EXTENSIONS OF DEMOCRACY–LIKE
PROPERTIES FOR SEQUENCES WITH GAPS

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Abstract. In [16], T. Oikhberg introduced and studied variants of the greedy and weak greedy algorithms for sequences with gaps. In this paper, we extend some of the notions that appear naturally in connection with these algorithms to the context of sequences with gaps. In particular, we will consider sequences of natural numbers for which the inequality $n_{k+1} \leqslant \mathbf{C} n_k$ or $n_{k+1} \leqslant \mathbf{C} + n_k$ holds for a positive constant $\mathbf{C}$ and all k, and find conditions under which the extended notions are equivalent their regular counterparts. In this context, we study an extension of democratic bases, proving that if $\mathbf{n} = (n_k)_{k=1}^{\infty}$ is an increasing sequence of natural numbers such that either $(n_{k+1} - n_k)_{k=1}^{\infty}$ is bounded and $(\mathbf{e}_k)_{k=1}^{\infty}$ is a Markushevich basis or $(n_{k+1}/n_k)_{k=1}^{\infty}$ is bounded and $(\mathbf{e}_k)_{k=1}^{\infty}$ is a Schauder basis, then $\mathbf{n}$-democracy is equivalent to democracy. Additionally, we give examples proving that these results are optimal, and we obtain similar results for some of the other properties that appear naturally in the study of the greedy algorithm.

1. Introduction

Let $\mathbb{X}$ be a separable, infinite dimensional Banach space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with dual space $\mathbb{X}^*$. A *fundamental minimal system* $\mathscr{B} = (\mathbf{e}_i)_{i \in \mathbb{N}} \subset \mathbb{X}$ is a sequence that satisfies the following:

- $\mathbb{X} = \overline{[\mathbf{e}_i : i \in \mathbb{N}]}$;

- there is a (unique) sequence $\mathscr{B}^* = (\mathbf{e}_i^*)_{i=1}^{\infty} \subset \mathbb{X}^*$ of biorthogonal functionals, that is, $\mathbf{e}_k^*(\mathbf{e}_i) = \delta_{k,i}$ for all $k, i \in \mathbb{N}$.

If $\mathscr{B}$ verifies the above conditions and

$$\mathbf{e}_i^*(x) = 0 \quad \forall i \in \mathbb{N} \Longrightarrow x = 0 \quad \text{(totality)},$$

we say that $\mathscr{B}$ is a *Markushevich basis*. If there is also a positive constant $\mathbf{C}$ such that

$$\|S_m(x)\| \leqslant \mathbf{C} \|x\| \qquad \forall x \in \mathbb{X}, \forall m \in \mathbb{N},$$

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where S_m is the mth partial sum $\sum_{i=1}^{m} \mathbf{e}_i^*(x)\mathbf{e}_i$, we say that $\mathscr{B}$ is a *Schauder basis*. Its basis constant $\mathbf{K}$ is the minimum $\mathbf{C}$ for which this inequality holds.

If, additionally, there is $\mathbf{C} > 0$ such that

$$\|P_A(x)\| \leqslant \mathbf{C}\|x\| \qquad \forall x \in \mathbb{X}, \forall A \subset \mathbb{N} : |A| < \infty,$$

where P_A is the projection on A with respect to $\mathscr{B}$ (that is $P_A(x) = \sum_{i \in A} \mathbf{e}_i^*(x)\mathbf{e}_i$), we say that $\mathscr{B}$ is *suppression unconditional*. The suppression unconditionality constant $\mathbf{C}_{su}$ is the minimum $\mathbf{C}$ for which the above holds. Equivalently (though not necessarily with the same constant), $\mathscr{B}$ is *unconditional* if

$$\|\sum_{j \in \mathbb{N}} a_j \mathbf{e}_j^*(x)\mathbf{e}_j\| \leqslant \mathbf{C}\|x\|, \quad \forall x \in \mathbb{X}, \forall (a_j)_{j \in \mathbb{N}} \subset \mathbb{F} : |a_j| \leqslant 1 \quad \forall j \in \mathbb{N},$$

for some $\mathbf{C} > 0$.

Hereinafter, by a *basis* for $\mathbb{X}$ we mean a fundamental minimal system $\mathscr{B}$ such that both $\mathscr{B}$ and $\mathscr{B}^*$ are semi-normalized, that is

$$0 < \inf_{i \in \mathbb{N}} \min\{\|\mathbf{e}_i\|, \|\mathbf{e}_i^*\|\} \leqslant \sup_{i \in \mathbb{N}} \max\{\|\mathbf{e}_i\|, \|\mathbf{e}_i^*\|\} < \infty.$$

We will use $\mathscr{B}$ to denote a basis, and we define positive constants α_1, α_2 as follows:

$$\alpha_1 := \sup_{i \in \mathbb{N}} \|\mathbf{e}_i\| \quad \text{and} \qquad \alpha_2 := \sup_{i \in \mathbb{N}} \|\mathbf{e}_i^*\|. \tag{1}$$

In 1999, S. V. Konyagin and V. N. Temlyakov introduced the Thresholding Greedy Algorithm (TGA), which has become one of the most important algorithms in the field of non-linear approximation, and has been studied by researchers such as F. Albiac, J. L. Ansorena, S. J. Dilworth, N. J. Kalton, D. Kutzarova, V. N. Temlyakov and P. Wojtaszczyk, among others. The algorithm essentially chooses for each $x \in \mathbb{X}$ the largest coefficients in modulus with respect to a basis. A relaxed version of this algorithm was introduced by V. N. Temlyakov in [18]. Fix $t \in (0, 1]$. We say that a set $A(x, t) := A$ is a t-*greedy set* for $x \in \mathbb{X}$ if

$$\min_{i \in A} |\mathbf{e}_i^*(x)| \geqslant t \max_{i \notin A} |\mathbf{e}_i^*(x)|.$$

A t-*greedy sum* of order m (or an m-term t-greedy sum) is the projection

$$\mathbf{G}_m^t(x) = \sum_{i \in A} \mathbf{e}_i^*(x)\mathbf{e}_i,$$

where A is a t-greedy set of cardinality m. The collection $(\mathbf{G}_m^t)_{m=1}^{\infty}$ is called the *Weak Thresholding Greedy Algorithm* (WTGA) (see [17, 18]), and we denote by $\mathscr{G}_m^t$ the collection of t-greedy sums $\mathbf{G}_m^t$ with $m \in \mathbb{N}$. If $t = 1$, we talk about greedy sets and greedy sums $\mathbf{G}_m$.

Different types of convergence of these algorithms have been studied in several papers, for instance [9, 10, 14]. For $t = 1$, a central concept in these studies is the notion of quasi-greediness ([14]).

DEFINITION 1. We say that $\mathscr{B}$ is quasi-greedy if there exists a positive constant $\mathbf{C}$ such that

$$\|\mathbf{G}_m(x)\| \leqslant \mathbf{C}\|x\|, \ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}.$$

The relation between quasi-greediness and the convergence of the algorithm was given by P. Wojtaszczyk in [19]: a basis is quasi-greedy if and only

$$\lim_n \mathbf{G}_n(x) = x, \ \forall x \in \mathbb{X}.$$

Recently, T. Oikhberg, in [16], introduced and studied a variant of the WTGA where only the t-greedy sums with order in a given increasing sequence of positive integers $\mathbf{n} = (n_k)_{k=1}^\infty$ are considered. In this context, we will consider two types of *gaps* of such a sequence: the *quotient gaps* of the sequence are the quotients $\left(\frac{n_{k+1}}{n_k}\right)_k$ when $n_{k+1} > n_k + 1$, whereas the *additive gaps* of the sequence are the differences $n_{k+1} - n_k$ in such cases (although it is the only sequence without gaps, for the sake of convenience we will allow $\mathbf{n} = \mathbb{N}$ in our proofs and definitions unless otherwise stated).

In our context, Oikhberg's central definition is as follows: given $\mathbf{n} = (n_k)_{k=1}^\infty \subset \mathbb{N}$ a strictly increasing sequence $n_1 < n_2 < \ldots$, a basis $\mathscr{B}$ is $\mathbf{n}$-t-quasi-greedy if

$$\lim_k \mathbf{G}_{n_k}^t(x) = x, \tag{2}$$

for any $x \in \mathbb{X}$ and any choice of t-greedy sums $\mathbf{G}_{n_k}^t(x)$. In [16, Theorem 2.1], the author shows that for sequences with gaps there is also a close connection between the boundedness of t-greedy sums and the convergence of the algorithm. Indeed, $\mathscr{B}$ is $\mathbf{n}$-t-quasi greedy if and only of there is $\mathbf{C} > 0$ such that

$$\|\mathbf{G}_n^t(x)\| \leqslant \mathbf{C}\|x\|, \ \forall x \in \mathbb{X}, \forall \mathbf{G}_n^t \in \mathscr{G}_n^t, \forall n \in \mathbf{n}. \tag{3}$$

We will use the notation $\mathbf{C}_{q,t}$ for the minimum $\mathbf{C}$ for which (3) holds, and we will say that $\mathscr{B}$ is $\mathbf{C}_{q,t}$-$\mathbf{n}$-t-quasi-greedy. Of course, if the basis is quasi-greedy, it is $\mathbf{n}$-quasi-greedy for any sequence $\mathbf{n}$ and, moreover, it is also $\mathbf{n}$-t-quasi greedy for all $0 < t \leqslant 1$ (see [16, Theorem 2.1], [15, Lemmas 2.1, 2.3], [13, Proposition 4.5], [12, Lemma 2.1, Lemma 6.3]). The reciprocal is false as [16, Proposition 3.1] shows and, in fact, this result shows that for any sequence $\mathbf{n}$ that has *arbitrarily large quotient gaps* (see Definition 2 below), there are Schauder bases that are $\mathbf{n}$-t-quasi greedy for all $0 < t \leqslant 1$, but not quasi-greedy. On the other hand, it was recently proven that if $\mathbf{n}$ has bounded quotient gaps, a Schauder basis that is $\mathbf{n}$-quasi-greedy is also quasi-greedy ([4, Theorem 5.2]).

DEFINITION 2. Let $\mathbf{n} = (n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with gaps. We say that $\mathbf{n}$ has arbitrarily large quotient gaps if

$$\limsup_{k \to \infty} \frac{n_{k+1}}{n_k} = \infty.$$

Alternatively, for $l \in \mathbb{N}_{>1}$, we say that $\mathbf{n}$ has l-bounded quotient gaps if

$$\frac{n_{k+1}}{n_k} \leqslant l,$$

for all $k \in \mathbb{N}$, and we say that it has bounded quotient gaps if it has l-bounded quotient gaps for some natural number $l \geqslant 2$.

We will also need the following classification:

DEFINITION 3. Let $\mathbf{n} = (n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with gaps. We say that $\mathbf{n}$ has arbitrarily large additive gaps if

$$\limsup_{k \to \infty} n_{k+1} - n_k = \infty.$$

Alternatively, for $l \in \mathbb{N}_{>1}$, we say that $\mathbf{n}$ has l-bounded additive gaps if $n_{k+1} - n_k \leqslant l$ for all $k \in \mathbb{N}$, and we say that it has bounded additive gaps if it has l-bounded additive gaps for some natural number $l \geqslant 2$.

Several properties that appear naturally in connection to these algorithms have been studied in the literature. In [14], Konyagin and Temlyakov characterized greedy bases (that is, bases where the greedy algorithm produces the best possible approximation) as those that are unconditional and *democratic*, where democratic bases are those bases such that there is $\mathbf{C} > 0$ such that

$$\|\sum_{j \in A} \mathbf{e}_j\| \leqslant \mathbf{C} \|\sum_{n \in B} \mathbf{e}_n\| \qquad \forall A, B \subset \mathbb{N} : |A| \leqslant |B| < \infty.$$

A similar characterization was proven in [10] for almost greedy bases, which are quasi-greedy and democratic. In papers such as [2, 11, 19], the authors studied properties such as symmetry for largest coefficients – which has been used to characterize 1-almost greediness and 1-greediness – and unconditionality for constant coefficients – which is used for example to characterize superdemocracy.

Here, motivated by the theory introduced by Oikhberg in [16] and by some of the examples from [6] and [7], we extend some of the aforementioned concepts to the context of sequences with gaps, and study their relations with their standard counterparts, that is the notions for $\mathbf{n} = \mathbb{N}$.

This paper is organized as follows: in Section 2 we introduce and study the notions of $\mathbf{n}$-unconditionality for constant coefficients and the $\mathbf{n}$-UL property. Section 3 focuses on the concepts of $\mathbf{n}$-democracy and other democracy-like properties, whereas Section 4 looks at $\mathbf{n}$-symmetry for largest coefficients and closely related properties. In Section 5, we consider two families of examples that are used throughtout the paper.

We will use the following notation throughout the paper – in addition to that already introduced: for A and B subsets of $\mathbb{N}$, we write $A < B$ to mean that $\max A < \min B$. If $m \in \mathbb{N}$, we write $m < A$ and $A < m$ for $\{m\} < A$ and $A < \{m\}$ respectively (and we use the symbols ">", "$\geqslant$" and "$\leqslant$" similarly). Also, $A \sqcup B$ means the union of A and B with $A \cap B = \emptyset$, and $\mathbb{N}_{>k}$ means the set $\mathbb{N} \setminus \{1, \ldots, k\}$.

For $A \subset \mathbb{N}$ finite and a basis $\mathscr{B}$, Ψ_A denotes the set of all collections of sequences $\boldsymbol{\varepsilon} = (\varepsilon_n)_{n \in A} \subset \mathbb{F}$ such that $|\varepsilon_n| = 1$ and

$$\mathbf{1}_{\boldsymbol{\varepsilon}A}[\mathscr{B}, \mathbb{X}] := \mathbf{1}_{\boldsymbol{\varepsilon}A} = \sum_{n \in A} \varepsilon_n \mathbf{e}_n.$$

If $\boldsymbol{\varepsilon} \equiv 1$, we just write $\mathbf{1}_A$. Also, every time we have index sets $A \subset B$ and $\boldsymbol{\varepsilon} \in \Psi_B$, we write $\mathbf{1}_{\boldsymbol{\varepsilon}A}$ considering the natural restriction of $\boldsymbol{\varepsilon}$ to A, with the convention that $\mathbf{1}_{\boldsymbol{\varepsilon}A} = 0$ if $A = \emptyset$.

As usual, by $\mathrm{supp}(x)$ we denote the support of $x \in \mathbb{X}$, that is the set $\{i \in \mathbb{N} : \mathbf{e}_i^*(x) \neq 0\}$. For $x \in X$ and $1 \leqslant p \leqslant \infty$, by $\|x\|_p$ we mean the ℓ_p-norm of $(\mathbf{e}_i^*(x))_i$ when it is well-defined. Finally, we set

$$\kappa := \begin{cases} 1 & \text{if } \mathbb{F} = \mathbb{R}; \\ 2 & \text{if } \mathbb{F} = \mathbb{C}. \end{cases} \tag{4}$$

2. Unconditionality for constant coefficients

In the literature, it is well known that every quasi-greedy basis is unconditional for constant coefficients, that is, for every finite set A and every sequence of signs $\varepsilon \in \Psi_A$,

$$\|\mathbf{1}_{\varepsilon A}\| \approx \|\mathbf{1}_A\|.$$

This condition was introduced by P. Wojtaszczyk in [19] and it is the key to characterize superdemocracy using democracy (see for instance [6, Lemma 3.5] for more details), among other applications. Here, we consider a natural extension for sequences with gaps.

DEFINITION 4. We say that $\mathscr{B}$ is $\mathbf{n}$-unconditional for constant coefficients if there is $\mathbf{C} > 0$ such that

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant \mathbf{C}\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\| \tag{5}$$

for all $A \subset \mathbb{N}$ with $|A| \in \mathbf{n}$ and all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Psi_A$. The smallest constant verifying (5) is denoted by $\mathbf{K}_u$ and we say that $\mathscr{B}$ is $\mathbf{K}_u$-$\mathbf{n}$-unconditional for constant coefficients. If $\mathbf{n} = \mathbb{N}$, we say that $\mathscr{B}$ is $\mathbf{K}_u$-unconditional for constant coefficients.

The following result gives sufficient conditions under which $\mathbf{n}$-unconditionality for constant coefficients entails unconditionality for constant coefficients; since the reverse implication is immediate, the notions are equivalent.

PROPOSITION 1. *Let $\mathscr{B}$ be a basis that is $\mathbf{K}_u$-$\mathbf{n}$-unconditional for constant coefficients. Then, if α_1 and α_2 are the constants defined in (1), that is,*

$$\alpha_1 := \sup_{i \in \mathbb{N}} \|\mathbf{e}_i\| \qquad \text{and} \qquad \alpha_2 := \sup_{i \in \mathbb{N}} \|\mathbf{e}_i^*\|,$$

we have:

i) *If* **n** *has* *l-bounded additive gaps, then $\mathscr{B}$ is* **C***-unconditional for constant coefficients with* $\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{K}_u + \mathbf{K}_u l\alpha_1\alpha_2 + l\alpha_1\alpha_2\}$.

ii) *If* **n** *has* *l-bounded quotient gaps and $\mathscr{B}$ is Schauder with constant* **K***, then $\mathscr{B}$ is* **C***-unconditional for constant coefficients with* $\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, (2l - 1)\mathbf{K}_u\mathbf{K}\}$.

Proof. i) Fix a finite set $A \subset \mathbb{N}$ with $|A| \notin \mathbf{n}$, and $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Psi_A$. If $|A| < n_1$, we have

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant |A|\alpha_1 \leqslant (n_1 - 1)\alpha_1 \leqslant (n_1 - 1)\alpha_1\alpha_2\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\|. \tag{6}$$

If $|A| > n_1$, let

$$k_0 := \max_{k \in \mathbb{N}}\{n_k < |A|\},$$

and choose $A_1 \subset A$ with $|A_1| = n_{k_0}$. We have

$$\begin{aligned}
\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| &\leqslant \|\mathbf{1}_{\boldsymbol{\varepsilon}A_1}\| + \|\mathbf{1}_{\boldsymbol{\varepsilon}A \setminus A_1}\| \leqslant \mathbf{K}_u\|\mathbf{1}_{\boldsymbol{\varepsilon}'A_1}\| + l\alpha_1\alpha_2\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\| \\
&\leqslant \mathbf{K}_u\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\| + \mathbf{K}_u\|\mathbf{1}_{\boldsymbol{\varepsilon}'A \setminus A_1}\| + l\alpha_1\alpha_2\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\| \leqslant (\mathbf{K}_u + \mathbf{K}_u l\alpha_1\alpha_2 + l\alpha_1\alpha_2)\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\|,
\end{aligned}$$

which, when combined with (6), gives i).

To show ii), fix $A \subset \mathbb{N}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Psi_A$ as before. If $|A| < n_1$, then by the same argument given above we have (6). On the other hand, if $|A| > n_1$, define k_0 as above. Since **n** has *l*-bounded quotient gaps and $n_{k_0} < |A| < n_{k_0+1} \leqslant ln_{k_0}$, there is $2 \leqslant m \leqslant l$ and a partition of A into nonempty disjoint sets $(A_j)_{1 \leqslant j \leqslant m}$ such that

$$|A_1| \leqslant n_{k_0}, \qquad |A_j| = n_{k_0}\forall 2 \leqslant j \leqslant m, \qquad A_j < A_{j+1}\forall 1 \leqslant j \leqslant m - 1.$$

For each $2 \leqslant j \leqslant m$, we get

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A_j}\| \leqslant \mathbf{K}_u\|\mathbf{1}_{\boldsymbol{\varepsilon}'A_j}\| \leqslant 2\mathbf{K}_u\mathbf{K}\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\|. \tag{7}$$

Let B be the (perhaps empty) set consisting of the first $n_{k_0} - |A_1|$ elements of $A \setminus A_1$. We have

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A_1}\| \leqslant \max_{\varepsilon \in \{-1,1\}}\|\mathbf{1}_{\boldsymbol{\varepsilon}A_1} + \varepsilon\mathbf{1}_B\| \leqslant \mathbf{K}_u\|\mathbf{1}_{\boldsymbol{\varepsilon}'A_1} + \mathbf{1}_{\boldsymbol{\varepsilon}'B}\| \leqslant \mathbf{K}_u\mathbf{K}\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\|.$$

From this and (7), it follows by the triangle inequality that

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant (2l - 1)\mathbf{K}_u\mathbf{K}\|\mathbf{1}_{\boldsymbol{\varepsilon}'A}\|. \tag{8}$$

The proof is completed combining (6) and (8). $\quad\square$

In the case $\mathbf{n} = \mathbb{N}$, it is is known that quasi-greediness implies a property that is stronger than unconditionality for constant coefficients, namely the UL property: if A is a finite set, then for any sequence $(a_i)_{i \in A}$,

$$\min_{i \in A}|a_i|\|\mathbf{1}_A\| \lesssim \|\sum_{i \in A} a_i\mathbf{e}_i\| \lesssim \max_{i \in A}|a_i|\|\mathbf{1}_A\|. \tag{9}$$

This relation was shown for the first time in [10] when $\mathbb{F} = \mathbb{R}$ and, for the complex case, the result was proved in [3]. Moreover, the UL property has own life since in [6, Section 5.5], the authors gave the first example in the literature of a basis in a Banach space such that (9) is satisfied but the basis is not quasi-greedy. Now, we extend this notion to the context of sequences with gaps.

DEFINITION 5. We say that a basis $\mathscr{B}$ has the $\mathbf{n}$-UL property if there are positive constants $\mathbf{C}_1, \mathbf{C}_2$ such that

$$\frac{1}{\mathbf{C}_1} \min_{i \in A} |a_i| \|\mathbf{1}_A\| \leqslant \|\sum_{i \in A} a_i \mathbf{e}_i\| \leqslant \mathbf{C}_2 \max_{i \in A} |a_i| \|\mathbf{1}_A\| \tag{10}$$

for all $A \subset \mathbb{N}$ with $|A| \in \mathbf{n}$ and all scalars $(a_i)_{i \in A}$. If $\mathbf{n} = \mathbb{N}$, we say that $\mathscr{B}$ has the UL property with constants $\mathbf{C}_1$ and $\mathbf{C}_2$.

For sequences with (in either sense) bounded gaps, we have the following result, similar to Lemma 1.

PROPOSITION 2. *Let $\mathscr{B}$ be a basis that has the $\mathbf{n}$-UL property with constants $\mathbf{C}_1$ and $\mathbf{C}_2$. The following hold:*

i) *If $\mathscr{B}$ has l-bounded additive gaps, $\mathscr{B}$ has the UL property with constants $\mathbf{C}_1', \mathbf{C}_2'$ verifying the following bounds:*

$$\mathbf{C}_1' \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{C}_1 + l\alpha_1\alpha_2 + \mathbf{C}_1 l\alpha_1\alpha_2\},$$

and

$$\mathbf{C}_2' \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{C}_2 + l\alpha_1\alpha_2 + \mathbf{C}_2 l\alpha_1\alpha_2\}.$$

ii) *If $\mathscr{B}$ has l-bounded quotient gaps and $\mathscr{B}$ is Schauder with constant $\mathbf{K}$, $\mathscr{B}$ has the UL property with constants $\mathbf{C}_1', \mathbf{C}_2'$ verifying the following bounds:*

$$\mathbf{C}_1' \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{K}^2\mathbf{C}_1 + 2(l-1)\mathbf{C}_1\mathbf{K}\},$$

and

$$\mathbf{C}_2' \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{K}^2\mathbf{C}_2 + 2(l-1)\mathbf{C}_2\mathbf{K}\}.$$

Proof. i) Fix a finite set $A \subset \mathbb{N}$ with $|A| \notin \mathbf{n}$, and scalars $(a_i)_{i \in A}$. If $|A| < n_1$, then

$$\min_{i \in A} |a_i| \|\mathbf{1}_A\| \leqslant \min_{i \in A} |a_i| |A| \alpha_1 \leqslant (n_1 - 1)\alpha_1\alpha_2 \|\sum_{i \in A} a_i \mathbf{e}_i\|, \tag{11}$$

and

$$\|\sum_{i \in A} a_i \mathbf{e}_i\| \leqslant \max_{i \in A} |a_i| n_1 \alpha_1 \leqslant (n_1 - 1)\alpha_1\alpha_2 \|\mathbf{1}_A\|. \tag{12}$$

On the other hand, if $|A| > n_1$, let

$$k_0 := \max_{k \in \mathbb{N}}\{n_k < |A|\},$$

and choose $A_1 \subset A$ a greedy set with $|A_1| = n_{k_0}$. We have

$$
\begin{aligned}
\min_{i \in A} |a_i| \|\mathbf{1}_A\| &\leqslant \min_{i \in A} |a_i| \|\mathbf{1}_{A_1}\| + \min_{i \in A} |a_i| \|\mathbf{1}_{A \setminus A_1}\| \leqslant \mathbf{C}_1 \left\| \sum_{i \in A_1} a_i \mathbf{e}_i \right\| + l \alpha_1 \alpha_2 \left\| \sum_{i \in A} a_i \mathbf{e}_i \right\| \\
&\leqslant (\mathbf{C}_1 + l \alpha_1 \alpha_2) \left\| \sum_{i \in A} a_i \mathbf{e}_i \right\| + \mathbf{C}_1 \left\| \sum_{i \in A \setminus A_1} a_i \mathbf{e}_i \right\| \\
&\leqslant (\mathbf{C}_1 + l \alpha_1 \alpha_2) \left\| \sum_{i \in A} a_i \mathbf{e}_i \right\| + \mathbf{C}_1 l \alpha_1 \max_{i \in A} |a_j| \\
&\leqslant (\mathbf{C}_1 + l \alpha_1 \alpha_2 + \mathbf{C}_1 l \alpha_1 \alpha_2) \left\| \sum_{i \in A} a_i \mathbf{e}_i \right\|,
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\left\| \sum_{i \in A} a_i \mathbf{e}_i \right\| &\leqslant \left\| \sum_{i \in A_1} a_i \mathbf{e}_i \right\| + \left\| \sum_{i \in A \setminus A_1} a_i \mathbf{e}_i \right\| \leqslant \mathbf{C}_2 \max_{i \in A_1} |a_i| \|\mathbf{1}_{A_1}\| + l \alpha_1 \max_{i \in A \setminus A_1} |a_i| \\
&\leqslant \mathbf{C}_2 \max_{i \in A_1} |a_i| \|\mathbf{1}_A\| + \mathbf{C}_2 \max_{i \in A_1} |a_i| \|\mathbf{1}_{A \setminus A_1}\| + l \alpha_1 \alpha_2 \max_{i \in A \setminus A_1} |a_i| \|\mathbf{1}_A\| \\
&\leqslant \max_{i \in A} |a_i| (\mathbf{C}_2 + \mathbf{C}_2 l \alpha_1 \alpha_2 + l \alpha_1 \alpha_2) \|\mathbf{1}_A\|.
\end{aligned}
\tag{14}
$$

The proof of i) is completed combining (11), (12), (13), and (14).

ii) Fix a finite set $A \subset \mathbb{N}$ with $|A| \notin \mathbf{n}$, and scalars $(a_i)_{i \in A}$. The case $|A| < n_1$ is handled as in the proof of i), so we assume $|A| > n_1$ and we set k_0 as before. Since $\mathbf{n}$ has l-bounded quotient gaps, there is $2 \leqslant m \leqslant l$ and a partition of A into nonempty sets $(A_j)_{1 \leqslant j \leqslant m}$ such that

$$
|A_1| \leqslant n_{k_0}, \qquad |A_k| = n_{k_0} \forall 2 \leqslant j \leqslant m, \qquad \text{and} \qquad A_j < A_{j+1} \forall 1 \leqslant j \leqslant m-1.
$$

For each $2 \leqslant j \leqslant m$, applying the $\mathbf{n}$-UL property and the Schauder condition we get

$$
\min_{i \in A} |a_i| \|\mathbf{1}_{A_j}\| \leqslant \min_{i \in A_j} |a_i| \|\mathbf{1}_{A_j}\| \leqslant \mathbf{C}_1 \| \sum_{i \in A_j} a_i \mathbf{e}_i \| \leqslant 2 \mathbf{C}_1 \mathbf{K} \| \sum_{i \in A} a_i \mathbf{e}_i \|.
\tag{15}
$$

Let B be the set consisting of the first n_{k_0} elements of A. Since A_1 is the set consisting of the first $|A_1| \leqslant n_{k_0}$ elements of A, we have

$$
\min_{i \in A} |a_i| \|\mathbf{1}_{A_1}\| \leqslant \mathbf{K} \min_{i \in B} |a_i| \|\mathbf{1}_B\| \leqslant \mathbf{K} \mathbf{C}_1 \| \sum_{i \in B} a_i \mathbf{e}_i \| \leqslant \mathbf{K}^2 \mathbf{C}_1 \| \sum_{i \in A} a_i \mathbf{e}_i \|.
$$

Combining this with (15), it follows by the triangle inequality that

$$
\min_{i \in A} |a_i| \|\mathbf{1}_A\| \leqslant (\mathbf{K}^2 \mathbf{C}_1 + 2(l-1) \mathbf{C}_1 \mathbf{K}) \| \sum_{i \in A} a_i \mathbf{e}_i \|.
\tag{16}
$$

Similarly, for each $2 \leqslant j \leqslant m$, applying the $\mathbf{n}$-UL and Schauder conditions we get

$$\| \sum_{i \in A_j} a_i \mathbf{e}_i \| \leqslant \mathbf{C}_2 \max_{i \in A_j} |a_i| \|\mathbf{1}_{A_j}\| \leqslant 2\mathbf{KC}_2 \max_{i \in A} |a_i| \|\mathbf{1}_A\|, \qquad (17)$$

and

$$\| \sum_{i \in A_1} a_i \mathbf{e}_i \| \leqslant \mathbf{K} \| \sum_{i \in B} a_i \mathbf{e}_i \| \leqslant \mathbf{KC}_2 \max_{i \in B} |a_i| \|\mathbf{1}_B\| \leqslant \mathbf{K}^2 \mathbf{C}_2 \max_{i \in A} |a_i| \|\mathbf{1}_A\|.$$

From this and (17), by the triangle inequality we obtain

$$\| \sum_{i \in A} a_i \mathbf{e}_i \| \leqslant (\mathbf{K}^2 \mathbf{C}_2 + 2(l-1)\mathbf{KC}_2) \max_{i \in A} |a_i| \|\mathbf{1}_A\|.$$

The proof is completed combining the above inequality with (11), (12) and (16). $\quad\square$

Propositions 1 and 2 give sufficient conditions on a sequence with gaps $\mathbf{n}$ under which $\mathbf{n}$-unconditionality for constant coefficients and the $\mathbf{n}$-UL property are equivalent to their standard counterparts. Our next result shows that these conditions are also necessary.

PROPOSITION 3. *Let $\mathbf{n}$ be a sequence. The following hold:*

- *If $\mathbf{n}$ has arbitrarily large additive gaps, there is a Banach space $\mathbb{X}$ with a Markushevich basis $\mathscr{B}$ that has the $\mathbf{n}$-UL property, but is not unconditional for constant coefficients.*

- *If $\mathbf{n}$ has arbitrarily large quotient gaps, there is a Banach space $\mathbb{X}$ with a Schauder basis $\mathscr{B}$ that has the $\mathbf{n}$-UL property, but is not unconditional for constant coefficients.*

Proof. See Examples 1 and 2. $\quad\square$

Summing up, we have the following equivalences.

COROLLARY 1. *Let $\mathbf{n}$ be a sequence with gaps. The following are equivalent:*

i) *$\mathbf{n}$ has bounded quotient gaps.*

ii) *Every Schauder basis that is $\mathbf{n}$-unconditional for constant coefficients is unconditional for constant coefficients.*

iii) *Every Schauder basis that has the $\mathbf{n}$-UL property has the UL property.*

COROLLARY 2. *Let $\mathbf{n}$ be a sequence with gaps. The following are equivalent:*

i) *$\mathbf{n}$ has bounded additive gaps.*

ii) *Every basis that is $\mathbf{n}$-unconditional for constant coefficients is unconditional for constant coefficients.*

   *iii) Every Markushevich basis that is **n**-unconditional for constant coefficients is unconditional for constant coefficients.*

   *iv) Every basis that has the **n**-UL property has the UL property.*

   *v) Every Markushevich basis that has the **n**-UL property has the UL property.*

Note that there is a significant difference between the behavior of the extensions to our context of the UL property and unconditionality for constant coefficients for general bases or Markushevich bases on one hand, and Schauder bases on the other hand. Similar differences occur when we consider democracy-like properties, as we shall see in the next section.

3. n-democracy and some democracy-like properties

In greedy approximation theory, democracy and several similar properties are widely used for the characterization of greedy-like bases (see for instance in [10, 12, 14]). Here, we study natural extensions of some of these properties to the general context of sequences with gaps. We begin our study with the extensions of two well-known properties.

DEFINITION 6. We say that $\mathscr{B}$ is **n**-superdemocratic if there exists a positive constant $\mathbf{C}$ such that

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{C}\|\mathbf{1}_{\varepsilon' B}\|, \tag{18}$$

for all A, B with $|A| \leqslant |B|$, $|A|, |B| \in \mathbf{n}$ and $\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B$. The smallest constant verifying (18) is denoted by Δ_s and we say that $\mathscr{B}$ is Δ_s-**n**-superdemocratic.

If (18) is satisfied for $\varepsilon \equiv \varepsilon' \equiv 1$, we say that $\mathscr{B}$ is Δ_d-**n**-democratic, where Δ_d is again the smallest constant for which the inequality holds. If $\mathbf{n} = \mathbb{N}$, we say that $\mathscr{B}$ is Δ_d-democratic and Δ_s-superdemocratic.

REMARK 1. As in the standard case ([10]), it is immediate that a basis is **n**-superdemocratic if and only if it is **n**-democratic and **n**-unconditional for constant coefficients.

REMARK 2. Note that a straightforward convexity argument gives that basis $\mathscr{B}$ is Δ_s-**n**-superdemocratic if and only if Δ_s is the minimum $\mathbf{C}$ for which

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{C}\|\mathbf{1}_{\varepsilon' B}\|,$$

for all A, B with $|A| = |B| \in \mathbf{n}$ and all $\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B$. Alternatively, this is equivalent to ask that $|A| \leqslant |B|$ and only that $|B| \in \mathbf{n}$.

As in the cases of the **n**-UL property and **n**-unconditionality for constant coefficients, a key distinction is whether the sequences have (in either sense) bounded gaps. We begin with the results for Schauder bases.

LEMMA 1. *Let* **n** *be a sequence with arbitrarily large quotient gaps. There is a Banach space* $\mathbb{X}$ *with a Schauder basis* $\mathscr{B}$ *that is* **n**-*superdemocratic but not democratic.*

Proof. See Example 2 and Remark 4. $\square$

Before we prove our next proposition, we prove a lemma that will be used throughout the paper.

LEMMA 2. *Let* $\mathbb{X}$ *be a Banach space,* **n** *a sequence with* l-*bounded quotient gaps, and* $A \subset \mathbb{N}$ *a finite nonempty set, and* $(x_j)_{j \in A} \subset \mathbb{X}$. *The following hold:*

i) *Either*

$$\max_{E \subset A} \left\| \sum_{j \in E} x_j \right\| \leqslant (n_1 - 1) \max_{j \in A} \|x_j\|$$

or there is $B \subset A$ *with* $|B| \in$ **n** *such that*

$$\max_{E \subset A} \left\| \sum_{j \in E} x_j \right\| \leqslant l \left\| \sum_{j \in B} x_j \right\|.$$

ii) *Given* $(b_j)_{j \in A}$ *with* $|b_j| \geqslant 1$ *for all* $j \in A$, *either*

$$\max_{\substack{(a_j)_{j \in A} \subset \mathbb{F} \\ |a_j| \leqslant 1 \forall j \in A}} \left\| \sum_{j \in A} a_j x_j \right\| \leqslant 2\kappa(n_1 - 1) \max_{j \in A} \|x_j\|$$

or there is $B \subset A$ *with* $|B| \in$ **n** *such that*

$$\max_{\substack{(a_j)_{j \in A} \subset \mathbb{F} \\ |a_j| \leqslant 1 \forall j \in A}} \left\| \sum_{j \in A} a_j x_j \right\| \leqslant 2\kappa l \left\| \sum_{j \in B} b_j x_j \right\|,$$

where κ *is the constant defined in* (4), *that is,*

$$\kappa := \begin{cases} 1 & \text{if } \mathbb{F} = \mathbb{R}; \\ 2 & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Proof. i) Define

$$\mathbb{Y} := \left\{ \sum_{j \in A} a_j x_j : a_j \in \mathbb{R} \, \forall j \in A \right\}.$$

It is immediate that $\mathbb{Y}$ is a finite dimensional Banach space over $\mathbb{R}$ with the norm inherited from $\mathbb{X}$. Since the norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$ are the same for elements of $\mathbb{Y}$,

we may work in $\mathbb{Y}$ to establish our result. We will denote the norm by $\|\cdot\|$ as in the statement.

Pick $D \subset A$ so that

$$\left\|\sum_{j \in D} x_j\right\| \geqslant \left\|\sum_{j \in E} x_j\right\|, \quad \forall E \subset A.$$

If $|D| < n_1$, then

$$\left\|\sum_{j \in E} x_j\right\| \leqslant \left\|\sum_{j \in D} x_j\right\| \leqslant (n_1 - 1) \max_{j \in A} \|x_j\|, \quad \forall E \subset A. \tag{19}$$

On the other hand, if $|D| \geqslant n_1$, set

$$k_0 := \max_{k \in \mathbb{N}} \{n_k \leqslant |D|\},$$

and choose $y^* \in \mathbb{Y}^*$ with $\|y^*\| = 1$ so that

$$y^*\left(\sum_{j \in D} x_j\right) = \left\|\sum_{j \in D} x_j\right\|.$$

Note that, if $\emptyset \subsetneq E \subset D$,

$$\left|\sum_{j \in E} y^*(x_j)\right| \leqslant \left\|\sum_{j \in E} x_j\right\| \leqslant \left\|\sum_{j \in D} x_j\right\| = \sum_{j \in D} y^*(x_j).$$

Hence,

$$y^*(x_j) \geqslant 0, \quad \forall j \in D.$$

Choose $B \subset D$ with $|B| = n_{k_0}$ so that

$$y^*(x_j) \geqslant y^*(x_i), \quad \forall j \in B, \forall i \in D \setminus B.$$

Given that $|D| \leqslant l|B|$, for each $E \subset A$ we have

$$\left\|\sum_{j \in E} x_j\right\| \leqslant \left\|\sum_{j \in D} x_j\right\| = \sum_{j \in D} y^*(x_j) \leqslant l \sum_{j \in B} y^*(x_j) \leqslant l \left\|\sum_{j \in B} x_j\right\|.$$

The proof of i) is completed combining the above inequality with (19).

ii) For each $j \in A$, let $y_j := b_j x_j$, and choose $D \subset A$ so that

$$\left\|\sum_{j \in D} y_j\right\| \geqslant \left\|\sum_{j \in E} y_j\right\|, \quad \forall E \subset A.$$

Using convexity we obtain

$$\max_{\substack{(a_j)_{j\in A}\subset \mathbb{F} \\ |a_j|\leqslant 1 \forall j\in A}} \left\| \sum_{j\in A} a_j x_j \right\| \leqslant \max_{\substack{(a_j)_{j\in A}\subset \mathbb{F} \\ |a_j|\leqslant 1 \forall j\in A}} \left\| \sum_{j\in A} a_j y_j \right\|$$

$$\leqslant \max_{\substack{(a_j)_{j\in A}\subset \mathbb{F} \\ |a_j|\leqslant 1 \forall j\in A}} \left(\left\| \sum_{j\in A} \mathrm{Re}(a_j) y_j \right\| + \left\| \sum_{j\in A} \mathrm{Im}(a_j) y_j \right\| \right)$$

$$\leqslant \kappa \max_{\substack{\lambda_j\in\{-1,1\} \\ \forall j\in E}} \left\| \sum_{j\in A} \lambda_j y_j \right\| \leqslant 2\kappa \left\| \sum_{j\in D} y_j \right\|.$$

The proof is completed by an application of i) to $(y_j)_{j\in D}$. $\quad\square$

PROPOSITION 4. *Suppose* $\mathbf{n}$ *has* l-*bounded quotient gaps, and let* $\mathscr{B}$ *be a Schauder basis with basis constant* $\mathbf{K}$. *Then:*

i) *If* $\mathscr{B}$ *is* Δ_d-$\mathbf{n}$-*democratic, it is* $\mathbf{C}$-*democratic with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}\Delta_d\}.$$

ii) *If* $\mathscr{B}$ *is* Δ_s-$\mathbf{n}$-*superdemocratic, it is* $\mathbf{C}$-*superdemocratic with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}\Delta_s\}.$$

Proof. i) Fix finite sets A, B with $|A| \leqslant |B|$. If $\|\mathbf{1}_A\| \leqslant (n_1-1)\alpha_1$, then

$$\|\mathbf{1}_A\| \leqslant (n_1-1)\alpha_1\alpha_2\|\mathbf{1}_B\|. \tag{20}$$

Otherwise, by Lemma 2 there is $A_0 \subset A$ with $|A_0| \in \mathbf{n}$ such that

$$\|\mathbf{1}_A\| \leqslant l\|\mathbf{1}_{A_0}\|.$$

Let B_0 be the set consisting in the first $n_{k_0} := |A_0|$ elements of B. We have

$$\|\mathbf{1}_{A_0}\| \leqslant \Delta_d\|\mathbf{1}_{B_0}\| \leqslant \Delta_d\mathbf{K}\|\mathbf{1}_B\|. \tag{21}$$

Thus,

$$\|\mathbf{1}_A\| \leqslant l\Delta_d\mathbf{K}\|\mathbf{1}_B\|.$$

Combining the above inequality with (20) we obtain that $\mathscr{B}$ is democratic with constant as in the statement.

ii) This is proved by the same argument as i). $\quad\square$

Note that the Schauder condition in Proposition 4 can be replaced with unconditionality for constant coefficients.

LEMMA 3. *Suppose* $\mathbf{n}$ *is a sequence with* l*-bounded quotient gaps, and* $\mathscr{B}$ *is a basis that is* $\mathbf{K}_u$*-unconditional for constant coefficients. Then:*

i) *If* $\mathscr{B}$ *is* Δ_d-$\mathbf{n}$*-democratic, then*

    1) $\mathscr{B}$ *is* $\mathbf{C}$*-democratic with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u\Delta_d\}.$$

    2) $\mathscr{B}$ *is* $\mathbf{M}$*-superdemocratic with*

$$\mathbf{M} \leqslant \min\{\max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u^2\Delta_d\}, 2\kappa\max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u\Delta_d\}\}.$$

ii) *If* $\mathscr{B}$ *is* Δ_s-$\mathbf{n}$*-superdemocratic, it is* $\mathbf{C}$*-superdemocratic with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u\Delta_s\}.$$

Proof. i) To show 1), we can use the same argument as Proposition 4, with the only difference that instead of (21) we get

$$\|\mathbf{1}_{A_0}\| \leqslant \Delta_d\|\mathbf{1}_{B_0}\| \leqslant \Delta_d \max_{\varepsilon\in\{-1,1\}} \|\mathbf{1}_{B_0} + \varepsilon\mathbf{1}_{B\setminus B_0}\| \leqslant \Delta_d\mathbf{K}_u\|\mathbf{1}_B\|.$$

For 2), fix finite sets A, B with $|A| \leqslant |B|$, $\varepsilon \in \Psi_A$ and $\varepsilon' \in \Psi_B$. If $\|\mathbf{1}_{\varepsilon A}\| \leqslant (n_1-1)\alpha_1$, then then

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant (n_1-1)\alpha_1\alpha_2\|\mathbf{1}_{\varepsilon' B}\|.$$

Otherwise, by Lemma 2i) there is $A_0 \subset A$ with $|A_0| \in \mathbf{n}$ such that

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant l\|\mathbf{1}_{\varepsilon A_0}\|.$$

Choose $B_0 \subset B$ with $|B_0| = |A_0|$. We have

$$\begin{aligned}
\|\mathbf{1}_{\varepsilon A}\| &\leqslant l\|\mathbf{1}_{\varepsilon A_0}\| \leqslant l\mathbf{K}_u\|\mathbf{1}_{A_0}\| \leqslant l\mathbf{K}_u\Delta_d\|\mathbf{1}_{B_0}\| \\
&\leqslant l\mathbf{K}_u\Delta_d \max_{\varepsilon\in\{-1.1\}} \|\mathbf{1}_{B_0} + \mathbf{1}_{B\setminus B_0}\| \\
&\leqslant l\mathbf{K}_u^2\Delta_d\|\mathbf{1}_{\varepsilon' B}\|.
\end{aligned}$$

Similarly, if $\|\mathbf{1}_{\varepsilon A}\| > 2\kappa(n_1-1)\alpha_1$, by Lemma 2ii) there is $A_0 \subset A$ with $|A_0| \in \mathbf{n}$ such that

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant 2\kappa l\|\mathbf{1}_{A_0}\|.$$

Thus, choosing B_0 as above we obtain

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant 2\kappa l\|\mathbf{1}_{A_0}\| \leqslant 2\kappa l\mathbf{K}_u\Delta_d\|\mathbf{1}_{\varepsilon' B}\|.$$

ii) This is proved in the same manner as 1) of i). □

For general bases, we have the following result.

PROPOSITION 5. *Let $\mathbf{n}$ be a sequence with gaps. Then:*

i) *If $\mathbf{n}$ has arbitrarily large additive gaps, there is a Banach space $\mathbb{X}$ with a Marku-shevich basis $\mathscr{B}$ that is $\mathbf{n}$-superdemocratic but not democratic.*

ii) *If $\mathbf{n}$ has l-bounded additive gaps and $\mathscr{B}$ is Δ_{sd}-$\mathbf{n}$-superdemocratic, it is $\mathbf{C}$-superdemocratic, with*

$$\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \Delta_{sd}(1 + l\alpha_1\alpha_2) + l\alpha_1\alpha_2\}.$$

iii) *If $\mathbf{n}$ has l-bounded additive gaps and $\mathscr{B}$ is Δ_d-$\mathbf{n}$-democratic, it is $\mathbf{C}$-democratic, with*

$$\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \Delta_d(1 + l\alpha_1\alpha_2) + l\alpha_1\alpha_2\}.$$

Proof. i). See Example 1.

ii). Choose $A, B, \varepsilon, \varepsilon'$ as in Definition 6. If $|A| \leqslant n_1 - 1$, we have

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \alpha_1\alpha_2(n_1 - 1)\|\mathbf{1}_{\varepsilon' B}\|.$$

Otherwise, there are $k_0 \in \mathbb{N}$ such that $n_{k_0} \leqslant |A| \leqslant n_{k_0+1}$ and $k_1 \geqslant k_0$ such that $n_{k_1} \leqslant |B| \leqslant n_{k_1+1}$. Choose $A_1 \subset A$ and $B_1 \subset B$ with $|A_1| = n_{k_0}$ and $|B_1| = n_{k_1}$. We have

$$
\begin{aligned}
\|\mathbf{1}_{\varepsilon A}\| &\leqslant \|\mathbf{1}_{\varepsilon A_1}\| + \|\mathbf{1}_{\varepsilon A \setminus A_1}\| \leqslant \Delta_{sd}\|\mathbf{1}_{\varepsilon' B_1}\| + l\alpha_1\alpha_2\|\mathbf{1}_{\varepsilon' B}\| \\
&\leqslant \Delta_{sd}\|\mathbf{1}_{\varepsilon' B}\| + \Delta_{sd}\|\mathbf{1}_{\varepsilon' B \setminus B_1}\| + l\alpha_1\alpha_2\|\mathbf{1}_{\varepsilon' B}\| \leqslant (\Delta_{sd}(1 + l\alpha_1\alpha_2) + l\alpha_1\alpha_2)\|\mathbf{1}_{\varepsilon' B}\|.
\end{aligned}
$$

iii) is proven in the same way as ii). $\square$

Next, we consider extensions of two other properties: conservativeness and super-conservativeness (see [5] and [10]).

DEFINITION 7. We say that a basis $\mathscr{B}$ is $\mathbf{n}$-superconservative if there exists a positive constant $\mathbf{C}$ such that

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{C}\|\mathbf{1}_{\varepsilon' B}\|, \tag{22}$$

for all $A, B \subset \mathbb{N}$ with $|A| \leqslant |B|$, $|A|, |B| \in \mathbf{n}$, $A < B$, and $\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B$. The smallest constant verifying (18) is denoted by Δ_{sc} and we say that $\mathscr{B}$ is Δ_{sc}-$\mathbf{n}$-superconservative.

If (22) is satisfied for $\varepsilon \equiv \varepsilon' \equiv 1$, we say that $\mathscr{B}$ is Δ_c-$\mathbf{n}$-conservative, where Δ_c is the smallest constant for which the inequality holds.

For $n = \mathbb{N}$, we say that $\mathscr{B}$ is Δ_{sc}-superconservative and Δ_c-conservative.

The extensions of these two properties to the context of sequences with gaps behave like the extensions of democracy and superdemocracy, in the sense shown in the following results, counterparts of the ones proven above.

LEMMA 4. *Suppose $\mathbf{n}$ has l-bounded quotient gaps, and let $\mathscr{B}$ be a Schauder basis with basis constant $\mathbf{K}$. Then:*

- If $\mathscr{B}$ is Δ_c-**n**-*conservative, it is conservative with constant no greater than* $\max\{(n_1-1)\alpha_1\alpha_2, l\Delta_c\mathbf{K}\}$.

- If $\mathscr{B}$ is Δ_{sc}-**n** *superconservative, it is superconservative with constant no greater than* $\max\{(n_1-1)\alpha_1\alpha_2, l\Delta_{sc}\mathbf{K}.\}$.

Proof. This is proved in the same manner as Proposition 4. □

LEMMA 5. *Let* **n** *be a sequence with arbitrarily large quotient gaps. There is a Banach space* $\mathbb{X}$ *with a Schauder basis* $\mathscr{B}$ *that is* **n**-*superconservative but not conservative.*

Proof. See Example 2. □

LEMMA 6. *Suppose* **n** *is a sequence with* l-*bounded quotient gaps, and* $\mathscr{B}$ *is a basis that is* $\mathbf{K}_u$-*unconditional for constant coefficients. Then:*

i) *If* $\mathscr{B}$ *is* Δ_c-**n**-*conservative, then*

  1) $\mathscr{B}$ *is* **C**-*conservative with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u\Delta_c\}$$

  2) $\mathscr{B}$ *is* **M**-*superconservative with*

$$\mathbf{M} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u^2\Delta_c\}$$

ii) *If* $\mathscr{B}$ *is* Δ_{sc}-**n**-*superconservative, it is* **C**-*superconservative with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, l\mathbf{K}_u\Delta_{sc}\}.$$

Proof. This Lemma is proved by the same arguments as Lemma 3, with only straightforward modifications. □

LEMMA 7. *Let* **n** *be a sequence with gaps. Then*

i) *If* **n** *has arbitrarily large additive gaps, there is a Banach space* $\mathbb{X}$ *with a Markushevich basis* $\mathscr{B}$ *that is* **n**-*superconservative but not conservative.*

ii) *If* **n** *has* l-*bounded additive gaps and* $\mathscr{B}$ *is* Δ_{sc}-**n**-*superconservative, it is* **C**-*superconservative, with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, \Delta_{sc}(1+l\alpha_1\alpha_2)+l\alpha_1\alpha_2\}.$$

iii) *If* **n** *has* l-*bounded additive gaps and* $\mathscr{B}$ *is* Δ_c-**n**-*conservative, it is* **C**-*conservative, with*

$$\mathbf{C} \leqslant \max\{(n_1-1)\alpha_1\alpha_2, \Delta_c(1+l\alpha_1\alpha_2)+l\alpha_1\alpha_2\}.$$

Proof. This is proved in the same manner as Proposition 5. □

4. n-symmetry and n-quasi-greediness for largest coefficients

In this section, we extend to the context of sequences with gaps the notions of quasi-greediness for largest coefficients and symmetry for largest coefficients. We also study an extension of suppression unconditionality for constant coefficients. We begin with the first of these properties, introduced in [1].

DEFINITION 8. We say that $\mathscr{B}$ is $\mathbf{n}$-quasi-greedy for largest coefficients if there exists a positive constant $\mathbf{C}$ such that

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{C}\|\mathbf{1}_{\varepsilon A} + x\| \tag{23}$$

for every $A \subset \mathbb{N}$ with $|A| \in \mathbf{n}$, $\boldsymbol{\varepsilon} \in \Psi_A$, and all $x \in \mathbb{X}$ such that $\operatorname{supp}(x) \cap A = \emptyset$ and $|\mathbf{e}_i^*(x)| \leqslant 1$ for all $i \in \mathbb{N}$. The smallest constant verifying (23) is denoted by $\mathbf{C}_{ql}$ and we say that $\mathscr{B}$ is $\mathbf{C}_{ql}$-$\mathbf{n}$-quasi-greedy for largest coefficients. When $\mathbf{n} = \mathbb{N}$, $\mathscr{B}$ is $\mathbf{C}_{ql}$-quasi-greedy for largest coefficients.

It is immediate that if $\mathscr{B}$ is $\mathbf{C}_{q,t}$-t-$\mathbf{n}$-quasi-greedy, it is also $\mathbf{C}_{ql}$-$\mathbf{n}$-quasi-greedy for largest coefficients with $\mathbf{C}_{ql} \leqslant \mathbf{C}_{q,t}$.

Note that it is enough to take x a finite linear combination of some of the $\mathbf{e}_j$'s in Definition 8. More precisely, we have the following elementary characterization.

LEMMA 8. *A basis $\mathscr{B}$ is $\mathbf{n}$-quasi-greedy for largest coefficients if and only if there exists a positive constant $\mathbf{L}$ such that*

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{L}\|x + \mathbf{1}_{\varepsilon A}\|, \tag{24}$$

for every $A \subset \mathbb{N}$ with $|A| \in \mathbf{n}$, $\boldsymbol{\varepsilon} \in \Psi_A$, and all $x \in [\mathbf{e}_j : j \in \mathbb{N}]$ such that $\operatorname{supp}(x) \cap A = \emptyset$ and $|\mathbf{e}_j^*(x)| \leqslant 1$ for all $j \in \mathbb{N}$. Moreover, if (24) holds, then $\mathbf{C}_{ql} \leqslant \mathbf{L}$.

Proof. Clearly we only need to show that if (24) holds, then it also holds for $x \in \mathbb{X} \setminus [\mathbf{e}_j : j \in \mathbb{N}]$, that is for x which is not a finite linear combination of some of the $\mathbf{e}_j$'s. Given such x, there is a sequence $(x_k)_{k \in \mathbb{N}} \subset [\mathbf{e}_j : j \in \mathbb{N}]$ such that

$$x_k \xrightarrow[k \to \infty]{} x.$$

For each k, let $y_k := x_k - P_A(x_k)$. Since $\mathbf{e}_j^*(x) = 0$ for all $j \in A$ and A is finite, we have

$$y_k \xrightarrow[k \to \infty]{} x,$$

so

$$\|y_k\|_\infty \xrightarrow[k \to \infty]{} \|x\|_\infty.$$

Hence, if $\|x\|_\infty < 1$, there is $k_0 \in \mathbb{N}$ such that $\|y_k\|_\infty \leqslant 1$ for all $k \geqslant k_0$, so

$$\|\mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{L}\|\mathbf{1}_{\varepsilon A} + y_{k+k_0}\| \xrightarrow[k \to \infty]{} \mathbf{L}\|\mathbf{1}_{\varepsilon A} + x\|.$$

On the other hand, if $\|x\|_\infty = 1$, define

$$z_k := \begin{cases} \|y_k\|_\infty^{-1} y_k & \text{if } y_k \neq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Since

$$z_k \xrightarrow[k\to\infty]{} x$$

and $\|z_k\|_\infty \leqslant 1$ for all $k \in \mathbb{N}$, the proof is completed by the same argument used in the case $\|x\|_\infty < 1$. $\quad\square$

If a basis is quasi-greedy for largest coefficients, it is unconditional for constant coefficients (this follows for example from the proof of [19, Proposition 3], or from [6, Remark 3.4]). Hence, Example 2 shows that, for $\mathbf{n}$ with arbitrarily large quotient gaps, $\mathbf{n}$-quasi-greediness for largest coefficients is not equivalent to its regular counterpart. On the other hand, the following proposition shows that equivalence holds in the remaining cases.

PROPOSITION 6. *Suppose* $\mathbf{n}$ *has* l-*bounded quotient gaps, and* $\mathscr{B}$ *is* $\mathbf{n}$-$\mathbf{C}_{ql}$-*quasi-greedy for largest coefficients. Then* $\mathscr{B}$ *is* $\mathbf{C}$-*quasi-greedy for largest coefficients with*

$$\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1 \alpha_2, l\mathbf{C}_{ql}\}$$

Proof. Fix a finite set $A \subset \mathbb{N}$ with $0 < |A| \notin \mathbf{n}$, and x, $\boldsymbol{\varepsilon} \in \Psi_A$ as in Definition 8. If $\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant (n_1 - 1)\alpha_1$, then

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant (n_1 - 1)\alpha_1 \alpha_2 \|\mathbf{1}_{\boldsymbol{\varepsilon}A} + x\|.$$

Otherwise, by Lemma 2, there is $B \subset A$ with $|B| \in \mathbf{n}$ such that $\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant l\|\mathbf{1}_{\boldsymbol{\varepsilon}B}\|$. Hence,

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant l\|\mathbf{1}_{\boldsymbol{\varepsilon}B}\| \leqslant l\mathbf{C}_{ql}\|\mathbf{1}_{\boldsymbol{\varepsilon}B} + \mathbf{1}_{\boldsymbol{\varepsilon}A\setminus B} + x\| = l\mathbf{C}_{ql}\|\mathbf{1}_{\boldsymbol{\varepsilon}A} + x\|,$$

and the proof is complete. $\quad\square$

Next, we consider an extension of suppression unconditionality for constant coefficients, a property studied in [1, 5, 6], among others. This property is equivalent to unconditionality for constant coefficients (see [6, Remark 3.4]) but, as we shall see, their extensions to our context behave differently and are not in general equivalent.

DEFINITION 9. We say that $\mathscr{B}$ is $\mathbf{n}$-*suppression unconditional for constant coefficients* if there is $\mathbf{C} > 0$ such that

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant \mathbf{C}\|\mathbf{1}_{\boldsymbol{\varepsilon}B}\|$$

for all $A \subset B \subset \mathbb{N}$ with $|A| \in \mathbf{n}$ and all $\boldsymbol{\varepsilon} \in \Psi_B$. The smallest constant verifying the above inequality is denoted by $\mathbf{K}_{suc}$ and we say that $\mathscr{B}$ is $\mathbf{K}_{suc}$-$\mathbf{n}$-*suppression unconditional for constant coefficients*. If $\mathbf{n} = \mathbb{N}$, we say that $\mathscr{B}$ is $\mathbf{K}_{suc}$-*suppression unconditional for constant coefficients*.

It is immediate from the definition that if $\mathscr{B}$ is $\mathbf{C}_{ql}$-$\mathbf{n}$-quasi-greedy for largest coefficients, it is $\mathbf{K}_{suc}$-$\mathbf{n}$-suppression unconditional for constant coefficients with $\mathbf{K}_{suc} \leqslant \mathbf{C}_{ql}$.

Unlike $\mathbf{n}$-unconditionality for constant coefficients (see Propositions 1 and 3), for sequences with bounded quotient gaps $\mathbf{n}$-suppression unconditionality for constant coefficients is equivalent to its regular counterpart.

PROPOSITION 7. *Suppose* $\mathbf{n}$ *has* l-*bounded quotient gaps, and* $\mathscr{B}$ *is* $\mathbf{K}_{suc}$-$\mathbf{n}$-*suppression unconditional for constant coefficients. Then* $\mathscr{B}$ *is* $\mathbf{C}$-*suppression unconditional for constant coefficients, with*

$$\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, l\mathbf{K}_{suc}\}.$$

Proof. This is proven by a simpler variant of the argument of Proposition 6, taking $x = \mathbf{1}_{\varepsilon'E}$ for some finite set $E \subset \mathbb{N}$ and $\varepsilon' \in \Psi_E$. $\quad\square$

Finally, we extend the property of being symmetric for largest coefficients to the context of sequences with gaps. This property was introduced in [2] (as Property (A)) and studied in [6, 8, 11, 6].

DEFINITION 10. We say that $\mathscr{B}$ is $\mathbf{n}$-symmetric for largest coefficients if there exists a positive constant $\mathbf{C}$ such that

$$\|x + \mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{C}\|x + \mathbf{1}_{\varepsilon'B}\|, \tag{26}$$

for any pair of sets A, B with $|A| \leqslant |B|$, $A \cap B = \emptyset$, $|A|, |B| \in \mathbf{n}$, for any $\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B$ and for any $x \in \mathbb{X}$ such that $|\mathbf{e}_i^*(x)| \leqslant 1 \,\forall i \in \mathbb{N}$ and $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$. The smallest constant verifying (26) is denoted by Δ and we say that $\mathscr{B}$ is Δ-$\mathbf{n}$-symmetric for largest coefficients. If $\mathbf{n} = \mathbb{N}$, we say that $\mathscr{B}$ is Δ-symmetric for largest coefficients.

Note that our definition is equivalent to only requiring that $|A| = |B| \in \mathbf{n}$ instead of $|A| \leqslant |B| \in \mathbf{n}$, and $x \in [\mathbf{e}_j : j \in \mathbb{N}]$. The following lemma proves these facts.

LEMMA 9. *A basis* $\mathscr{B}$ *is* $\mathbf{n}$-*symmetric for largest coefficients if and only if there exists a positive constant* $\mathbf{L}$ *such that*

$$\|x + \mathbf{1}_{\varepsilon A}\| \leqslant \mathbf{L}\|x + \mathbf{1}_{\varepsilon'B}\|, \tag{27}$$

for any pair of sets A, B *with* $|A| = |B|$, $A \cap B = \emptyset$, $|B| \in \mathbf{n}$, *for any* $\varepsilon \in \Psi_A, \varepsilon' \in \Psi_B$ *and for any* $x \in [\mathbf{e}_j : j \in \mathbb{N}]$ *such that* $|\mathbf{e}_i^*(x)| \leqslant 1 \,\forall i \in \mathbb{N}$ *and* $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$. *Moreover,* Δ *is the minimum* $\mathbf{L}$ *for which* (27) *holds.*

Proof. Of course, we only have to show that (27) implies $\mathbf{n}$-symmetry for largest coefficients with constant no greater than $\mathbf{L}$. Let $x, A, B, \varepsilon, \varepsilon'$ be as in Definition 10, with the additional condition that $x \in [\mathbf{e}_j : j \in \mathbb{N}]$. If $|A| = |B| \in \mathbf{n}$, there is nothing to prove. Else, choose a set $C > \operatorname{supp}(x) \cup A \cup B$ such that $|A| + |C| = |B| \in \mathbf{n}$. We have

$$\|x + \mathbf{1}_{\varepsilon A}\| \leqslant \frac{1}{2}\left(\|x + \mathbf{1}_{\varepsilon A} + \mathbf{1}_C\| + \|x + \mathbf{1}_{\varepsilon A} - \mathbf{1}_C\|\right) \leqslant \mathbf{L}\|x + \mathbf{1}_{\varepsilon'B}\|.$$

To prove the result for $x \notin [\mathbf{e}_j : j \in \mathbb{N}]$, apply the argument of Lemma 8. $\quad\square$

REMARK 3. Note that for Markushevich bases, $x \in [\mathbf{e}_j : j \in \mathbb{N}]$ if and only if x has finite support, so for such bases Lemmas 8 and 9 can be proved using [8, Lemma 3.2] (a result that can also be extended to bases that are not total, with only a slight modification of the proof).

Next, we study the relation between $\mathbf{n}$-symmetry for largest coefficients and $\mathbf{n}$-superdemocracy.

LEMMA 10. Let $\mathscr{B}$ be a basis. If $\mathscr{B}$ is Δ-$\mathbf{n}$-symmetric for largest coefficients, it is Δ_s-$\mathbf{n}$-superdemocratic with $\Delta_s \leqslant \Delta^2$.

Proof. Consider two sets A, B with cardinality in $\mathbf{n}$ and $|A| \leqslant |B|$, and a set $C > A \cup B$ such that $|C| = |A|$. Then,

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|} = \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_C\|} \frac{\|\mathbf{1}_C\|}{\|\mathbf{1}_{\varepsilon' B}\|} \leqslant \Delta^2. \quad \square \tag{28}$$

In the case $\mathbf{n} = \mathbb{N}$, it is known that if $\mathscr{B}$ is Δ-symmetric for largest coefficients, then it is Δ_s-superdemocratic with $\Delta_s \leqslant 2\kappa\Delta$ ([6, Proposition 1.1]). In Lemma 10, for a general sequence $\mathbf{n}$, we have shown that if $\mathscr{B}$ is Δ-$\mathbf{n}$-symmetric for largest coefficients, it is Δ_s-$\mathbf{n}$-superdemocratic with $\Delta_s \leqslant \Delta^2$. This suggests the question of whether the latter estimate can be improved in the sense that $\Delta_s \lesssim \Delta$. Our next result shows that this is not possible. In fact, it is not even possible to obtain $\Delta_s \lesssim \Delta^p$ for any $1 < p < 2$.

PROPOSITION 8. *Let $0 < \delta < 1$ and $M > 1$. There is a sequence $\mathbf{n}$ and a Banach space $\mathbb{X}$ with a Schauder basis $\mathscr{B}$ that is Δ-$\mathbf{n}$-symmetric for largest coefficients and Δ_s-$\mathbf{n}$-superdemocratic with*

$$\Delta > M \qquad and \qquad \Delta_s \geqslant \Delta^{2-\delta}.$$

Proof. Fix $0 < \varepsilon < 1 < q < p$ so that the following hold:

$$1 - \frac{1}{q} \leqslant \frac{1}{q} - \frac{1}{p+\varepsilon}, \tag{29}$$

$$1 - \frac{1}{p} \geqslant (2-\delta)\left(\frac{1}{q} - \frac{1}{p+\varepsilon}\right). \tag{30}$$

For example, one can take $q = \frac{8}{5}$ and $p = 4 - \varepsilon$ for a sufficiently small ε. Now choose $m \in \mathbb{N}$ an even number sufficiently large so that

$$m^{\frac{1}{q} - \frac{1}{p+\varepsilon}} > 2 + 2^{\frac{1}{p}} m^{\frac{1}{q} - \frac{1}{p}} \qquad and \qquad m^{1 - \frac{1}{p}} > M^2. \tag{31}$$

Define $\mathbb{X}$ as the completion of $\mathfrak{c}_{00}$ with the norm

$$\|(a_i)_i\| := \max\left\{ \left|\sum_{i=1}^m a_i\right|, \left(\sum_{i=1}^m |a_i|^p\right)^{\frac{1}{p}}, \left(\sum_{i=m+1}^\infty |a_i|^q\right)^{\frac{1}{q}} \right\},$$

and let $\mathbf{n}$ be the sequence $\{m\} \cup \mathbb{N}_{>m^q+m}$, and $B_m := \{1,\ldots,m\}$.

As the norm $\|\cdot\|$, when restricted to $(\mathbf{e}_i)_{i \geqslant m+1}$, coincides with the usual norm on ℓ_q, it follows easily that the unit vector basis $\mathscr{B} = (\mathbf{e}_i)_{i \in \mathbb{N}}$ is a symmetric basis for $\mathbb{X}$, and thus it is symmetric for largest coefficients. Hence, in particular there are constants $\Delta > 0$ and $\Delta_s > 0$ such that $\mathscr{B}$ is Δ-$\mathbf{n}$-symmetric for largest coefficients and Δ_s-$\mathbf{n}$-superdemocratic.

To estimate Δ, by Lemma 9 it is enough to consider sets $A, B \subset \mathbb{N}$ with $|A| = |B| \in \mathbf{n}$, $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$, and $x \in \mathbb{X}$ with finite support such that $|\mathbf{e}_i^*(x)| \leqslant 1 \, \forall i \in \mathbb{N}$ and $\operatorname{supp}(x) \cap (A \cup B) = \emptyset$.

First we consider the case $|A| = |B| > m^q + m$. Take $D > A \cup B \cup \operatorname{supp}(x)$ with $|D| = m$. By (29) we have

$$\|P_{B_m}(x + \mathbf{1}_{\varepsilon A})\| \leqslant m \leqslant \frac{m}{(|B| - m)^{\frac{1}{q}}} \left(\sum_{i > m} |\mathbf{e}_i^*(\mathbf{1}_{\varepsilon A \setminus B_m})|^q \right)^{\frac{1}{q}} \leqslant \|x + \mathbf{1}_{\varepsilon' B}\|. \qquad (32)$$

On the other hand,

$$\begin{aligned}
\|P_{B_m^c}(x + \mathbf{1}_{\varepsilon A})\| &= \left(\left(\sum_{i=m+1}^{\infty} |\mathbf{e}_i^*(x)|^q \right) + |A \setminus B_m| \right)^{\frac{1}{q}} \\
&\leqslant \left(\left(\sum_{i=m+1}^{\infty} |\mathbf{e}_i^*(x)|^q \right) + |B \setminus B_m| + |D| \right)^{\frac{1}{q}} \\
&\leqslant \|P_{B_m^c}(x + \mathbf{1}_{\varepsilon B}) + \mathbf{1}_D\| \leqslant \|P_{B_m^c}(x + \mathbf{1}_{\varepsilon B})\| + \|\mathbf{1}_D\| \\
&\leqslant \|x + \mathbf{1}_{\varepsilon B}\| + \|P_{B_m^c}(\mathbf{1}_{\varepsilon' B})\| \leqslant 2\|x + \mathbf{1}_{\varepsilon' B}\|.
\end{aligned} \qquad (33)$$

Combining (32) and (33) we obtain

$$\|x + \mathbf{1}_{\varepsilon A}\| = \max \left\{ \|P_{B_m}(x + \mathbf{1}_{\varepsilon A})\|, \|P_{B_m^c}(x + \mathbf{1}_{\varepsilon A})\| \right\} \leqslant 2\|x + \mathbf{1}_{\varepsilon' B}\|. \qquad (34)$$

Now we consider the case $|A| = |B| = m$. As $|\operatorname{supp}(P_{B_m}(x + \mathbf{1}_{\varepsilon A}))| \leqslant |B_m \setminus B| = |B \setminus B_m|$, by (29) we have

$$\begin{aligned}
\|P_{B_m}(x + \mathbf{1}_{\varepsilon A})\| &\leqslant |\operatorname{supp}(P_{B_m}(x + \mathbf{1}_{\varepsilon A}))| \leqslant |B \setminus B_m|^{\frac{1}{q} - \frac{1}{p+\varepsilon}} |B \setminus B_m|^{\frac{1}{q}} \\
&\leqslant m^{\frac{1}{q} - \frac{1}{p+\varepsilon}} \|P_{B_m^c}(\mathbf{1}_{\varepsilon' B})\| \leqslant m^{\frac{1}{q} - \frac{1}{p+\varepsilon}} \|x + \mathbf{1}_{\varepsilon' B}\|,
\end{aligned} \qquad (35)$$

and

$$\begin{aligned}
\|P_{B_m^c}(\mathbf{1}_{\varepsilon A})\| &\leqslant \frac{m^{\frac{1}{q}}}{\max\left\{ |B \setminus B_m|^{\frac{1}{q}}, |B \cap B_m|^{\frac{1}{p}} \right\}} \max\left\{ |B \setminus B_m|^{\frac{1}{q}}, |B \cap B_m|^{\frac{1}{p}} \right\} \\
&\leqslant \frac{m^{\frac{1}{q}}}{\left(\frac{m}{2}\right)^{\frac{1}{p}}} \max\left\{ \|P_{B_m^c}(x + \mathbf{1}_{\varepsilon' B})\|, \|P_{B_m}(x + \mathbf{1}_{\varepsilon' B})\| \right\} \\
&= 2^{\frac{1}{p}} m^{\frac{1}{q} - \frac{1}{p}} \|x + \mathbf{1}_{\varepsilon' B}\|.
\end{aligned} \qquad (36)$$

As $\|P_{B^c_m}(x)\| \leqslant \|x + \mathbf{1}_{\varepsilon'B}\|$, from (35), (36) and the triangle inequality we obtain

$$
\begin{aligned}
\|x + \mathbf{1}_{\varepsilon A}\| &= \max\left\{\|P_{B_m}(x + \mathbf{1}_{\varepsilon A})\|, \|P^c_{B_m}(x + \mathbf{1}_{\varepsilon A})\|\right\} \\
&\leqslant \max\left\{m^{\frac{1}{q} - \frac{1}{p} + \varepsilon}, 1 + 2^{\frac{1}{p}} m^{\frac{1}{q} - \frac{1}{p}}\right\} \|x + \mathbf{1}_{\varepsilon'B}\| \\
&= m^{\frac{1}{q} - \frac{1}{p+\varepsilon}} \|x + \mathbf{1}_{\varepsilon'B}\|,
\end{aligned}
\tag{37}
$$

where we used (31) for the last estimate. From (34) and (37), using (31) we deduce that

$$
\Delta \leqslant m^{\frac{1}{q} - \frac{1}{p+\varepsilon}}.
\tag{38}
$$

Now let $\varepsilon' \in \Psi_{B_m}$ be any sequence of alternating signs. As m is even, we have

$$
\sum_{i=1}^{m} \mathbf{e}_i^*(\mathbf{1}_{\varepsilon'B_m}) = 0.
$$

Thus,

$$
\|\mathbf{1}_{\varepsilon'B_m}\| = m^{\frac{1}{p}}.
$$

Since $\|\mathbf{1}_{B_m}\| = m$, we conclude (using (30) and (38)) that

$$
\Delta_s \geqslant m^{1 - \frac{1}{p}} \geqslant \left(m^{\frac{1}{q} - \frac{1}{p+\varepsilon}}\right)^{2-\delta} \geqslant \Delta^{2-\delta}.
$$

Finally, from this result and (31), by Lemma 10 we get

$$
\Delta \geqslant \Delta_s^{\frac{1}{2}} \geqslant \left(m^{1 - \frac{1}{p}}\right)^{\frac{1}{2}} > M. \quad \square
$$

Our next result shows that $\mathbf{n}$ symmetry for largest coefficients can be characterized in terms of $\mathbf{n}$-superdemocracy and $\mathbf{n}$-quasi-greediness for largest coefficients (see [1]).

PROPOSITION 9. *A basis $\mathscr{B}$ is $\mathbf{n}$-symmetric for largest coefficients if and only if $\mathscr{B}$ is $\mathbf{n}$-superdemocratic and $\mathbf{n}$-quasi-greedy for largest coefficients. Moreover,*

$$
\mathbf{C}_{ql} \leqslant 1 + \Delta, \ \Delta \leqslant 1 + \mathbf{C}_{ql}(1 + \Delta_s).
$$

Proof. To show that $\mathbf{n}$-superdemocracy and $\mathbf{n}$-quasi-greediness together imply $\mathbf{n}$-symmetry for largest coefficients, just follow the proof of [1, Proposition 4.3]. Assume now that $\mathscr{B}$ is $\mathbf{n}$-symmetric for largest coefficients. By Lemma 10, $\mathscr{B}$ is $\mathbf{n}$-superdemocratic. Given $x \in [\mathbf{e}_j : j \in \mathbb{N}]$, $|A| \in \mathbf{n}$ with $A \cap \mathrm{supp}(x) = \emptyset$ and $\varepsilon \in \Psi_A$, choose $C > \mathrm{supp}(x) \cup A$ so that $|C| = |A|$. We have

$$
\begin{aligned}
\|\mathbf{1}_{\varepsilon A}\| &\leqslant \|x + \mathbf{1}_{\varepsilon A}\| + \|x\| \\
&\leqslant \|x + \mathbf{1}_{\varepsilon A}\| + \frac{1}{2}(\|x + \mathbf{1}_C\| + \|x - \mathbf{1}_C\|) \\
&\leqslant \|x + \mathbf{1}_{\varepsilon A}\| + \Delta\|x + \mathbf{1}_{\varepsilon A}\|.
\end{aligned}
\tag{39}
$$

By (39) and Lemma 8, $\mathscr{B}$ is $\mathbf{C}_{ql}$-$\mathbf{n}$-quasi-greedy for largest coefficients with $\mathbf{C}_{ql} \leqslant 1 + \Delta$. $\square$

Our next two results characterize the sequences $\mathbf{n}$ for which $\mathbf{n}$-symmetry for largest coefficients is equivalent to symmetry for largest coefficients.

PROPOSITION 10. *Let $\mathbf{n}$ be a sequence with arbitrarily large quotient gaps. There is a Banach space $\mathbb{X}$ with a Schauder basis $\mathscr{B}$ that is $\mathbf{n}$-symmetric for largest coefficients but not democratic.*

Proof. See Example 2 and Remark 4. $\square$

THEOREM 1. *Let $\mathscr{B}$ be a basis and assume that $\mathbf{n}$ has l-bounded quotient gaps. If $\mathscr{B}$ is Δ-$\mathbf{n}$-symmetric for largest coefficients, then $\mathscr{B}$ is $\mathbf{C}$-symmetric for largest coefficients with $\mathbf{C} \leqslant \max\{1 + 2(n_1 - 1)\alpha_1\alpha_2, \, 1 + 2\Delta^2(1 + l)\}$.*

Proof. To show that $\mathscr{B}$ is symmetric for largest coefficients we use Lemma 9: Take $x \in [\mathbf{e}_j : j \in \mathbb{N}]$ so that $\max_{j \in \mathbb{N}} |\mathbf{e}_j^*(x)| \leqslant 1$, and two finite sets $A, B \subset \mathbb{N}$ so that $A \cap B = \emptyset$, $|A| = |B|$, and $\mathrm{supp}(x) \cap (A \cup B) = \emptyset$.

Assume first that there exists $i \in \mathbb{N}$ such that $n_i \leqslant m \leqslant n_{i+1}$ with $n_i, n_{i+1} \in \mathbf{n}$. Then, we can decompose $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ with $|A_0| = |B_0| = n_i \in \mathbf{n}$. Thus,

$$\|x + \mathbf{1}_{\varepsilon A}\| \leqslant \|x + \mathbf{1}_{\varepsilon'B}\| + \|\mathbf{1}_{\varepsilon A_0}\| + \|\mathbf{1}_{\varepsilon A_1}\| + \|\mathbf{1}_{\varepsilon'B_0}\| + \|\mathbf{1}_{\varepsilon'B_1}\|. \tag{40}$$

Take $C > \mathrm{supp}(x) \cup A \cup B$ such that $|C| = |A_0|$. Hence,

$$\begin{aligned}
\|\mathbf{1}_{\varepsilon A_0}\| \leqslant \Delta\|\mathbf{1}_C\| &\leqslant \frac{\Delta}{2}(\|x + \mathbf{1}_{\varepsilon'B_1} + \mathbf{1}_C\| + \|x + \mathbf{1}_{\varepsilon'B_1} - \mathbf{1}_C\|) \\
&\leqslant \Delta \max\{\|x + \mathbf{1}_{\varepsilon'B_1} + \mathbf{1}_C\|, \|x + \mathbf{1}_{\varepsilon'B_1} - \mathbf{1}_C\|\} \\
&\leqslant \Delta^2 \|x + \mathbf{1}_{\varepsilon'B_1} + \mathbf{1}_{\varepsilon'B_0}\| = \Delta^2 \|x + \mathbf{1}_{\varepsilon'B}\|.
\end{aligned} \tag{41}$$

Thus, the same argument for (41) can be used to estimate $\|\mathbf{1}_{\varepsilon'B_0}\|$, and we obtain that

$$\max\{\|\mathbf{1}_{\varepsilon A_0}\|, \|\mathbf{1}_{\varepsilon'B_0}\|\} \leqslant \Delta^2 \|x + \mathbf{1}_{\varepsilon'B}\|. \tag{42}$$

To estimate $\|\mathbf{1}_{\varepsilon A_1}\|$, take now a set $F > \mathrm{supp}(x) \cup A \cup B \cup C$ such that $|F| + |A_1| = ln_i$, and write

$$\mathbf{1}_{\varepsilon A_1} \pm \mathbf{1}_F = \sum_{j=1}^{l} \mathbf{1}_{\eta T_j},$$

where $T_k \cap T_i = \emptyset$ for $i \neq k$, $|T_j| = n_i$ for all $j = 1, \ldots, l$ and η the corresponding sign. Hence, since $\|\mathbf{1}_{\eta T_j}\| \leqslant \Delta\|\mathbf{1}_C\|$ for all $j = 1, \ldots, l$,

$$\begin{aligned}
\|\mathbf{1}_{\varepsilon A_1}\| &\leqslant \frac{1}{2}(\|\mathbf{1}_{\varepsilon A_1} + \mathbf{1}_F\| + \|\mathbf{1}_{\varepsilon A_1} - \mathbf{1}_F\|) \\
&\leqslant \max\{\|\mathbf{1}_{\varepsilon A_1} + \mathbf{1}_F\|, \|\mathbf{1}_{\varepsilon A_1} - \mathbf{1}_F\|\} \\
&\leqslant l\Delta\|\mathbf{1}_C\| \overset{(41)}{\leqslant} l\Delta^2 \|x + \mathbf{1}_{\varepsilon'B}\|.
\end{aligned} \tag{43}$$

Applying (43) to estimate $\|\mathbf{1}_{\varepsilon' B_1}\|$, we obtain

$$\max\{\|\mathbf{1}_{\varepsilon A_1}\|, \|\mathbf{1}_{\varepsilon' B_1}\|\} \leqslant \Delta^2 \|x + \mathbf{1}_{\varepsilon' B}\|. \tag{44}$$

Thus, applying (42) and (44) in (40),

$$\|x + \mathbf{1}_{\varepsilon A}\| \leqslant (1 + 2\Delta^2 + 2l\Delta^2)\|x + \mathbf{1}_{\varepsilon' B}\|,$$

for sets A and B with cardinality equal to or greater than n_1. Assume now $|A| < n_1$. In that case,

$$\begin{aligned}
\|x + \mathbf{1}_{\varepsilon A}\| &\leqslant \|x + \mathbf{1}_{\varepsilon' B}\| + \|\mathbf{1}_{\varepsilon A}\| + \|\mathbf{1}_{\varepsilon' B}\| \\
&\leqslant \|x + \mathbf{1}_{\varepsilon' B}\| + 2\alpha_1(n_1 - 1) \\
&\leqslant (1 + 2(n_1 - 1)\alpha_1\alpha_2)\|x + \mathbf{1}_{\varepsilon' B}\|.
\end{aligned}$$

Thus, the basis is $\mathbf{C}$-symmetric for largest coefficients with

$$\mathbf{C} \leqslant \max\{1 + 2(n_1 - 1)\alpha_1\alpha_2, \, 1 + 2\Delta^2(1 + l)\}. \quad \square$$

To close this section, we use $\mathbf{n}$-democracy and the $\mathbf{n}$-UL property as an alternative to the Schauder condition in [4, Theorem 5.2] – where it is proven that if $\mathbf{n}$ has bounded quotient gaps, every $\mathbf{n}$-quasi-greedy Schauder basis is quasi-greedy – and we also obtain symmetry for largest coefficients.

PROPOSITION 11. *Suppose $\mathbf{n}$ is a sequence with l-bounded quotient gaps, and $\mathscr{B}$ is a basis that is $\mathbf{C}_{q,t}$-t-$\mathbf{n}$-quasi-greedy and has the $\mathbf{n}$-UL-property with constants $\mathbf{C}_1$ and $\mathbf{C}_2$. Then, the following hold:*

i) *If $\mathscr{B}$ is Δ_d-$\mathbf{n}$-democratic, it is $\mathbf{C}$-t-quasi-greedy with*

$$\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{C}_{q,t}(1 + (l-1)\mathbf{C}_1\mathbf{C}_2\Delta_d)\},$$

and is Δ-symmetric for largest coefficients with

$$\Delta \leqslant \max\{1 + 2(n_1 - 1)\alpha_1\alpha_2, 1 + 2(1+l)(1 + \mathbf{C}_{q,t}(1 + \mathbf{C}_1\mathbf{C}_2\Delta_d))^2\}.$$

ii) *If $\mathscr{B}$ is Δ_s-$\mathbf{n}$-superdemocratic, it is $\mathbf{C}$-t-quasi-greedy with*

$$\mathbf{C} \leqslant \max\{(n_1 - 1)\alpha_1\alpha_2, \mathbf{C}_{q,t}(1 + (l-1)\mathbf{C}_1\Delta_s)\},$$

and is Δ-symmetric for largest coefficients with

$$\Delta \leqslant \max\{1 + 2(n_1 - 1)\alpha_1\alpha_2, 1 + 2(1+l)(1 + \mathbf{C}_{q,t}(1 + \Delta_s))^2\}.$$

Proof. i) Fix $x \in \mathbb{X}$ and A a t-greedy set for x with $|A| \notin \mathbf{n}$. If $|A| < n_1$, then

$$\|P_A(x)\| \leqslant \sum_{i \in A} |\mathbf{e}_i^*(x)| \, \|\mathbf{e}_i\| \leqslant \alpha_1\alpha_2(n_1 - 1)\|x\|.$$

If $|A| > n_1$, define
$$k_0 := \max_{k \in \mathbb{N}} \{n_k < |A|\},$$

and let $\{A_i\}_{1 \leqslant i \leqslant j}$ be a partition of A such that $2 \leqslant j \leqslant l$, A_1 is an n_{k_0}-greedy set for $P_A(x)$, and $|A_i| \leqslant n_{k_0}$ for all $2 \leqslant i \leqslant j$. Since A_1 is a t-greedy set for x of cardinality n_{k_0}, we have
$$||P_{A_1}(x)|| \leqslant \mathbf{C}_{q,t} ||x||. \tag{45}$$

For every $2 \leqslant i \leqslant j$, choose $A_i \subset D_i$ such that $|D_i| = n_{k_0}$. Given that for every $2 \leqslant i \leqslant j$,
$$\max_{m \in A_i} |\mathbf{e}_m^*(x)| \leqslant \min_{m \in A_1} |\mathbf{e}_m^*(x)|,$$

using convexity and the $\mathbf{n}$-UL and the $\mathbf{n}$-democracy properties we obtain
$$\begin{aligned}
||P_{A_i}(x)|| &\leqslant \max_{m \in A_i} |\mathbf{e}_m^*(x)| \sup_{\boldsymbol{\varepsilon} \in \Psi_{A_i}} ||\mathbf{1}_{\boldsymbol{\varepsilon} A_i}|| \leqslant \min_{m \in A_1} |\mathbf{e}_m^*(x)| \sup_{\boldsymbol{\varepsilon} \in \Psi_{D_i}} ||\mathbf{1}_{\boldsymbol{\varepsilon} D_i}|| \\
&\leqslant \min_{m \in A_1} |\mathbf{e}_m^*(x)| \mathbf{C}_2 ||\mathbf{1}_{D_i}|| \leqslant \mathbf{C}_2 \Delta_d \min_{m \in A_1} |\mathbf{e}_m^*(x)| ||\mathbf{1}_{A_1}|| \\
&\leqslant \mathbf{C}_1 \mathbf{C}_2 \Delta_d ||P_{A_1}(x)||.
\end{aligned} \tag{46}$$

Combining this result with (45) and using the triangle inequality, we get
$$||P_A(x)|| \leqslant \sum_{i=1}^{j} ||P_{A_i}(x)|| \leqslant \mathbf{C}_{q,t} \left(1 + (l-1) \mathbf{C}_1 \mathbf{C}_2 \Delta_d\right) ||x||.$$

This proves that $\mathscr{B}$ is t-quasi-greedy with constant as in the statement. To prove that it is symmetric for largest coefficients, we apply Proposition 9 and Theorem 1, considering that $\mathscr{B}$ is $\mathbf{C}_{ql}$-$\mathbf{n}$-quasi-greedy for largest coefficients and Δ_s-$\mathbf{n}$-superdemocratic, with $\mathbf{C}_{ql} \leqslant \mathbf{C}_{q,t}$, and $\Delta_s \leqslant \mathbf{C}_1 \mathbf{C}_2 \Delta_d$.

ii) This is proven by essentially the same argument as the previous case. The only differences are that instead of (46), we obtain
$$\begin{aligned}
||P_{A_i}(x)|| &= \max_{m \in A_i} |\mathbf{e}_m^*(x)| \sup_{\boldsymbol{\varepsilon} \in \Psi_{A_i}} ||\mathbf{1}_{\boldsymbol{\varepsilon} A_i}|| \leqslant \min_{m \in A_1} |\mathbf{e}_m^*(x)| \sup_{\boldsymbol{\varepsilon} \in \Psi_{D_i}} ||\mathbf{1}_{\boldsymbol{\varepsilon} D_i}|| \\
&\leqslant \min_{m \in A_1} |\mathbf{e}_m^*(x)| \Delta_s ||\mathbf{1}_{A_1}|| \\
&\leqslant \mathbf{C}_1 \Delta_s ||P_{A_1}(x)||
\end{aligned}$$

(and thus, we also get Δ_s instead of $\mathbf{C}_2 \Delta_d$ in the upper bound for $\mathbf{C}$), and that we apply Proposition 9 using the hypothesis that $\mathscr{B}$ is Δ_s-$\mathbf{n}$-superdemocratic. $\quad\square$

5. Examples

In this section, we consider two families of examples that are used throughout the paper, and study the relevant properties of the bases. First, we construct a family of examples that proves that for sequences with arbitrarily large additive gaps, $\mathbf{n}$-unconditionality for constant coefficients, the $\mathbf{n}$-UL property, $\mathbf{n}$-(super)democracy and $\mathbf{n}$-(super)-conservativeness are not equivalent to their standard counterparts.

EXAMPLE 1. Given $\mathbf{n}$ with arbitrarily large additive gaps, choose recursively a subsequence $(n_{k_i})_{i \in \mathbb{N}_0}$ and $(m_i)_{i \in \mathbb{N}}$ a sequence of positive integers with $m_1 > 4$ so that for every $i \in \mathbb{N}$,

$$m_i n_{k_i+1}^3 < m_{i+1}, \qquad m_i^2 < n_{k_i} \qquad n_{k_i} + 2m_i < n_{k_i+1} \qquad \text{and} \qquad n_{k_i+1}^2 < n_{k_{i+1}}, \quad (47)$$

and choose a sequence of sets of positive integers $(B_i)_{i \in \mathbb{N}}$ so that

$$m_i < B_i < B_{i+1} \qquad \text{and} \qquad |B_i| =_{n_{k_i}+m_i} \qquad \forall i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, define

$$\mathscr{F}_i := \left\{ f = (f_j)_{j \in \mathbb{N}} : |\mathrm{supp}(f)| \leqslant n_{k_i+1}, \|f\|_\infty \leqslant 1, \|f\|_1 \leqslant m_i, \sum_{j \in B_i} f_j = 0 \right\}$$

and, for $(a_j)_{j \in \mathbb{N}} \in c_{00}$,

$$\left\| (a_j)_{j \in \mathbb{N}} \right\|_{\diamond,i} := \frac{1}{n_{k_{i-1}+1}^2} \sup_{f \in \mathscr{F}_i} \left| \sum_{j \in \mathbb{N}} f_j a_j \right|.$$

Let $\mathbb{X}$ be the completion of c_{00} with the norm

$$\|x\| = \max \left\{ \|x\|_\infty, \|x\|_\diamond := \sup_{i \in \mathbb{N}} \|x\|_{\diamond,i} \right\},$$

and let $\mathscr{B}$ be the canonical unit vector system of $\mathbb{X}$. Then, the following hold:

a) $\mathscr{B}$ is a normalized Markushevich basis for $\mathbb{X}$ with normalized biorthogonal funcitionals $\mathscr{B}^*$.

b) $\mathscr{B}$ is $\mathbf{C}$-$\mathbf{n}$-superdemocratic (and hence also $\mathbf{C}$-$\mathbf{n}$-conservative) with $\mathbf{C} \leqslant 2$.

c) $\mathscr{B}$ has the $\mathbf{n}$-UL property, with $\max\{\mathbf{C}_1, \mathbf{C}_2\} \leqslant 2$

d) $\mathscr{B}$ is not $\mathbf{n}$-suppression unconditional for constant coefficients, and thus not unconditional for constant coefficients and it does not have the UL-Property.

e) $\mathscr{B}$ is not conservative and thus not democratic.

Proof. a) It is clear that $\mathscr{B}$ and $\mathscr{B}^*$ are normalized. To see that $\mathscr{B}$ is a Markushevich basis, fix $x \in \mathbb{X}$ such that $e_j^*(x) = 0$ for all $j \in \mathbb{N}$, and choose a sequence $(x_l)_{l \in \mathbb{N}}$ with $x_l \in [e_j : 1 \leqslant l \leqslant s(l)]$ for some $s(l) \in \mathbb{N}$, and

$$x_l \xrightarrow[l \to \infty]{} x.$$

Given $v > 0$, choose $l_0 \in \mathbb{N}$ so that

$$\|x_l - x\| \leqslant v \qquad \forall l \geqslant l_0.$$

Now pick i_0 and $f \in \mathscr{F}_{i_0}$ so that

$$\left\| x_{l_0} \right\|_\diamond \leqslant v + \left\| x_{l_0} \right\|_{\diamond,i_0} \leqslant 2v + \sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(x_{l_0} \right).$$

Since f has finite support and $\mathbf{e}_j^*(x) = 0$ for all $j \in \mathbb{N}$, there is $l_1 > l_0$ such that

$$\sum_{1 \leqslant j \leqslant \max(\operatorname{supp}(f)) + s(l_0)} \left| \mathbf{e}_j^* \left(x_{l_1} \right) \right| \leqslant v. \tag{48}$$

Hence,

$$\sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(x_{l_0} \right) \leqslant \left| \sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(x_{l_1} \right) \right| + \left| \sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(x_{l_0} - x_{l_1} \right) \right| \leqslant v + \left\| x_{l_0} - x_{l_1} \right\| \leqslant 3v.$$

It follows that

$$\left\| x_{l_0} \right\|_\diamond \leqslant 5v.$$

Also by (48),

$$\left\| x_{l_0} \right\|_\infty = \sup_{j \in \operatorname{supp}(x_{l_0})} \left| \mathbf{e}_j^* \left(x_{l_0} \right) \right| \leqslant \sup_{j \in \operatorname{supp}(x_{l_0})} \left| \mathbf{e}_j^* \left(x_{l_1} \right) \right| + \sup_{j \in \operatorname{supp}(x_{l_0})} \left| \mathbf{e}_j^* \left(x_{l_1} - x_{l_0} \right) \right| \leqslant 3v.$$

We deduce that

$$\| x \| \leqslant v + \left\| x_{l_0} \right\| \leqslant 6v.$$

Since v is arbitrary, this entails that $x = 0$ and completes the proof of a).

To prove the rest of the statements, first we show the following:

i. For all $i \in \mathbb{N}$, all sets $A \subset \mathbb{N}$ with $1 \leqslant |A| \leqslant n_{k_i+1}$ and all scalars $(a_j)_{j \in A}$,

$$\left\| \sum_{j \in A} a_j \mathbf{e}_j \right\|_\diamond \leqslant \max_{j \in A} |a_j| \frac{m_i}{n_{k_{i-1}+1}^2}.$$

ii. For all $i \in \mathbb{N}$, all sets $A \subset \mathbb{N}$ with $m_i \leqslant |A| \leqslant n_{k_i}$ and all scalars $(a_j)_{j \in A}$,

$$\left\| \sum_{j \in A} a_j \mathbf{e}_j \right\|_\diamond \geqslant \frac{1}{2} \min_{j \in A} |a_j| \frac{m_i}{n_{k_{i-1}+1}^2}.$$

iii. For all $i \in \mathbb{N}_{\geqslant 2}$, all sets $A \subset \mathbb{N}$ with $n_{k_{i-1}+1} \leqslant |A| \leqslant m_i$ and all scalars $(a_j)_{j \in A}$,

$$\min_{j \in A} |a_j| \max \left\{ \frac{|A|}{2n_{k_{i-1}+1}^2}, \frac{m_{i-1}}{n_{k_{i-2}+1}^2} \right\} \leqslant \left\| \sum_{j \in A} a_j \mathbf{e}_j \right\|_\diamond$$

$$\leqslant \max_{j \in A} |a_j| \max \left\{ \frac{|A|}{n_{k_{i-1}+1}^2}, \frac{m_{i-1}}{n_{k_{i-2}+1}^2} \right\}.$$

iv. For all $A \subset \mathbb{N}$ with $1 \leqslant |A| \leqslant m_1$ and all scalars $(a_j)_{j \in A}$,

$$\min_{j \in A} |a_j| \frac{|A|}{2n_{k_0+1}^2} \leqslant \left\| \sum_{j \in A} a_j \mathbf{e}_j \right\|_{\diamond} \leqslant \max_{j \in A} |a_j| \frac{|A|}{n_{k_0+1}^2}.$$

To prove i., suppose first that $l > i$ and $f \in \mathscr{F}_l$. By (47) we get

$$\frac{1}{n_{k_{l-1}+1}^2} \left| \sum_{j \in A} f_j a_j \right| \leqslant \max_{j \in A} |a_j| \frac{|A|}{n_{k_{l-1}+1}^2} \leqslant \max_{j \in A} |a_j| \frac{n_{k_i+1}}{n_{k_i+1}^2} \leqslant \max_{j \in A} |a_j| \frac{m_i}{n_{k_{i-1}+1}^2}.$$

Similarly, for each $l < i$ and each $f \in \mathscr{F}_l$ we have

$$\frac{1}{n_{k_{l-1}+1}^2} \left| \sum_{j \in A} f_j a_j \right| \leqslant \max_{j \in A} |a_j| \frac{m_l}{n_{k_{l-1}+1}^2} \leqslant \max_{j \in A} |a_j| \frac{m_i}{n_{k_{i-1}+1}^2}.$$

Since, for $f \in \mathscr{F}_i$,

$$\frac{1}{n_{k_{i-1}+1}^2} \left| \sum_{j \in A} f_j a_j \right| \leqslant \max_{j \in A} |a_j| \frac{1}{n_{k_{i-1}+1}^2} \sum_{j \in \mathbb{N}} |f_j| \leqslant \max_{j \in A} |a_j| \frac{m_i}{n_{k_{i-1}+1}^2},$$

taking supremum we complete the proof of i.

Next, we prove ii. Assume $a_j \neq 0$ for all $j \in A$, choose $A_1 \subset A$ and $A_2 \subset B_i \setminus A$ with $|A_1| = |A_2| = m_i$, and let

$$f_j := \begin{cases} \frac{|a_j|}{2a_j} & \text{if } j \in A_1; \\ -\frac{1}{m_i} \sum_{l \in A_1 \cap B_i} f_l & \text{if } j \in A_2; \\ 0 & \text{in any other case.} \end{cases}$$

Then $f = (f_j)_{j \in \mathbb{N}} \in \mathscr{F}_i$, and

$$\left\| \sum_{l \in A} a_l \mathbf{e}_l \right\|_{\diamond} \geqslant \frac{1}{n_{k_{i-1}+1}^2} \left| \sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(\sum_{l \in A} a_l \mathbf{e}_l \right) \right| = \frac{1}{2n_{k_{i-1}+1}^2} \sum_{j \in A_1} |a_j| \geqslant \frac{m_i}{2n_{k_{i-1}+1}^2} \min_{j \in A} |a_j|.$$

To prove iii., by a density argument we may assume $a_j \neq 0$ for all $j \in A$. For every $l \geqslant i$ and every $f \in \mathscr{F}_l$,

$$\frac{1}{n_{k_{l-1}+1}^2} \left| \sum_{j \in A} f_j a_j \right| \leqslant \max_{j \in A} |a_j| \frac{|A|}{n_{k_{i-1}+1}^2}.$$

Hence,

$$\sup_{l \geqslant i} \left\| \sum_{j \in A} a_j \mathbf{e}_j \right\|_{\diamond, l} \leqslant \max_{j \in A} |a_j| \frac{|A|}{n_{k_{i-1}+1}^2}. \tag{49}$$

Now pick $B \subset B_i \setminus A$ with $|B| = |A|$, and define

$$f_j = \begin{cases} \frac{|a_j|}{2a_j} & \text{if } j \in A; \\ -\frac{1}{|A|} \sum_{l \in A \cap B_i} f_l & \text{if } j \in B; \\ 0 & \text{in any other case.} \end{cases}$$

Then $f = (f_j)_{j \in \mathbb{N}} \in \mathscr{F}_i$, and

$$\sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(\sum_{l \in A} a_l \mathbf{e}_l \right) = \sum_{j \in A} f_j a_j = \frac{1}{2} \sum_{j \in A} |a_j|.$$

It follows from this and (49) that

$$\frac{|A|}{2n_{k_{i-1}+1}^2} \min_{j \in A} |a_j| \leqslant \frac{1}{2n_{k_{i-1}+1}^2} \sum_{j \in A} |a_j| \leqslant \sup_{l \geqslant i} \left\| \sum_{j \in A} f_j \mathbf{e}_j \right\|_{\diamond, i} \leqslant \max_{j \in A} |a_j| \frac{|A|}{n_{k_{i-1}+1}^2}. \qquad (50)$$

On the other hand, if $1 \leqslant l < i$, using (47) we obtain

$$\frac{1}{n_{k_{l-1}+1}^2} \left| \sum_{j \in A} f_j a_j \right| \leqslant \max_{j \in A} |a_j| \frac{m_l}{n_{k_{l-1}+1}^2} \leqslant \max_{j \in A} |a_j| \frac{m_{i-1}}{n_{k_{i-2}+1}^2}. \qquad (51)$$

Now pick $A_1 \subset A \setminus B_{i-1}$ with $|A_1| = m_{i-1}$, and set

$$f_j = \begin{cases} \frac{|a_j|}{a_j} & \text{if } j \in A_1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = (f_j)_{j \in \mathbb{N}} \in \mathscr{F}_{i-1}$, and

$$\sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^* \left(\sum_{l \in A} a_l \mathbf{e}_l \right) = \sum_{j \in A} f_j a_j = \sum_{j \in A_1} |a_j| \geqslant m_{i-1} \min_{j \in A} |a_j|$$

which, when combined with (51) gives

$$\min_{j \in A} |a_j| \frac{m_{i-1}}{n_{k_{i-2}+1}^2} \leqslant \sup_{1 \leqslant l < i} \left\| \sum_{j \in A} f_j \mathbf{e}_j \right\|_{\diamond, i} \leqslant \max_{j \in A} |a_j| \frac{m_{i-1}}{n_{k_{i-2}+1}^2}.$$

The proof of iii. is now completed combining the above inequality with (50), whereas iv. is proven by the same argument that gives (50).

b) $\mathbf{n}$-superdemocracy: fix $A, B \subset \mathbb{N}$ with $|A| = |B| = n \in \mathbf{n}$, and $\varepsilon \in A$, $\varepsilon' \in B$. Then $\|\mathbf{1}_{\varepsilon A}\| \leqslant 2\|\mathbf{1}_{\varepsilon' B}\|$ is obtained as follows:

- If there is $l \in \mathbb{N}$ such that $n_{k_l+1} \leqslant n \leqslant m_{l+1}$, apply iii. with $i = l + 1$.

- If there is $l \in \mathbb{N}$ such that $m_l \leqslant n \leqslant n_{k_l}$, combine i. and ii.

- If $n \leqslant m_1$, apply iv.

c) **n**-*UL property:* This is proven in the same manner as Step 1.

d) **n**-*suppression unconditionality for constant coefficients:* Fix $i > 1$, and choose sets $D_i \subset B_i$ with $|D_i| = n_{k_i}$. Then by ii.,

$$\|\mathbf{1}_{D_i}\| \geqslant \frac{m_i}{2n_{k_{i-1}+1}^2}. \tag{52}$$

Let us show that

$$\|\mathbf{1}_{B_i}\| \leqslant \frac{m_{i-1}}{n_{k_{i-2}+1}^2}. \tag{53}$$

For $1 \leqslant l < i$ and $f \in \mathscr{F}_l$, we have

$$\frac{1}{n_{k_{l-1}+1}^2} \left| \sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^*(\mathbf{1}_{B_i}) \right| \leqslant \frac{m_l}{n_{k_{l-1}+1}^2}.$$

Hence,

$$\sup_{1 \leqslant l < i} \|\mathbf{1}_{B_i}\|_{\diamond,i} \leqslant \max_{1 \leqslant l \leqslant i-1} \frac{m_l}{n_{k_{l-1}+1}^2} = \frac{m_{i-1}}{n_{k_{i-2}+1}^2} \qquad \text{(by (47))}.$$

On the other hand, for $l > i$ and $f \in \mathscr{F}_l$,

$$\frac{1}{n_{k_{l-1}+1}^2} \left| \sum_{j \in \mathbb{N}} f_j \mathbf{e}_j^*(\mathbf{1}_{B_i}) \right| \leqslant \frac{|B_i|}{n_{k_{l-1}+1}^2} \leqslant \frac{1}{n_{k_{l-1}+1}} \leqslant 1.$$

Thus,

$$\sup_{l > i} \|\mathbf{1}_{B_i}\|_{\diamond,i} \leqslant 1.$$

Given that by construction $\|\mathbf{1}_{B_i}\|_{\diamond,i} = 0$, (53) is proven, and it follows from that, (47) and (52) that

$$\frac{\|\mathbf{1}_{D_i}\|}{\|\mathbf{1}_{B_i}\|} \xrightarrow[i \to \infty]{} \infty,$$

so $\mathscr{B}$ is not **n**-suppression unconditional for constant coefficients.

e) *conservativeness:* For each $i \geqslant 2$, choose $E_i < B_i$ with $|E_i| = m_i$. By iii.,

$$\|\mathbf{1}_{E_i}\| \geqslant \frac{m_i}{2n_{k_{i-1}+1}^2}.$$

From this, (47) and (53) it follows that

$$\frac{\|\mathbf{1}_{E_i}\|}{\|\mathbf{1}_{B_i}\|} \xrightarrow[i \to \infty]{} \infty,$$

so $\mathscr{B}$ is not conservative. □

Next, we consider a family of examples from [16, Proposition 3.1], with a slight modification for our purposes.

EXAMPLE 2. Suppose $\mathbf{n}$ has arbitrarily large gaps, write $\mathbf{n} = (n_k)_{k=1}^{\infty}$ and find $k_1 < k_2 < \dots$ such that the sequence $(n_{k_i+1}/n_{k_i})_{i=1}^{\infty}$ increases without a bound and $n_{k_1} > 4$. For $i \in \mathbb{N}$, write

$$c_i = \left(\frac{n_{k_i+1}}{n_{k_i}} \right)^{1/4}, \qquad m_i = \left\lfloor \sqrt{n_{k_i+1}n_{k_i}} \right\rfloor.$$

Let $\tilde{m}_i = \sum_{j<i} m_i$ (so that $\tilde{m}_1 = 0$ and $\tilde{m}_{i+1} = \tilde{m}_i + m_i$ for $i \geqslant 1$), $\beta_i := \left\lfloor \frac{m_i}{2} \right\rfloor$, and let $\mathbb{X}$ be completion of c_{00} with the norm:

$$\left\| \sum_j a_j \mathbf{e}_j \right\| = \max \left\{ \|(a_j)_j\|_2, \sup_{i \in \mathbb{N}} \frac{c_i}{\sqrt{m_i}} \max_{i \leqslant l \leqslant m_i} \left| \sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+l} (-1)^{\theta(j)} a_j \right| \right\},$$

where

$$\theta(j) = \begin{cases} 2j & \text{if } \tilde{m}_i + 1 \leqslant j \leqslant \tilde{m}_i + \beta_i \\ j & \text{if } \tilde{m}_i + \beta_i + 1 \leqslant j \leqslant \tilde{m}_i + m_i. \end{cases}$$

The unit vector basis $\mathscr{B} = (\mathbf{e}_i)_{i \in \mathbb{N}}$ is a monotone Schauder basis with the following properties.

a) $\mathscr{B}$ is $\mathbf{n}$-t-quasi-greedy with $\mathbf{C}_{q,t} \leqslant \frac{2}{t}$ for all $0 < t \leqslant 1$, and not quasi-greedy.

b) $\mathscr{B}$ is Δ_s-$\mathbf{n}$-superdemocratic (and hence also Δ_s-$\mathbf{n}$-superconservartive) with $\Delta_s \leqslant \sqrt{2}$.

c) $\mathscr{B}$ is Δ-$\mathbf{n}$-symmetric for largest coefficients with $\Delta \leqslant 3 + 2\sqrt{2}$.

d) $\mathscr{B}$ has the $\mathbf{n}$-UL property with $\max\{\mathbf{C}_1, \mathbf{C}_2\} \leqslant \sqrt{2}$.

e) $\mathscr{B}$ is not conservative and hence not democratic.

f) $\mathscr{B}$ is not unconditional for constant coefficients. Hence, it does not have the UL property.

Proof. It is clear from the definition that $\mathscr{B}$ is a monotone Schauder basis.

a) $\mathbf{n}$-quasi-greediness: This was proven in [16, Proposition 3.1]: The only modification introduced in our construction is that for some $j \in \mathbb{N}$, $\mathbf{e}_j$ is replaced with $-\mathbf{e}_j$, and it is clear that this change does not affect the $\mathbf{n}$-quasi-greedy or quasi-greedy properties.

b) $\mathbf{n}$-superdemocracy: Note that for every $m \in \mathbb{N}$, $2\lfloor \sqrt{m} \rfloor \geqslant \sqrt{m}$, so

$$\frac{1}{\sqrt{\lfloor \sqrt{m} \rfloor}} = \frac{\sqrt{2}}{\sqrt[4]{m}}.$$

Now fix $B \subset \mathbb{N}$ with $|B| \in \mathbf{n}$, and $\boldsymbol{\varepsilon} \in \Psi_B$. For every $i \in \mathbb{N}$ with $|B| \leqslant n_{k_i}$, we have

$$\frac{c_i}{\sqrt{m_i}} \max_{1 \leqslant l \leqslant m_i} \left| \sum_{j=\widetilde{m}_i+1}^{\widetilde{m}_i+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{\boldsymbol{\varepsilon}B}) \right| \leqslant \frac{c_i}{\sqrt{m_i}} |B| = \sqrt[4]{\frac{n_{k_i+1}}{n_{k_i}}} \frac{|B|}{\sqrt{\lfloor \sqrt{n_{k_i} n_{k_i+1}} \rfloor}}$$

$$\leqslant \sqrt{2} \sqrt[4]{\frac{n_{k_i+1}}{n_{k_i}}} \frac{|B|}{\sqrt[4]{n_{k_i} n_{k_i+1}}} = \frac{\sqrt{2}|B|}{\sqrt{n_{k_i}}} \leqslant \frac{\sqrt{2}|B|}{\sqrt{|B|}}$$

$$= \sqrt{2}\sqrt{|B|}. \tag{54}$$

On the other hand, if $|B| \geqslant n_{k_i+1}$, then

$$\frac{c_i}{\sqrt{m_i}} \max_{1 \leqslant k \leqslant m_i} \left| \sum_{j=\widetilde{m}_i+1}^{\widetilde{m}_i+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{\boldsymbol{\varepsilon}B}) \right| \leqslant \frac{c_i}{\sqrt{m_i}} m_i = \sqrt[4]{\frac{n_{k_i+1}}{n_{k_i}}} \sqrt{\lfloor \sqrt{n_{k_i} n_{k_i+1}} \rfloor}$$

$$\leqslant \sqrt[4]{\frac{n_{k_i+1}}{n_{k_i}}} \sqrt{\sqrt{n_{k_i} n_{k_i+1}}} = \sqrt{n_{k_i+1}} \leqslant \sqrt{|B|}. \tag{55}$$

Taking supremum in (54) and (55) we get

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}B}\| \leqslant \sqrt{2}\sqrt{|B|}. \tag{56}$$

As

$$\|\mathbf{1}_{\boldsymbol{\varepsilon}B}\| \geqslant \|\mathbf{1}_{\boldsymbol{\varepsilon}B}\|_2 = \sqrt{|B|},$$

it follows that $\mathscr{B}$ is Δ_s-$\mathbf{n}$-superdemocratic with $\Delta_s \leqslant \sqrt{2}$.

  c) **n**-*symmetry for largest coefficients:* It follows by 2 that $\mathscr{B}$ is $\mathbf{C}_{ql}$-$\mathbf{n}$-quasi-greedy for largest coefficients with $\mathbf{C}_{ql} \leqslant 2$. From that and 2, an application of Proposition 9 gives the desired result.

  d) **n**-*UL property:* Fix $A \subset \mathbb{N}$ with $|A| \in \mathbf{n}$, and scalars $(a_i)_{i \in A}$. By convexity and using that the basis is $\sqrt{2}$-$\mathbf{n}$-superdemocratic,

$$\left\| \sum_{i \in A} a_i \mathbf{e}_i \right\| \leqslant \max_{i \in A} |a_i| \max_{\boldsymbol{\varepsilon} \in \Psi_A} \|\mathbf{1}_{\boldsymbol{\varepsilon}A}\| \leqslant \sqrt{2} \max_{i \in A} |a_i| \|\mathbf{1}_A\|. \tag{57}$$

On the other hand, using (56) we get

$$\left\| \sum_{i \in A} a_i \mathbf{e}_i \right\| \geqslant \left\| \sum_{i \in A} a_i \mathbf{e}_i \right\|_2 = \sqrt{\sum_{i \in A} |a_i|^2} \geqslant \min_{j \in A} |a_j| \sqrt{|A|}$$

$$\geqslant \frac{1}{\sqrt{2}} \min_{i \in A} |a_i| \|\mathbf{1}_A\|. \tag{58}$$

  e) *conservativeness:* To see that $\mathscr{B}$ is not conservative, for each $i \in \mathbb{N}$ let

$$B_i := \{\widetilde{m}_i+1, \ldots, \widetilde{m}_i+\beta_i\} \qquad \text{and} \qquad D_i := \{\widetilde{m}_i+\beta_i+1, \ldots, \widetilde{m}_i+2\beta_i\}.$$

We have

$$\|\mathbf{1}_{B_i}\| \geqslant \left| \frac{c_i}{\sqrt{m_i}} \sum_{j=\widetilde{m}_i+1}^{\widetilde{m}_i+\beta_i} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{B_i}) \right| = \frac{c_i}{\sqrt{m_i}} \beta_i \geqslant \frac{c_i}{\sqrt{m_i}} \frac{m_i}{3} = \frac{c_i \sqrt{m_i}}{3}. \tag{59}$$

On the other hand, for each $1 \leqslant l \leqslant \beta_i$,

$$\sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{D_i}) = 0,$$

whereas for $\beta_i + 1 \leqslant l \leqslant m_i$,

$$\left| \sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{D_i}) \right| = \left| \sum_{j=\tilde{m}_i+\beta_i+1}^{\tilde{m}_i+l} (-1)^j \mathbf{e}_j^*(\mathbf{1}_{D_i}) \right| \leqslant 1.$$

Since

$$\sum_{j=\tilde{m}_{i'}+1}^{\tilde{m}_{i'}+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{D_i}) = 0 \qquad \forall i' \neq i \forall 1 \leqslant l \leqslant m_{i'},$$

we deduce that

$$\|\mathbf{1}_{D_i}\| = \|\mathbf{1}_{D_i}\|_2 \leqslant \sqrt{\beta_i + 1} \leqslant \sqrt{m_i}. \tag{60}$$

Given that $(c_i)_i$ is unbounded, $B_i < D_i$ and $|B_i| \leqslant |D_i|$ for all i, it follows from (59) and (60) that $\mathscr{B}$ is not conservative.

f) Unconditionality for constant coefficients: This can be proven using the argument given in [16, Proposition 3.2] to prove that the basis is not quasi-greedy. We give a proof for the sake of completion: Fix $i \in \mathbb{N}$, and consider again the set B_i. By (59), we have

$$\|\mathbf{1}_{B_i}\| \geqslant \frac{c_i \sqrt{m_i}}{3}.$$

Now let $\varepsilon \in \Psi_{B_i}$ be a sequence of alternating signs. Then for all $1 \leqslant l \leqslant m_i$ we have

$$\frac{c_i}{\sqrt{m_i}} \left| \sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{\varepsilon B_i}) \right| = \frac{c_i}{\sqrt{m_i}} \left| \sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+\max\{l,\beta_i\}} \varepsilon_j \right| \leqslant \frac{c_i}{\sqrt{m_i}} \leqslant 2.$$

As

$$\frac{c_{i'}}{\sqrt{m_{i'}}} \left| \sum_{j=\tilde{m}_l+1}^{\tilde{m}_l+l} (-1)^{\theta(j)} \mathbf{e}_j^*(\mathbf{1}_{\varepsilon B_i}) \right| = 0 \qquad \forall i' \neq i \forall 1 \leqslant l \leqslant m_{i'},$$

it follows that

$$\|\mathbf{1}_{\varepsilon B_i}\| = \|\mathbf{1}_{\varepsilon B_i}\|_2 = \sqrt{\beta_i} \leqslant \sqrt{m_i}.$$

As before, using the fact that $(c_i)_i$ is unbounded we conclude that $\mathscr{B}$ is not unconditional for constant coefficients. $\square$

REMARK 4. A slight modification of Example 2 shows that even for unconditional Schauder bases, $\mathbf{n}$-superdemocracy does not entail democracy, or even conservativeness. Indeed, if we replace the norm in Example 2 by the norm

$$\left\| \sum_j a_j \mathbf{e}_j \right\|_{\diamond} = \max \left\{ \|(a_j)_j\|_2, \sup_{i \in \mathbb{N}} \frac{c_i}{\sqrt{m_i}} \sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+m_i} |a_j| \right\},$$

the resulting basis is 1-unconditional, and the proof of **n**-superdemocracy holds with only minor, strightforward modifications. Since $(c_i)_i$ is unbounded,

$$\|\mathbf{1}_{B_i}\|_\diamond \geqslant \|\mathbf{1}_{B_i}\| \geqslant \frac{c_i \sqrt{m_i}}{3} \geqslant \frac{c_i}{3} \sqrt{|B_i|},$$

and the subsequence $(\mathbf{e}_{\widetilde{m}_i+1})_{i\in\mathbb{N}}$ is clearly equivalent to the unit vector basis of ℓ_2, $\mathscr{B}$ is not conservative.

Annex: Summary of some important constants

Symbol	Name of constant	Ref. equation
$\mathbf{C}_q$	Quasi-greedy constant	(3)
$\mathbf{K}_u$	Unconditionality for constant coeff. constant	(5)
Δ_d	Democracy constant	(18)
Δ_s	Superdemocracy constant	(18)
Δ	Symmetry for largest coeff. constant	(26)

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