;

(ii) There exists a constant $M > 0$ such that $\int_{\mathbb{R}} \int_{\mathbb{R}^d} |\psi_{j,k}(x, y)| dx dy \leqslant M$ for all $j, k \in \mathscr{J}$;

(iii) $supp\, \psi_{j,k} \subset B(\gamma_{j,k}, a)$ for some $a > 0$.

Note that the first sampling scheme requires that the sampling functionals are compact support. Here, we also consider the second average scheme which is defined as

$$\langle f, \psi_a(\cdot - \gamma_{j,k}) \rangle = f * \psi_a^*(\gamma_{j,k}), \ j, k \in \mathscr{J},$$

where $\psi \in L^1(\mathbb{R}^{d+1})$ satisfies $\int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi(x, y) dx dy = 1$, $\psi_a(\cdot) = \frac{1}{a^{d+1}} \psi(\frac{\cdot}{a})$ and $\psi_a^*(\cdot) = \overline{\psi_a(-\cdot)}$.

This paper is organized as follows. In section 2, some lemmas are given. In section 3, the sampling stability for two kinds of average sampling functionals are established. In section 4, two kinds of iterative approximation projection reconstruction algorithms are considered for recovering the signals in $V_{p,q}(\varphi)$ from the corresponding average samples. In section 5, three kinds of error estimations under different conditions are also provided.

2. Some lemmas

In this section, we will give some lemmas which are the basis of the subsequent sections.

LEMMA 1. [9] *Let* $\varphi \in W(L^{1,1})(\mathbb{R}^{d+1})$. *Then* φ *satisfies*

$$\sum_{k \in \mathbb{Z}^{d+1}} |\widehat{\varphi}(\xi + 2k\pi)|^2 > 0, \; \xi \in \mathbb{R}^{d+1}$$

if and only if there exists a function

$$g(x,y) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} d(k_1,k_2)\varphi(x-k_1,y-k_2) \tag{4}$$

such that

$$\langle \varphi(\cdot - \alpha), g \rangle = \delta_{0,\alpha},$$

where $d = \{d(k_1,k_2) : k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^d\} \in \ell^1(\mathbb{Z}^{d+1})$.

It has been proved in [8] that $g \in W(L^{1,1})(\mathbb{R}^{d+1})$.

LEMMA 2. [8] *Let* $1 \leqslant p,q \leqslant \infty$. *Suppose that* φ *satisfies the assumptions* $(A1)$ *and* $(A3)$, *then any signal* $f(x,y) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c(k_1,k_2)\varphi(x-k_1,y-k_2) \in V_{p,q}(\varphi)$ *belongs to* $L^{p,q}(\mathbb{R}^{d+1})$. *Moreover, there is the following norm equivalent*

$$\|g\|_{W(L^{1,1})}^{-1}\|c\|_{\ell^{p,q}} := C_{\varphi}\|c\|_{\ell^{p,q}} \leqslant \|f\|_{L^{p,q}} \leqslant \|\varphi\|_{W(L^{1,1})}\|c\|_{\ell^{p,q}}. \tag{5}$$

In fact, [8] only established the lower bound of (5) for $1 < p,q < \infty$. However, the above result still holds for $p,q = 1$ or ∞, which contain eight cases needed to be verified. In the following, we only give the proof of four cases, the remained can be proved similarly.

Proof. According to Lemma 1, we know

$$c(k_1,k_2) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(x,y)\overline{g(x-k_1,y-k_2)}dxdy, \; k_1 \in \mathbb{Z}, \; k_2 \in \mathbb{Z}^d.$$

(i) $p = 1$, $q = 1$

$$\|c\|_{\ell^{1,1}} \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x,y)| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| dxdy$$

$$\leqslant \left(\sup_{x \in [0,1]} \sup_{y \in [0,1]^d} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| \right) \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x,y)| dxdy$$

$$\leqslant \|g\|_{W(L^{1,1})} \|f\|_{L^{1,1}}.$$

(ii) $p = 1$, $1 < q < \infty$

For the fixed $k_1 \in \mathbb{Z}$, let $b_{k_1}(k_2) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x,y)| |g(x-k_1, y-k_2)| dxdy$, $k_2 \in \mathbb{Z}^d$. Taking $\alpha = \{\alpha(k_2)\}_{k_2 \in \mathbb{Z}^d} \in \ell^{q'}(\mathbb{Z}^d)$, $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$|\langle \alpha, b_{k_1} \rangle| = \left| \sum_{k_2 \in \mathbb{Z}^d} \alpha(k_2) \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x,y)| |g(x-k_1, y-k_2)| dxdy \right|$$

$$\leqslant \int_{\mathbb{R}} \|f(x, \cdot)\|_{L^q} \left\| \sum_{k_2 \in \mathbb{Z}^d} \alpha(k_2) |g(x-k_1, \cdot - k_2)| \right\|_{L^{q'}} dx. \tag{6}$$

Furthermore, we have

$$\left\| \sum_{k_2 \in \mathbb{Z}^d} \alpha(k_2) |g(x-k_1, \cdot - k_2)| \right\|_{L^{q'}}^{q'}$$

$$\leqslant \int_{\mathbb{R}^d} \left(\sum_{k_2 \in \mathbb{Z}^d} |\alpha(k_2)| |g(x-k_1, y-k_2)| \right)^{q'} dy$$

$$\leqslant \int_{\mathbb{R}^d} \left(\sum_{k_2 \in \mathbb{Z}^d} |\alpha(k_2)|^{q'} |g(x-k_1, y-k_2)| \right) \left(\sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| \right)^{q'/q} dy$$

$$\leqslant \left(\sup_{y \in [0,1]^d} \sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| \right)^{q'/q} \sum_{k_2 \in \mathbb{Z}^d} |\alpha(k_2)|^{q'} \int_{\mathbb{R}^d} |g(x-k_1, y-k_2)| dy$$

$$\leqslant \|\alpha\|_{\ell^{q'}}^{q'} \left(\sup_{y \in [0,1]^d} \sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| \right)^{1 + \frac{q'}{q}}.$$

This together with (6) gives

$$|\langle \alpha, b_{k_1} \rangle| \leqslant \|\alpha\|_{\ell^{q'}} \int_{\mathbb{R}} \|f(x, \cdot)\|_{L^q} \left(\sup_{y \in [0,1]^d} \sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| \right) dx.$$

Therefore,

$$\|b_{k_1}\|_{\ell^q} \leqslant \int_{\mathbb{R}} \|f(x, \cdot)\|_{L^q} \left(\sup_{y \in [0,1]^d} \sum_{k_2 \in \mathbb{Z}^d} |g(x-k_1, y-k_2)| \right) dx$$

and

$$\|c\|_{\ell^{1,q}} = \sum_{k_1 \in \mathbb{Z}} \|b_{k_1}\|_{\ell^q}$$

$$\leqslant \int_{\mathbb{R}} \|f(x,\cdot)\|_{L^q} \sum_{k_1 \in \mathbb{Z}} \Big(\sup_{y \in [0,1]^d} \sum_{k_2 \in \mathbb{Z}^d} |g(x - k_1, y - k_2)| \Big) dx$$

$$\leqslant \|g\|_{W(L^{1,1})} \|f\|_{L^{1,q}}.$$

(iii) $p = 1$, $q = \infty$

$$\|c\|_{\ell^{1,\infty}} \leqslant \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} \sup \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x,y)||g(x - k_1, y - k_2)| dx dy$$

$$\leqslant \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \Big(\sup_{y \in \mathbb{R}^d} |f(x,y)| \Big) \Big(\int_{[0,1]^d} \sum_{l \in \mathbb{Z}^d} |g(x - k_1, y - l)| dy \Big) dx$$

$$\leqslant \int_{\mathbb{R}} \Big(\sup_{y \in \mathbb{R}^d} |f(x,y)| \Big) \Big(\sum_{k_1 \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x - k_1, y - l)| \Big) dx$$

$$= \sum_{l' \in \mathbb{Z}} \int_{[0,1]} \Big(\sup_{y \in \mathbb{R}^d} |f(x + l', y)| \Big) \Big(\sum_{k_1 \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x + l' - k_1, y - l)| \Big) dx$$

$$\leqslant \int_{[0,1]} \Big(\sum_{l' \in \mathbb{Z}} \sup_{y \in \mathbb{R}^d} |f(x + l', y)| \Big) \Big(\sum_{l' \in \mathbb{Z}} \sum_{k_1 \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x + l' - k_1, y - l)| \Big) dx$$

$$\leqslant \Big(\sum_{k_1 \in \mathbb{Z}} \sup_{x \in [0,1]} \sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x - k_1, y - l)| \Big) \sum_{l' \in \mathbb{Z}} \int_{[0,1]} \sup_{y \in \mathbb{R}^d} |f(x + l', y)| dx$$

$$= \|g\|_{W(L^{1,1})} \|f\|_{L^{1,\infty}}.$$

(iv) $p = \infty$, $q = \infty$

$$\|c\|_{\ell^{\infty,\infty}} \leqslant \sup_{k_1 \in \mathbb{Z}} \sup_{k_2 \in \mathbb{Z}^d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x,y)||g(x - k_1, y - k_2)| dx dy$$

$$\leqslant \sup_{k_1 \in \mathbb{Z}} \sup_{k_2 \in \mathbb{Z}^d} \int_{\mathbb{R}} \Big(\sup_{y \in \mathbb{R}^d} |f(x,y)| \Big) \Big(\sum_{l \in \mathbb{Z}^d} \int_{[0,1]^d} |g(x - k_1, y - k_2 + l)| dy \Big) dx$$

$$\leqslant \sup_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \Big(\sup_{y \in \mathbb{R}^d} |f(x,y)| \Big) \Big(\sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x - k_1, y - l)| \Big) dx$$

$$\leqslant \|f\|_{L^{\infty,\infty}} \sum_{l' \in \mathbb{Z}} \int_{[0,1]} \Big(\sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x - l', y - l)| \Big) dx$$

$$\leqslant \|f\|_{L^{\infty,\infty}} \sum_{l' \in \mathbb{Z}} \sup_{x \in [0,1]} \sum_{l \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |g(x - l', y - l)|$$

$$= \|f\|_{L^{\infty,\infty}} \|g\|_{W(L^{1,1})}. \quad \square$$

LEMMA 3. *Suppose that* φ *satisfies the assumptions* (A1), (A2) *and* (A4). *If* $\psi \in L^1(\mathbb{R}^{d+1})$ *satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi(x,y) dx dy = 1,$$

then $\lim_{a\to 0}\|\varphi^a\|_{W(L^{1,1})}=0$, *where* $\varphi^a(x,y):=\varphi(x,y)-\varphi*\psi_a^*(x,y)$.

Proof. Since $\lim_{\delta\to 0}\|\omega_\delta(\varphi)\|_{W(L^{1,1})}=0$, then for any $\varepsilon>0$, there exists a $\delta'>0$ $(\delta'<\delta_0)$, such that

$$\|\omega_\delta(\varphi)\|_{W(L^{1,1})}<\varepsilon,\ \forall\,\delta\leqslant\delta'.$$

Note that

$$\|\varphi^a\|_{W(L^{1,1})}$$
$$\leqslant\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\varphi(x+k_1,y+k_2)-\varphi(x+k_1+s,y+k_2+t)||\psi_a(s,t)|dsdt$$
$$\leqslant\left(\iint_{\sqrt{s^2+|t|^2}\leqslant\delta'}+\iint_{\delta'\leqslant\sqrt{s^2+|t|^2}\leqslant\delta_0}+\iint_{\sqrt{s^2+|t|^2}\geqslant\delta_0}\right)$$
$$\times\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}|\varphi(x+k_1,y+k_2)-\varphi(x+k_1+s,y+k_2+t)||\psi_a(s,t)|dsdt$$
$$:=I+II+III.\tag{7}$$

Now we respectively estimate I, II and III.

$$I=\iint_{\sqrt{s^2+|t|^2}\leqslant\delta'}\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}|\varphi(x+k_1,y+k_2)-\varphi(x+k_1+s,y+k_2+t)|$$
$$\times|\psi_a(s,t)|dsdt$$
$$\leqslant\iint_{\sqrt{s^2+|t|^2}\leqslant\delta'}\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\omega_{\delta'}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|dsdt$$
$$=\iint_{\sqrt{s^2+|t|^2}\leqslant\delta'}\|\omega_{\delta'}(\varphi)\|_{W(L^{1,1})}|\psi_a(s,t)|dsdt$$
$$\leqslant\varepsilon\iint_{\sqrt{s^2+|t|^2}\leqslant\delta'}|\psi_a(s,t)|dsdt\leqslant\varepsilon\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\psi(s,t)|dsdt\leqslant\varepsilon\|\psi\|_{L^1}.\tag{8}$$

$$II=\iint_{\delta'\leqslant\sqrt{s^2+|t|^2}\leqslant\delta_0}\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}|\varphi(x+k_1,y+k_2)-\varphi(x+k_1+s,y+k_2+t)|$$
$$\times|\psi_a(s,t)|dsdt$$
$$\leqslant\iint_{\delta'\leqslant\sqrt{s^2+|t|^2}\leqslant\delta_0}\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\omega_{\delta_0}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|dsdt$$
$$\leqslant\iint_{\delta'\leqslant\sqrt{s^2+|t|^2}}\|\omega_{\delta_0}(\varphi)\|_{W(L^{1,1})}|\psi_a(s,t)|dsdt\to 0\quad\text{as}\quad a\to 0.\tag{9}$$

Let

$$\mathscr{Z}_0^{d+1} := \left\{ (\ell_1, \ell_2) \in \mathbb{Z} \times \mathbb{Z}^d : \right.$$
$$\left. \sqrt{(\ell_1+s)^2 + |\ell_2+t|^2} \geqslant \delta_0, \text{ for all } s \in [0,1] \text{ and } t \in [0,1]^d \right\}.$$

$$III = \iint\limits_{\sqrt{s^2+|t|^2} \geqslant \delta_0} \sum_{k_1 \in \mathbb{Z}} \sup_{x \in [0,1]} \sum_{k_2 \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |\varphi(x+k_1, y+k_2) - \varphi(x+k_1+s, y+k_2+t)|$$
$$\times |\psi_a(s,t)| \, ds \, dt$$

$$\leqslant \iint\limits_{\sqrt{s^2+|t|^2} \geqslant \delta_0} \sum_{k_1 \in \mathbb{Z}} \sup_{x \in [0,1]} \sum_{k_2 \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |\varphi(x+k_1, y+k_2)| |\psi_a(s,t)| \, ds \, dt$$

$$+ \sum_{(\ell_1,\ell_2) \in \mathscr{Z}_0^{d+1}} \int_{[0,1]} \int_{[0,1]^d} \left(\sum_{k_1 \in \mathbb{Z}} \sup_{x \in [0,1]} \sum_{k_2 \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} |\varphi(x+k_1+\ell_1+s, y+k_2+\ell_2+t)| \right)$$
$$\times |\psi_a(s+\ell_1, t+\ell_2)| \, ds \, dt$$

$$\leqslant \|\varphi\|_{W(L^{1,1})} \iint\limits_{\sqrt{s^2+|t|^2} \geqslant \delta_0} |\psi_a(s,t)| \, ds \, dt$$

$$+ \sum_{(\ell_1,\ell_2) \in \mathscr{Z}_0^{d+1}} \left(\sum_{k_1 \in \mathbb{Z}} \sup_{x \in [0,1]} \sup_{s \in [0,1]} \sum_{k_2 \in \mathbb{Z}^d} \sup_{y \in [0,1]^d} \sup_{t \in [0,1]^d} |\varphi(x+k_1+s, y+k_2+t)| \right)$$
$$\times \int_{[0,1]} \int_{[0,1]^d} |\psi_a(s+\ell_1, t+\ell_2)| \, ds \, dt$$

$$\leqslant (1+2^{d+1}) \|\varphi\|_{W(L^{1,1})} \iint\limits_{\sqrt{s^2+|t|^2} \geqslant \delta_0/a} |\psi(s,t)| \, ds \, dt \to 0 \quad \text{as} \quad a \to 0. \tag{10}$$

Combining (8), (9) and (10), we could obtain $\lim\limits_{a \to 0} \|\varphi^a\|_{W(L^{1,1})} = 0$. $\quad\square$

LEMMA 4. *Let φ and ψ be as in Lemma 3. Then*

$$\lim_{\delta \to 0} \|\omega_\delta(\varphi^a)\|_{W(L^{1,1})} = 0.$$

Proof. Direct computation gives

$$\omega_\delta(\varphi^a)(x,y)$$
$$\leqslant \omega_\delta(\varphi)(x,y) + \int_{\mathbb{R}} \int_{\mathbb{R}^d} \sup_{|s| \leqslant \delta, |t| \leqslant \delta} |\varphi(x+s-u, y+t-v) - \varphi(x-u, y-v)| |\psi_a^*(u,v)| \, du \, dv$$
$$\leqslant \omega_\delta(\varphi)(x,y) + \int_{\mathbb{R}} \int_{\mathbb{R}^d} \omega_\delta(\varphi)(x-u, y-v) |\psi_a^*(u,v)| \, du \, dv$$
$$= \omega_\delta(\varphi)(x,y) + \omega_\delta(\varphi) * |\psi_a^*(\cdot,\cdot)|(x,y).$$

Now, it is enough to prove $\lim\limits_{\delta\to 0}\|\omega_\delta(\varphi)*|\psi_a^*|\|_{W(L^{1,1})}=0$. In fact, one has

$$\|\omega_\delta(\varphi)*|\psi_a^*|\|_{W(L^{1,1})}$$

$$\leqslant \sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\omega_\delta(\varphi)(x+k_1-s,y+k_2-t)\big|\psi_a^*(s,t)\big|dsdt$$

$$\leqslant \sum_{\ell_1\in\mathbb{Z}}\sum_{\ell_2\in\mathbb{Z}^d}\int_{[0,1]}\int_{[0,1]^d}\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\omega_\delta(\varphi)(x+k_1-s-\ell_1,y+k_2-t-\ell_2)$$

$$\times \big|\psi_a^*(s+\ell_1,t+\ell_2)\big|dsdt$$

$$\leqslant \sum_{\ell_1\in\mathbb{Z}}\sum_{\ell_2\in\mathbb{Z}^d}\sup_{s\in[0,1]}\sup_{t\in[0,1]^d}\Big(\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\omega_\delta(\varphi)(x+k_1-s-\ell_1,y+k_2-t-\ell_2)\Big)$$

$$\times \int_{[0,1]}\int_{[0,1]^d}|\psi_a^*(s+\ell_1,t+\ell_2)|dsdt$$

$$\leqslant \sum_{\ell_1\in\mathbb{Z}}\sum_{\ell_2\in\mathbb{Z}^d}\Big(\sum_{k_1\in\mathbb{Z}}\sup_{x\in[0,1]}\sup_{s\in[0,1]}\sum_{k_2\in\mathbb{Z}^d}\sup_{y\in[0,1]^d}\sup_{t\in[0,1]^d}\omega_\delta(\varphi)(x+k_1-s,y+k_2-t)\Big)$$

$$\times \int_{[0,1]}\int_{[0,1]^d}|\psi_a^*(s+\ell_1,t+\ell_2)|dsdt$$

$$\leqslant 2^{d+1}\|\omega_\delta(\varphi)\|_{W(L^{1,1})}\|\psi\|_{L^1}\to 0 \quad\text{as}\quad \delta\to 0. \quad\square$$

3. Sampling stability

In this section, we will establish the results of sampling stability for two kinds of average sampling functionals.

THEOREM 1. *Let* $1\leqslant p,q\leqslant\infty$ *and* φ *satisfy the assumptions* $(A1)-(A4)$. *Suppose that* $\Gamma=\{\gamma_{j,k}=(x_j,y_k):x_j\in\mathbb{R},y_k\in\mathbb{R}^d,j,k\in\mathscr{J}\}$ *is a relatively-separated set with gaps* δ_1 *and* δ_2 *for both variables and* $\{\psi_{j,k}:j,k\in\mathscr{J}\}$ *is the first average sampling functional with support radius* a. *If* $\rho_1=max\{\delta_1,\delta_2\}$ *and* a *are chosen such that*

$$r_1:=MC_\varphi^{-1}\|\omega_{a+\rho_1}(\varphi)\|_{W(L^{1,1})}<1, \tag{11}$$

then signals $f\in V_{p,q}(\varphi)$ *can be stably reconstructed from the average samples* $\{\langle f,\psi_{j,k}\rangle\}_{j,k\in\mathscr{J}}$. *Furthermore,*

$$\Big(\frac{2\delta_1}{A_{\Gamma,x}(\delta_1)}\Big)^{-1/p}\Big(\frac{V_d\delta_2^d}{A_{\Gamma,y}(\delta_2)}\Big)^{-1/q}(1-r_1)\|f\|_{L^{p,q}}\leqslant\big\|\{\langle f,\psi_{j,k}\rangle\}_{j,k\in\mathscr{J}}\big\|_{\ell^{p,q}}$$

$$\leqslant\Big(\frac{2\delta_1}{B_{\Gamma,x}(\delta_1)}\Big)^{-1/p}\Big(\frac{V_d\delta_2^d}{B_{\Gamma,y}(\delta_2)}\Big)^{-1/q}(1+r_1)\|f\|_{L^{p,q}}, \tag{12}$$

where $V_d=\frac{\pi^{d/2}}{\Gamma(d/2+1)}$ *is the volume of* d-*dimensional unit sphere.*

Proof. For any $x \in B(x_j, \delta_1)$ and $y \in B(y_k, \delta_2)$, one has

$$|\langle f, \psi_{j,k} \rangle - f(x,y)| \leqslant \iint_{B(\gamma_{j,k},a)} |f(t,s) - f(x,y)||\psi_{j,k}(t,s)|dtds$$
$$\leqslant M\omega_{a+\rho_1}(f)(x,y) := F_1(x,y). \tag{13}$$

Furthermore, for any $f(x,y) = \sum\limits_{k_1 \in \mathbb{Z}} \sum\limits_{k_2 \in \mathbb{Z}^d} c(k_1, k_2)\varphi(x-k_1, y-k_2) \in V_{p,q}(\varphi)$,

$$\|F_1\|_{L^{p,q}} \leqslant M\|\sum\limits_{k_1 \in \mathbb{Z}} \sum\limits_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_{a+\rho_1}(\varphi)(x-k_1, y-k_2)\|_{L^{p,q}}$$
$$\leqslant M\|c\|_{\ell^{p,q}}\|\omega_{a+\rho_1}(\varphi)\|_{W(L^{1,1})}$$
$$\leqslant MC_\varphi^{-1}\|\omega_{a+\rho_1}(\varphi)\|_{W(L^{1,1})}\|f\|_{L^{p,q}}$$
$$= r_1\|f\|_{L^{p,q}}. \tag{14}$$

Define

$$u_{j,k}(x,y) := \alpha_j(x)\beta_k(y) = \frac{\chi_{B(x_j,\delta_1)}(x)}{\sum\limits_{j' \in \mathscr{J}} \chi_{B(x_{j'},\delta_1)}(x)} \cdot \frac{\chi_{B(y_k,\delta_2)}(y)}{\sum\limits_{k' \in \mathscr{J}} \chi_{B(y_{k'},\delta_2)}(y)}. \tag{15}$$

In the following, we just consider $1 \leqslant p,q < \infty$, the cases for $p = \infty$ or $q = \infty$ can be proved similarly. It follows from (13) that

$$|\langle f, \psi_{j,k} \rangle|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) \leqslant |f(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) + |F_1(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y). \tag{16}$$

Taking ℓ^q-norm for variable $k \in \mathscr{J}$ on both sides of (16) and then taking L^q-norm for variable $y \in \mathbb{R}^d$, one has

$$\left(\sum\limits_{k \in \mathscr{J}} |\langle f, \psi_{j,k} \rangle|^q \|\beta_k\|_{L^1}\right)^{1/q} \alpha_j^{1/p}(x) \leqslant \alpha_j^{1/p}(x)\|f(x,\cdot)\|_{L^q} + \alpha_j^{1/p}(x)\|F_1(x,\cdot)\|_{L^q}. \tag{17}$$

Taking ℓ^p-norm for variable $j \in \mathscr{J}$ on both sides of (17) and then taking L^p-norm for variable x, we can obtain

$$\left[\sum\limits_{j \in \mathscr{J}} \left(\sum\limits_{k \in \mathscr{J}} |\langle f, \psi_{j,k} \rangle|^q \|\beta_k\|_{L^1}\right)^{p/q} \|\alpha_j\|_{L^1}\right]^{1/p} \leqslant \|f\|_{L^{p,q}} + \|F_1\|_{L^{p,q}} \leqslant (1+r_1)\|f\|_{L^{p,q}}. \tag{18}$$

It is easy to verify

$$2\delta_1 B_{\Gamma,x}^{-1}(\delta_1) \leqslant \|\alpha_j\|_{L^1} \leqslant 2\delta_1 A_{\Gamma,x}^{-1}(\delta_1)$$

and

$$V_d \delta_2^d B_{\Gamma,y}^{-1}(\delta_2) \leqslant \|\beta_k\|_{L^1} \leqslant V_d \delta_2^d A_{\Gamma,y}^{-1}(\delta_2).$$

Then the right hand side of (12) follows from (18). The left side can be obtained by the same method from

$$|f(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) \leqslant |\langle f, \psi_{j,k} \rangle|\alpha_j^{1/p}(x)\beta_k^{1/q}(y) + |F_1(x,y)|\alpha_j^{1/p}(x)\beta_k^{1/q}(y). \quad \square$$

THEOREM 2. *Let* $1 \leqslant p, q \leqslant \infty$, φ *satisfy the assumptions* $(A1)-(A4)$. *Suppose that* $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j, k \in \mathscr{J}\}$ *is a relatively-separated set with gaps* δ_3 *and* δ_4 *for both variables and* $\psi \in L^1(\mathbb{R}^{d+1})$ *satisfies* $\int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi(x,y) dx dy = 1$. *If* $\rho_2 = max\{\delta_3, \delta_4\}$ *and* a *are chosen such that*

$$r_2 := C_{\varphi}^{-1} \left(\|\omega_{\rho_2}(\varphi)\|_{W(L^{1,1})} + \|\omega_{\rho_2}(\varphi^a)\|_{W(L^{1,1})} + \|\varphi^a\|_{W(L^{1,1})} \right) < 1, \quad (19)$$

then any signals $f \in V_{p,q}(\varphi)$ *can be stably reconstructed from the average samples* $\{\langle f, \psi_a(\cdot - \gamma_{j,k}) \rangle\}_{j,k \in \mathscr{J}}$, *and*

$$\left(\frac{2\delta_3}{A_{\Gamma,x}(\delta_3)} \right)^{-1/p} \left(\frac{V_d \delta_4^d}{A_{\Gamma,y}(\delta_4)} \right)^{-1/q} (1 - r_2) \|f\|_{L^{p,q}} \leqslant \left\| \{\langle f, \psi_a(\cdot - \gamma_{j,k}) \rangle\}_{j,k \in \mathscr{J}} \right\|_{\ell^{p,q}}$$

$$\leqslant \left(\frac{2\delta_3}{B_{\Gamma,x}(\delta_3)} \right)^{-1/p} \left(\frac{V_d \delta_4^d}{B_{\Gamma,y}(\delta_4)} \right)^{-1/q} (1 + r_2) \|f\|_{L^{p,q}}, \quad (20)$$

where $V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ *is the volume of d-dimensional unit sphere.*

Proof. For any $x \in B(x_j, \delta_3)$ and $y \in B(y_k, \delta_4)$, if $f(x,y) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c(k_1, k_2)$ $\varphi(x - k_1, y - k_2) \in V_{p,q}(\varphi)$, then we can obtain

$$|\langle f, \psi_a(\cdot - \gamma_{j,k}) \rangle - f(x,y)|$$

$$\leqslant \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1, k_2)| |\varphi(x - k_1, y - k_2) - \varphi * \psi_a^*(x_j - k_1, y_k - k_2)|$$

$$\leqslant \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1, k_2)| \Big(\omega_{\rho_2}(\varphi)(x - k_1, y - k_2)$$

$$+ \omega_{\rho_2}(\varphi^a)(x - k_1, y - k_2) + \varphi^a(x - k_1, y - k_2) \Big)$$

$$:= F_2(x,y).$$

Furthermore, one has

$$\|F_2\|_{L^{p,q}} \leqslant \|c\|_{\ell^{p,q}} \left(\|\omega_{\rho_2}(\varphi)\|_{W(L^{1,1})} + \|\omega_{\rho_2}(\varphi^a)\|_{W(L^{1,1})} + \|\varphi^a\|_{W(L^{1,1})} \right)$$

$$\leqslant C_{\varphi}^{-1} \left(\|\omega_{\rho_2}(\varphi)\|_{W(L^{1,1})} + \|\omega_{\rho_2}(\varphi^a)\|_{W(L^{1,1})} + \|\varphi^a\|_{W(L^{1,1})} \right) \|f\|_{L^{p,q}}$$

$$= r_2 \|f\|_{L^{p,q}}.$$

The remained proof is similar to that of Theorem 1. $\square$

4. Iterative approximation projection reconstruction algorithms

In this section, we will give the iterative approximation projection reconstruction algorithms for recovering the signals in $V_{p,q}(\varphi)$ from two kinds of average samples. For two kinds of average sampling functionals, we define the pre-reconstruction operators

$$A_\Gamma f := \sum_{j,k \in \mathscr{J}} \langle f, \psi_{j,k} \rangle u_{j,k}$$

and

$$A_{\Gamma,a}f := \sum_{j,k\in\mathscr{J}} \langle f, \psi_a(\cdot - \gamma_{j,k})\rangle u_{j,k},$$

where $u_{j,k}$ is defined in (15). Let P be a bounded projection from $L^{p,q}(\mathbb{R}^{d+1})$ onto $V_{p,q}(\varphi)$. Then the corresponding iterative approximation projection reconstruction algorithms are given as

$$\begin{cases} f_0 = P\Big(\sum_{j,k\in\mathscr{J}} c_0(j,k)u_{j,k}\Big) \\ f_n = f_0 + f_{n-1} - PA_\Gamma f_{n-1}, \ n \geqslant 1 \end{cases} \tag{21}$$

and

$$\begin{cases} f_0 = P\Big(\sum_{j,k\in\mathscr{J}} c_0(j,k)u_{j,k}\Big) \\ f_n = f_0 + f_{n-1} - PA_{\Gamma,a}f_{n-1}, \ n \geqslant 1. \end{cases} \tag{22}$$

LEMMA 5. *Let* $1 \leqslant p,q \leqslant \infty$, *and* φ *satisfy* (A1) *and* (A3). *Define*

$$Pf(x,y) := \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}^d} \langle f, g(\cdot - k_1, \cdot - k_2)\rangle\varphi(x - k_1, y - k_2), \tag{23}$$

where $g(x,y)$ *is given in* (4). *Then* P *is a bounded projection from* $L^{p,q}(\mathbb{R}^{d+1})$ *onto* $V_{p,q}(\varphi)$.

Proof. Let $b(k_1,k_2) = \langle f, g(\cdot - k_1, \cdot - k_2)\rangle$. Then it follows from the proof of Lemma 2 that

$$\|b\|_{\ell^{p,q}} \leqslant \|g\|_{W(L^{1,1})}\|f\|_{L^{p,q}},$$

which means that $Pf \in V_{p,q}(\varphi)$. Moreover, one has

$$\begin{aligned} \|Pf\|_{L^{p,q}} &\leqslant \|\varphi\|_{W(L^{1,1})}\|b\|_{\ell^{p,q}} \\ &\leqslant \|\varphi\|_{W(L^{1,1})}\|g\|_{W(L^{1,1})}\|f\|_{L^{p,q}}. \end{aligned}$$

Therefore, P is a bounded projection from $L^{p,q}(\mathbb{R}^{d+1})$ onto $V_{p,q}(\varphi)$. $\quad\square$

THEOREM 3. *Let* p, q, φ *and* $\{\psi_{j,k} : j,k \in \mathscr{J}\}$ *be as in Theorem* 1. *Suppose that* $\Gamma = \{\gamma_{j,k} = (x_j,y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j,k \in \mathscr{J}\}$ *is a relatively-separated set with gaps* δ_5 *and* δ_6 *for both variables. If* $\rho_3 = max\{\delta_5,\delta_6\}$ *and* a *are chosen such that*

$$r_3 := C_\varphi^{-1}\|P\|_{op}\Big[2M\|\omega_a(\varphi)\|_{W(L^{1,1})} + (1+M)\|\omega_{\rho_3}(\varphi)\|_{W(L^{1,1})}$$

$$+ M\|\omega_{a+\rho_3}(\varphi)\|_{W(L^{1,1})}\Big] < 1, \tag{24}$$

then the algorithm (21) *exponentially converges to some* $f_\infty \in V_{p,q}(\varphi)$, *and*

$$\|f_n - f_\infty\|_{L^{p,q}} \leqslant \frac{r_3^{n+1}}{1 - r_3}\|f_0\|_{L^{p,q}}. \tag{25}$$

If $c_0(j,k) = \langle f, \psi_{j,k}\rangle$, $j,k \in \mathscr{J}$ *for* $f \in V_{p,q}(\varphi)$, *then* $f_\infty = f$.

Proof. Note that $f_{n+1} - f_n = (I - PA_\Gamma)(f_n - f_{n-1})$, $n \geqslant 1$. For any $f(x,y) \in V_{p,q}(\varphi)$,

$$\|f - PA_\Gamma f\|_{L^{p,q}} \leqslant \|P\|_{op}\|f - A_\Gamma f\|_{L^{p,q}}$$
$$\leqslant \|P\|_{op}\big(\|f - Q_\Gamma f\|_{L^{p,q}} + \|Q_\Gamma f - A_\Gamma f\|_{L^{p,q}}\big), \tag{26}$$

where $Q_\Gamma f(x,y) := \sum\limits_{j,k \in \mathscr{J}} f(x_j,y_k)u_{j,k}(x,y)$. Furthermore,

$$|f(x,y) - Q_\Gamma f(x,y)| \leqslant \sum_{j,k \in \mathscr{J}} |f(x,y) - f(x_j,y_k)|u_{j,k}(x,y)$$
$$\leqslant \omega_{\rho_3}(f)(x,y)$$
$$\leqslant \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_{\rho_3}(\varphi)(x-k_1,y-k_2).$$

By Lemma 2, we can obtain

$$\|f - Q_\Gamma f\|_{L^{p,q}} \leqslant C_\varphi^{-1}\|\omega_{\rho_3}(\varphi)\|_{W(L^{1,1})}\|f\|_{L^{p,q}}. \tag{27}$$

On the other hand, we have

$$|Q_\Gamma f(x,y) - A_\Gamma f(x,y)|$$
$$\leqslant \sum_{j,k \in \mathscr{J}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(x_j,y_k) - f(s,t)||\psi_{j,k}(s,t)|u_{j,k}(x,y)dsdt$$
$$\leqslant M \sum_{j,k \in \mathscr{J}} \omega_a(f)(x_j,y_k)u_{j,k}(x,y)$$
$$\leqslant M \sum_{j,k \in \mathscr{J}} \Big(\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_a(\varphi)(x_j-k_1,y_k-k_2)\Big)u_{j,k}(x,y)$$
$$= MQ_\Gamma\Big(\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_a(\varphi)(\cdot - k_1, \cdot - k_2)\Big)(x,y)$$
$$\leqslant M\Big|\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_a(\varphi)(x-k_1,y-k_2)$$
$$- Q_\Gamma\Big(\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_a(\varphi)(\cdot - k_1, \cdot - k_2)\Big)(x,y)\Big|$$
$$+ M\Big|\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} |c(k_1,k_2)|\omega_a(\varphi)(x-k_1,y-k_2)\Big|.$$

By the same method as (27), we obtain

$$\|Q_\Gamma f - A_\Gamma f\|_{L^{p,q}} \leqslant MC_\varphi^{-1}\Big(\big\|\omega_{\rho_3}(\omega_a(\varphi))\big\|_{W(L^{1,1})} + \|\omega_a(\varphi)\|_{W(L^{1,1})}\Big)\|f\|_{L^{p,q}}$$
$$\leqslant MC_\varphi^{-1}\Big(\big\|\omega_{a+\rho_3}(\varphi)\big\|_{W(L^{1,1})} + \|\omega_{\rho_3}(\varphi)\|_{W(L^{1,1})}$$
$$+ 2\|\omega_a(\varphi)\|_{W(L^{1,1})}\Big)\|f\|_{L^{p,q}}.$$

This together with (26) and (27) gives

$$\|f - PA_\Gamma f\|_{L^{p,q}} \leqslant r_3 \|f\|_{L^{p,q}}. \tag{28}$$

Therefore, (25) is proved. Define

$$R_\Gamma := I + \sum_{n=1}^{\infty} (I - PA_\Gamma)^n.$$

Then $R_\Gamma PA_\Gamma = PA_\Gamma R_\Gamma = I$ on $V_{p,q}(\varphi)$. If $c_0(j,k) = \langle f, \psi_{j,k} \rangle$ for $f \in V_{p,q}(\varphi)$, then

$$f_\infty = R_\Gamma f_0 = R_\Gamma PA_\Gamma f = If = f. \quad \square \tag{29}$$

THEOREM 4. *Let p, q, φ and ψ be as in Theorem 2. Suppose that $\Gamma = \{\gamma_{j,k} = (x_j, y_k) : x_j \in \mathbb{R}, y_k \in \mathbb{R}^d, j,k \in \mathscr{J}\}$ is a relatively-separated set with gaps δ_7 and δ_8 for both variables. If $\rho_4 = max\{\delta_7, \delta_8\}$ and a are chosen such that*

$$r_4 := C_\varphi^{-1} \|P\|_{op} \Big[\|\omega_{\rho_4}(\varphi)\|_{W(L^{1,1})} + \|\omega_{\rho_4}(\varphi^a)\|_{W(L^{1,1})} + \|\varphi^a\|_{W(L^{1,1})} \Big] < 1, \tag{30}$$

then the algorithm (22) exponentially converges to some $f_\infty \in V_{p,q}(\varphi)$, and

$$\|f_n - f_\infty\|_{L^{p,q}} \leqslant \frac{r_4^{n+1}}{1 - r_4} \|f_0\|_{L^{p,q}}. \tag{31}$$

If $c_0(j,k) = \langle f, \psi_a(\cdot - \gamma_{j,k}) \rangle$, $j,k \in \mathscr{J}$ for $f \in V_{p,q}(\varphi)$, then $f_\infty = f$.

Proof. For any $f(x,y) = \sum\limits_{k_1 \in \mathbb{Z}} \sum\limits_{k_2 \in \mathbb{Z}^d} c(k_1, k_2) \varphi(x - k_1, y - k_2) \in V_{p,q}(\varphi)$, one has

$$\|f - PA_{\Gamma,a} f\|_{L^{p,q}} \leqslant \|P\|_{op} \big(\|f - Q_\Gamma f\|_{L^{p,q}} + \|Q_\Gamma f - Q_\Gamma(f * \psi_a^*)\|_{L^{p,q}} \big). \tag{32}$$

By (27), we know

$$\|f - Q_\Gamma f\|_{L^{p,q}} \leqslant C_\varphi^{-1} \|\omega_{\rho_4}(\varphi)\|_{W(L^{1,1})} \|f\|_{L^{p,q}}. \tag{33}$$

Since $Q_\Gamma f(x,y) - Q_\Gamma(f * \psi_a^*)(x,y) = Q_\Gamma \big(\sum\limits_{k_1 \in \mathbb{Z}} \sum\limits_{k_2 \in \mathbb{Z}^d} c(k_1,k_2) \varphi^a(\cdot - k_1, \cdot - k_2) \big)(x,y)$,

then

$$
\begin{aligned}
\|Q_\Gamma f - Q_\Gamma(f * \psi_a^*)\|_{L^{p,q}} &\leqslant \Big\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c(k_1,k_2) \varphi^a(x - k_1, y - k_2) \Big\|_{L^{p,q}} \\
&\quad + \Big\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c(k_1,k_2) \varphi^a(x - k_1, y - k_2) \\
&\quad - Q_\Gamma \big(\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c(k_1,k_2) \varphi^a(\cdot - k_1, \cdot - k_2) \big)(x,y) \Big\|_{L^{p,q}} \\
&\leqslant C_\varphi^{-1} \big(\|\varphi^a\|_{W(L^{1,1})} + \|\omega_{\rho_4}(\varphi^a)\|_{W(L^{1,1})} \big) \|f\|_{L^{p,q}}.
\end{aligned}
$$

This together with (32) and (33) obtains

$$\|f - PA_{\Gamma,a}f\|_{L^{p,q}} \leqslant r_4\|f\|_{L^{p,q}}.$$

Therefore, (31) is proved. Define

$$R_{\Gamma,a} := I + \sum_{n=1}^{\infty}(I - PA_{\Gamma,a})^n.$$

Then $R_{\Gamma,a}PA_{\Gamma,a} = PA_{\Gamma,a}R_{\Gamma,a} = I$ on $V_{p,q}(\varphi)$. If $c_0(j,k) = \langle f, \psi_a(\cdot - \gamma_{j,k})\rangle$ for $f \in V_{p,q}(\varphi)$, then

$$f_\infty = R_{\Gamma,a}f_0 = R_{\Gamma,a}PA_{\Gamma,a}f = If = f. \tag{34}$$

$\square$

5. Error estimations

Generally, the perfect recovering of the signal is difficult to come true, since there are many factors which will contribute to the error. In this section, we will provide error estimations under three kinds of conditions. For brevity, the constants r_3, r_4 are assumed to satisfy the specific definitions in Theorem 3 and Theorem 4 in the following section.

THEOREM 5. *If the average sampling values in Theorem 3 and Theorem 4 are of the form* $\{\langle f, \psi_{j,k}\rangle + \varepsilon_{j,k}\}_{j,k\in\mathscr{J}}$ *and* $\{\langle f, \psi_a(\cdot - \gamma_{j,k})\rangle + \varepsilon_{j,k}\}_{j,k\in\mathscr{J}}$, *where* $\{\varepsilon_{j,k}\}_{j,k\in\mathscr{J}} \in \ell^{p,q}$, *then for Theorem 3,*

$$\|f_\infty - f\|_{L^{p,q}} \leqslant \frac{1}{1-r_3}\|P\|_{op}\Big\|\{\varepsilon_{j,k}\}_{j,k\in\mathscr{J}}\Big\|_{\ell^{p,q}}\Big(\sup_{k\in\mathscr{J}}\Big\|\Big(\sup_{j\in\mathscr{J}}\|u_{j,k}\|_{L_y^1}\Big)^{1/q}\Big\|_{L_x^1}\Big)^{1/p} \tag{35}$$

and for Theorem 4,

$$\|f_\infty - f\|_{L^{p,q}} \leqslant \frac{1}{1-r_4}\|P\|_{op}\Big\|\{\varepsilon_{j,k}\}_{j,k\in\mathscr{J}}\Big\|_{\ell^{p,q}}\Big(\sup_{k\in\mathscr{J}}\Big\|\Big(\sup_{j\in\mathscr{J}}\|u_{j,k}\|_{L_y^1}\Big)^{1/q}\Big\|_{L_x^1}\Big)^{1/p}. \tag{36}$$

Proof. Write $h_0(x,y) := \sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\varepsilon_{j,k}Pu_{j,k}(x,y)$. Then for Theorem 3,

$$\|f_\infty - f\|_{L^{p,q}} = \|R_\Gamma h_0\|_{L^{p,q}} \leqslant \|I + \sum_{n=1}^{\infty}(I - PA_\Gamma)^n\|\|h_0\|_{L^{p,q}}$$

$$\leqslant \frac{1}{1-r_3}\|h_0\|_{L^{p,q}} \leqslant \frac{1}{1-r_3}\|P\|_{op}\|\sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\varepsilon_{j,k}u_{j,k}\|_{L^{p,q}}$$

$$\leqslant \frac{1}{1-r_3}\|P\|_{op}\Big\|\{\varepsilon_{j,k}\}_{j,k\in\mathscr{J}}\Big\|_{\ell^{p,q}}\Big(\sup_{k\in\mathscr{J}}\Big\|\Big(\sup_{j\in\mathscr{J}}\|u_{j,k}\|_{L_y^1}\Big)^{1/q}\Big\|_{L_x^1}\Big)^{1/p}.$$

The inequality (36) could also be obtained by the similar way, we will omit it. $\square$

THEOREM 6. *If the average sampling values in the Theorem* 3 *and Theorem* 4 *are of the form* $\{\langle f, \psi_{j,k}\rangle + \varepsilon_{j,k}\}_{j,k\in\mathscr{J}}$ *and* $\{\langle f, \psi_a(\cdot - \gamma_{j,k})\rangle + \varepsilon_{j,k}\}_{j,k\in\mathscr{J}}$, *where* $\{\varepsilon_{j,k}\}_{j,k\in\mathscr{J}}$ *are bounded and independent identically distributed random variables with zeros mean and variance* σ^2, *that is*

$$\varepsilon_{j,k} \in [-B,B], \quad \mathbb{E}(\varepsilon_{j,k}) = 0, \quad Var(\varepsilon_{j,k}) = \sigma^2. \tag{37}$$

Furthermore, if there exists constants $M_1 > 0$ *and* $M_2 > 0$ *such that*

$$\sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\left|R_{\Gamma}Pu_{j,k}(x,y)\right|^2 \leqslant M_1, \qquad \forall(x,y)\in\mathbb{R}^{d+1} \tag{38}$$

and

$$\sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\left|R_{\Gamma,a}Pu_{j,k}(x,y)\right|^2 \leqslant M_2, \qquad \forall(x,y)\in\mathbb{R}^{d+1}, \tag{39}$$

then for Theorem 3,

$$\mathbb{E}(f_\infty(x,y) - f(x,y)) = 0 \tag{40}$$

and

$$Var(f_\infty(x,y) - f(x,y)) \leqslant \sigma^2 M_1. \tag{41}$$

For Theorem 4,

$$\mathbb{E}(f_\infty(x,y) - f(x,y)) = 0 \tag{42}$$

and

$$Var(f_\infty(x,y) - f(x,y)) \leqslant \sigma^2 M_2. \tag{43}$$

Proof. For Theorem 3,

$$\mathbb{E}\left(f_\infty(x,y) - f(x,y)\right) = \mathbb{E}\left(R_{\Gamma}P\sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\varepsilon_{j,k}u_{j,k}(x,y)\right) = 0. \tag{44}$$

and by the independence of random variables of $\{\varepsilon_{j,k}\}_{j,k\in\mathscr{J}}$

$$\begin{aligned}
Var(f_\infty(x,y) - f(x,y)) &= \mathbb{E}\left(R_{\Gamma}P\sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\varepsilon_{j,k}u_{j,k}(x,y)\right)^2 \\
&\leqslant \sum_{j\in\mathscr{J}}\sum_{k\in\mathscr{J}}\mathbb{E}(\varepsilon_{j,k})^2\left(R_{\Gamma}Pu_{j,k}(x,y)\right)^2 \\
&\leqslant \sigma^2 M_1. \tag{45}
\end{aligned}$$

The inequalities (42) and (43) could be obtained by the similar way, we will omit it. $\square$

THEOREM 7. *If the numerical error in the ith iterative step of the algorithm* (21) *and* (22) *is* $\varepsilon_i \in V_{p,q}(\varphi)$, $i \geqslant 0$, *i.e.*

$$\begin{cases}
\widetilde{f_0} = P\left(\sum_{j,k\in\mathscr{J}}c_0(j,k)u_{j,k}\right) + \varepsilon_0 \\
\widetilde{f_n} = \widetilde{f_0} + \widetilde{f_{n-1}} - PA_{\Gamma}\widetilde{f_{n-1}} + \varepsilon_n, \ n \geqslant 1
\end{cases} \tag{46}$$

and

$$\begin{cases} \widetilde{f_0} = P\Big(\sum_{j,k \in \mathscr{J}} c_0(j,k) u_{j,k} \Big) + \varepsilon_0 \\ \widetilde{f_n} = \widetilde{f_0} + \widetilde{f_{n-1}} - PA_{\Gamma,a}\widetilde{f_{n-1}} + \varepsilon_n, \ n \geqslant 1, \end{cases} \tag{47}$$

with $c_0(j,k) = \langle f, \psi_{j,k} \rangle$ *in the* (46) *and* $c_0(j,k) = \langle f, \psi_a(\cdot - \gamma_{j,k}) \rangle$ *in the* (47). *Then for the algorithm* (46),

$$\|\widetilde{f_n} - f\|_{L^{p,q}} \leqslant \frac{r_3^{n+1}(1+r_3)\|f\|_{L^{p,q}}}{1-r_3} + \frac{2}{1-r_3}\Big[n\|\varepsilon_0\|_{L^{p,q}} + \sum_{i=0}^{n} \|\varepsilon_i\|_{L^{p,q}} \Big] \tag{48}$$

and for the algorithm (47),

$$\|\widetilde{f_n} - f\|_{L^{p,q}} \leqslant \frac{r_4^{n+1}(1+r_4)\|f\|_{L^{p,q}}}{1-r_4} + \frac{2}{1-r_4}\Big[n\|\varepsilon_0\|_{L^{p,q}} + \sum_{i=0}^{n} \|\varepsilon_i\|_{L^{p,q}} \Big]. \tag{49}$$

Proof. For the algorithm (46),

$$\widetilde{f_0} - f_0 = \widetilde{\varepsilon_0} \tag{50}$$

and

$$\widetilde{f_n} - f_n = -\sum_{k=0}^{n-1}(I - PA_\Gamma)^{n-1-k}PA_\Gamma\widetilde{\varepsilon_k} + \widetilde{\varepsilon_n}, \quad n \geqslant 1, \tag{51}$$

where $\widetilde{\varepsilon_0} = \varepsilon_0$ and $\widetilde{\varepsilon_k} = (k+1)\varepsilon_0 + \sum_{i=1}^{k}\varepsilon_i$, $k \geqslant 1$. Thus,

$$\|\widetilde{f_0} - f\|_{L^{p,q}} \leqslant \|\widetilde{f_0} - f_0\|_{L^{p,q}} + \|f_0 - f\|_{L^{p,q}} \leqslant \|\widetilde{\varepsilon_0}\|_{L^{p,q}} + r_3\|f\|_{L^{p,q}} \tag{52}$$

and for $n \geqslant 1$,

$$\begin{aligned} \|\widetilde{f_n} - f\|_{L^{p,q}} &\leqslant \|f - f_n\|_{L^{p,q}} + \|f_n - \widetilde{f_n}\|_{L^{p,q}} \\ &\leqslant \|\sum_{k=n+1}^{\infty}(I - PA_\Gamma)^k f_0\|_{L^{p,q}} + \|\sum_{k=0}^{n-1}(I - PA_\Gamma)^{n-1-k}PA_\Gamma\widetilde{\varepsilon_k}\|_{L^{p,q}} + \|\widetilde{\varepsilon_n}\|_{L^{p,q}} \\ &\leqslant \frac{r_3^{n+1}\|f_0\|_{L^{p,q}}}{1-r_3} + \sum_{k=0}^{n-1}r_3^{n-1-k}\|PA_\Gamma\widetilde{\varepsilon_k}\|_{L^{p,q}} + \|\widetilde{\varepsilon_n}\|_{L^{p,q}} \\ &\leqslant \frac{r_3^{n+1}(1+r_3)\|f\|_{L^{p,q}}}{1-r_3} + \sum_{k=0}^{n-1}r_3^{n-1-k}(1+r_3)\|\widetilde{\varepsilon_k}\|_{L^{p,q}} + \|\widetilde{\varepsilon_n}\|_{L^{p,q}} \\ &\leqslant \frac{r_3^{n+1}(1+r_3)\|f\|_{L^{p,q}}}{1-r_3} + (1+r_3)\sum_{k=0}^{n-1}r_3^{n-1-k}\|k\varepsilon_0 + \sum_{i=0}^{k}\varepsilon_i\|_{L^{p,q}} + \|n\varepsilon_0 + \sum_{i=0}^{n}\varepsilon_i\|_{L^{p,q}} \\ &\leqslant \frac{r_3^{n+1}(1+r_3)\|f\|_{L^{p,q}}}{1-r_3} + \frac{2}{1-r_3}\Big[n\|\varepsilon_0\|_{L^{p,q}} + \sum_{i=0}^{n}\|\varepsilon_i\|_{L^{p,q}} \Big]. \end{aligned} \tag{53}$$

Combining the inequalities (52) and (53), the result (48) is proved. The inequality (49) is followed by the similar way, we will omit it. □

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