Mathematical
Inequalities
& Applications
Volume 26, Number 1 (2023), 1–15 doi:10.7153/mia-2023-26-01

HARDY–LITTLEWOOD–STEIN INEQUALITIES FOR DOUBLE TRIGONOMETRIC SERIES

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Dedicated to M. A. Sadybekov on the occasion of his 60th birthday

(*Communicated by I. Perić*)

Abstract. In the present paper, we obtain sharper analogues of the Hardy-Littlewood-Stein inequalities for double trigonometric series. We also establish a new unified version of the Hardy-Littlewood-Stein inequalities for Fourier series in regular systems, which covers the whole range $1 < p < \infty$ including the critical case $p = 2$.

1. Introduction

The Hardy-Littlewood inequality plays an important role in the theory of orthogonal Fourier series. Indeed it reveals the essence of the connection between integral properties of functions and summability of its Fourier coefficients.

Historically, Hardy and Littlewood established an estimate for the norm of a function $f \in L_p[0,2\pi]$, $2 < p < \infty$, in terms of its Fourier coefficients in the trigonometric system. Subsequently, Paley extended the Hardy-Littlewood inequality to arbitrary bounded complete orthonormal systems $\{\varphi_k\}_{k\in\mathbb{N}}$.

Let us briefly recall these results. Let $\Phi = \{\varphi_k(x)\}_{k=1}^{\infty}$ be a bounded complete orthogonal system in $L_2[0,1]^n$. Let $a = \{a_k\}_{k\in\mathbb{N}}$ be a sequence such that for its non-increasing rearrangement $a^* = \{a_k^*\}_{k=1}^{\infty}$ the following series converges

$$\sum_{k=1}^{\infty} k^{p-2} a_k^{*p} < \infty, \quad 2 \leqslant p < \infty. \tag{1}$$

Then there exists $f \in L_p[0,1]^n$ with its Fourier coefficients

$$a_k = \int_{[0,1]^n} f(x)\varphi_k(x)dx$$

Mathematics subject classification (2020): 42A16, 42B05.
Keywords and phrases: Hardy-Littlewood-Stein inequality, double trigonometric series, Fourier series, Lorentz space.
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corresponding to the system Φ and

$$\|f\|^p_{L_p[0,1]^n} \leqslant c\sum_{k=1}^{\infty} k^{p-2}(a_k^*)^p. \tag{2}$$

On the other hand, if $1 < p \leqslant 2$ and f is from the space $L_p[0,1]^n$, then the sequence of its Fourier coefficients $\{a_k\}_{k=1}^{\infty}$ satisfies (1) and

$$\sum_{k=1}^{\infty} k^{p-2}(a_k^*)^p \leqslant c\|f\|^p_{L_p[0,1]^n}. \tag{3}$$

Let f be a measurable function on $[0,1]^n$. A function

$$f^*(t) = \inf\{\sigma : \ \mu\{x\in\Omega : \ |f(x)| > \sigma\} \leqslant t\}$$

is called a non-increasing rearrangement of the function f.

A Lorentz space $L_{p,q}[0,1]^n$, $0 < p < \infty$, is a set of all measurable functions f such that

$$\|f\|_{L_{p,q}} = \left(\int_0^1 (t^{1/p}f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty$$

for $1 \leqslant q < \infty$, and

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty$$

for $q = \infty$.

Correspondingly, for the discrete Lorentz space we use the notation $l_{p,q}$, which is a set of sequences $a = \{a_k\}_{k=1}^{\infty}$ such that

$$\|a\|_{l_{p,q}} = \left(\sum_{k=1}^{\infty} k^{q/p-1}(a_k^*)^q\right)^{1/q} < \infty,$$

where $\{a_k^*\}_{k=1}^{\infty}$ is a non-increasing rearrangement of the sequence $\{|a_k|\}_{k=1}^{\infty}$. When $q = \infty$, we have

$$\|a\|_{l_{p,\infty}} = \sup_{k\in\mathbb{N}} k^{1/p} a_k^* < \infty.$$

The Hardy-Littlewood inequality was extended to the Lorentz space by Stein [18]:

Let $f \sim \sum_{k=1}^{\infty} a_k\varphi_k$ and $0 < q \leqslant \infty$. Then we have

$$\|f\|_{L_{p,q}} \leqslant c\|a\|_{l_{p',q}} \tag{4}$$

for $2 < p < \infty$, and

$$\|f\|_{L_{p,q}} \geqslant c\|a\|_{l_{p',q}} \tag{5}$$

for $1 < p < 2$.

Note that in the critical case $p = 2$ analogues of the Hardy-Littlewood-Stein inequalities have different forms [3, 8, 19]. For more general Lorentz spaces, the Hardy-Littlewood-Stein inequalities were obtained in [4, 7, 12, 11, 14, 15].

Let both $\{\varphi_k(x_1)\}_{k=1}^{\infty}$ and $\{\psi_k(x_2)\}_{k=1}^{\infty}$ be bounded complete orthonormal systems in $L_2[0,1]$. Then the system $\{\zeta_{k_1k_2}(x_1,x_2)\}_{k_1=1,k_2=1}^{\infty,\infty}$ with

$$\zeta_{k_1k_2}(x_1,x_2) = \varphi_{k_1}(x_1)\psi_{k_2}(x_2), \quad k_1,k_2 \in \mathbb{N},$$

is a bounded complete orthonormal system in $L_2[0,1]^2$.

For multiple Fourier series, analogues of the Hardy-Littlewood-Stein inequalities in anisotropic Lorentz spaces $L_{\mathbf{p},\mathbf{q}}[0,1]^n$, $\mathbf{p} = (p_1,\dots,p_n)$, $\mathbf{q} = (q_1,\dots,q_n)$, (see [2]) were obtained by the first author in [9, 10]. In a particular case, for the double series

$$f \sim \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} a_{k_1k_2}\varphi_{k_1}(x_1)\psi_{k_2}(x_2),$$

we have

$$\left(\sum_{k_2=1}^{\infty}\sum_{k_1=1}^{\infty} k_1^{p-2}k_2^{p-2}(a_{k_1k_2}^{*_1*_2})^p\right)^{1/p} \leqslant c\|f\|_{L_p[0,1]^2} \tag{6}$$

for $1<p<2$, and

$$\|f\|_{L_p[0,1]^2} \leqslant c\left(\sum_{k_2=1}^{\infty}\sum_{k_1=1}^{\infty} k_1^{p-2}k_2^{p-2}(a_{k_1k_2}^{*_1*_2})^p\right)^{1/p} \tag{7}$$

for $2<p<\infty$, where $a^* = \{a_{k_1k_2}^{*_1*_2}\}_{(k_1,k_2)\in\mathbb{N}^2}$ is a repeated non-increasing rearrangement of $a = \{a_{m_1m_2}\}_{(m_1,m_2)\in\mathbb{N}^2}$.

For multiple Fourier series, inequalities (6) and (7) are sharper than (2) and (3), i.e. from (6) and (7) follow (2) and (3), respectively, but the converse argument does not hold.

The main aim of this paper is to obtain inequalities, which are sharper than the Hardy-Littlewood-Stein inequalities (4) and (5) for the double trigonometric series. We also establish a new inequality of the form (5) for double Fourier series in regular systems, but for the whole range $1<p<\infty$.

The present paper is organized as follows: In Section 2 we develop preliminary tools for our analysis. In Section 3, we prove sharper analogues of the Hardy-Littlewood-Stein inequalities for double trigonometric series. In Section 4, we discuss a new unified version of the Hardy-Littlewood-Stein inequalities for Fourier series in regular systems, which covers the whole range $1<p<\infty$ including the critical case $p=2$.

2. $l_{p,q}(\Lambda)$ and $n_{p,q}(G)$ spaces

Let $a = \{a_{m_1m_2}\}_{(m_1,m_2)\in\mathbb{N}^2}$ be a bounded sequence with a unique limit point 0, that is, every neighborhood around 0 contains infinitely many terms of the sequence. By $\{a_{k_1m_2}^{*_1}\}_{k_1}^{\infty}$ we denote a non-increasing rearrangement of the sequence $\{|a_{m_1m_2}|\}_{m_1=1}^{\infty}$ for a fixed parameter $m_2\in\mathbb{N}$. Now let us fix a parameter $k_1\in\mathbb{N}$, then $\{a_{k_1k_2}^{*_1*_2}\}_{k_2\in\mathbb{N}}$ is a non-increasing rearrangement of the sequence $\{a_{k_1m_2}^{*_1}\}_{m_2\in\mathbb{N}}$.

We define the systems of sets $\Lambda = \{\Lambda_n\}_{n=0}^{\infty}$ and $G = \{G_n\}_{n=0}^{\infty}$ by

$$\Lambda_n = \cup_{\substack{m_1+m_2=n\\ 0\leqslant m_i}}\{(k_1,k_2) : 2^{m_i} \leqslant k_i < 2^{m_i+1},\ \ i=1,2\}$$

and

$$G_n = \cup_{r=0}^{n}\Lambda_r = \cup_{\substack{m_1+m_2\leqslant n\\ 0\leqslant m_i}}\{(k_1,k_2) : 2^{m_i} \leqslant k_i < 2^{m_i+1},\ \ i=1,2\},$$

respectively. Note that G_n is called a step hyperbolic gross of order n.

LEMMA 1. *Let $a = \{a_{m_1m_2}\}_{(m_1,m_2)\in\mathbb{N}^2}$ be a sequence such that $|a_{m_1m_2}| \to 0$ when $m_1+m_2 \to \infty$. Then for any $n \in \mathbb{N}$ we have*

$$\sum_{k=1}^{2^n}(a_k^*)^2 \leqslant \sum_{(r_1,r_2)\in G_n}(a_{r_1r_2}^{*_1*_2})^2, \tag{8}$$

and

$$\sum_{(r_1,r_2)\in\Lambda_n}(a_{r_1r_2}^{*_1*_2})^2 \leqslant \sum_{k=2^{n-1}}^{\infty}(a_k^*)^2, \tag{9}$$

where $\{a_k^*\}_{k=1}^{\infty}$ is a non-increasing rearrangement of the sequence $\{|a_{m_1m_2}|\}_{(m_1,m_2)\in\mathbb{N}^2}$.

Proof. Note that all the sets

$$\{a_k^*\}_{k=1}^{\infty}, \quad \{a_{m_1m_2}^{*_1*_2}\}_{(m_1,m_2)\in\mathbb{N}^2} \quad \text{and} \quad \{|a_{m_1m_2}|\}_{(m_1,m_2)\in\mathbb{N}^2}$$

consist of the same elements.

Let $(k_1,k_2) \notin G_n$. Then $k_1\cdot k_2 \geqslant 2^n$, and we get $a_{k_1k_2}^{*_1*_2} \leqslant a_{m_1m_2}^{*_1*_2}$ for $m_1 \leqslant k_1, m_2 \leqslant k_2$. The number of the elements $a_{k_1k_2}^{*_1*_2}$ is at least $k_1\cdot k_2$, so we have $a_{k_1k_2}^{*_1*_2} \leqslant a_{2^n}^*$. This means that the sum $\sum_{(r_1,r_2)\in G_n}(a_{r_1r_2}^{*_1*_2})^2$ contains all terms of $\sum_{k=1}^{2^n}(a_k^*)^2$. Hence, inequality (8) is valid.

For the same reason, the sum $\sum_{2^{n-1}}^{\infty}(a_k^*)^2$ contains all terms of the sum

$$\sum_{(r_1,r_2)\in\Lambda_n}(a_{r_1r_2}^{*_1*_2})^2.$$

Thus, inequality (9) also holds. □

We define sets $a(\Lambda) = \{a_k(\Lambda)\}$ and $a(G) = \{\bar{a}_k(G)\}$ for double sequence $a = \{a_{m_1m_2}\}_{(m_1,m_2)\in\mathbb{N}^2}$ by

$$a_n(\Lambda) = \left(\frac{1}{2^n}\sum_{(m_1,m_2)\in\Lambda_n}\left(a_{m_1m_2}^{*_1*_2}\right)^2\right)^{\frac{1}{2}}, \quad n\in\mathbb{N},$$

and

$$\bar{a}_n(G) = \sup_{r\geqslant n}\frac{1}{2^r}\left|\sum_{(m_1,m_2)\in G_r}a_{m_1m_2}\right|, \quad n\in\mathbb{N},$$

respectively.

For $0<q,p\leqslant\infty$, the spaces $l_{p,q}(\Lambda)$ and $n_{p,q}(G)$ can be defined in terms of quasi-norms

$$\|a\|_{l_{p,q}(\Lambda)}=\left(\sum_{n=1}^{\infty}\left(2^{\frac{n}{p}}a_n(\Lambda)\right)^q\right)^{\frac{1}{q}},$$

and

$$\|a\|_{n_{p,q}(G)}=\left(\sum_{n=1}^{\infty}\left(d_n^{\frac{1}{p}}\,\bar{a}_n(G)\right)^q\right)^{\frac{1}{q}},$$

respectively. Here $d_n=\frac{2^n}{n}$.

For $q=\infty$, $0<p\leqslant\infty$, we have

$$\|a\|_{l_{p,\infty}(\Lambda)}=\sup_{n\in\mathbb{N}}2^{\frac{n}{p}}a_n(\Lambda),$$

and

$$\|a\|_{n_{p,\infty}(G)}=\sup_{n\in\mathbb{N}}d_n^{\frac{1}{p}}\,\bar{a}_n(G).$$

Note that in the one-dimensional case $n=1$ the space $l_{p,q}(\Lambda)$ coincides with the Lorentz space $l_{p,q}$.

Although the Lorentz-type space $l_{p,q}(\Lambda)$ and the net-type space $n_{p,q}(G)$ may look similar, these spaces are essentially different (see, e.g. [13]):

EXAMPLE 1. Let $1<p<\infty$ and $0<q\leqslant\infty$. The sequence

$$a=\{(-1)^{m_1+m_2}\}_{(m_1,m_2)\in\mathbb{N}^2}$$

is an element of $n_{p,q}(G)$, at the same time the sequence $|a|=\{|(-1)^{m_1+m_2}|\}=\{1\}$ does belong to $n_{p,q}(G)$.

The following version of Hardy's inequality from [16] will be useful in our analysis. For further discussions related to Hardy's inequality, we refer to the recent book [17] (see, e.g. the preface of the book).

LEMMA 2. [16] *Let* $\alpha>0$ *and* $0<q,h\leqslant\infty$*. Assume that a sequence* $\{d_k\}_{k\in\mathbb{N}}$ *satisfies*

$$\frac{d_{k+1}}{d_k}\geqslant\delta,\quad k=2,3,\dots,\tag{10}$$

for some $\delta>1$*. Then we have the Hardy inequalities*

$$\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}\left(\sum_{r=0}^{k}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}}\leqslant c_{\alpha,q}\left(\sum_{k=0}^{\infty}\left(d_k^{-\alpha}|b_k|\right)^q\right)^{\frac{1}{q}}$$

and

$$\left(\sum_{k=0}^{\infty}\left(d_k^{\alpha}\left(\sum_{r=k}^{\infty}|b_r|^h\right)^{\frac{1}{h}}\right)^q\right)^{\frac{1}{q}} \leqslant c_{\alpha,q}\left(\sum_{k=0}^{\infty}(d_k^{\alpha}|b_k|)^q\right)^{\frac{1}{q}}.$$

Now we present some useful properties of spaces $l_{p,q}(\Lambda)$ and $n_{p,q}(G)$.

LEMMA 3. *a) If* $0<q<q_1\leqslant\infty$ *and* $1<p\leqslant\infty$, *then*

$$l_{p,q}(\Lambda)\hookrightarrow l_{p,q_1}(\Lambda) \quad \textit{and} \quad n_{p,q}(G)\hookrightarrow n_{p,q_1}(G).$$

b) If $0<p<p_1<\infty$ *and* $0<q,q_1\leqslant\infty$, *then*

$$l_{p,q}(\Lambda)\hookrightarrow l_{p_1,q_1}(\Lambda) \quad \textit{and} \quad n_{p,q}(G)\hookrightarrow n_{p_1,q_1}(G).$$

Proof. These properties are proved similarly as the proofs of analogous properties of the (classical) Lorentz spaces $l_{p,q}$ (see [13]). □

LEMMA 4. *Let* $0<q\leqslant\infty$. *We have the embedding*

$$l_{p,q}\hookrightarrow l_{p,q}(\Lambda) \tag{11}$$

for $1<p<2$, *and*

$$l_{p,q}(\Lambda)\hookrightarrow l_{p,q} \tag{12}$$

for $2<p<\infty$.

Proof. Let $1<p<2$, $a\in l_{p,q}$. By using the relation (9) from Lemma 1 we get

$$\|a\|_{l_{p,q}(\Lambda)} = \left(\sum_{n=0}^{\infty}\left(2^{\frac{n}{p}}\left(\frac{1}{2^n}\sum_{(m_1,m_2)\in\Lambda_n}(a_{m_1m_2}^{*_1*_2})^2\right)^{\frac{1}{2}}\right)^q\right)^{\frac{1}{q}}$$

$$\leqslant c\left(\sum_{n=0}^{\infty}\left(2^{\frac{n}{p}}\left(\frac{1}{2^n}\sum_{r=2^n}^{\infty}(a_r^*)^2\right)^{\frac{1}{2}}\right)^q\right)^{\frac{1}{q}}$$

$$= c\left(\sum_{n=0}^{\infty}\left(2^{n(\frac{1}{p}-\frac{1}{2})}\left(\sum_{r=n}^{\infty}(b_r)^2\right)^{\frac{1}{2}}\right)^q\right)^{\frac{1}{q}},$$

where

$$b_r = \left(\sum_{m=2^r}^{2^{r+1}-1}(a_m^*)^2\right)^{\frac{1}{2}}.$$

Furthermore, by using Hardy's inequality (Lemma 2) we obtain (11).

Let $2 < p < \infty$, $a \in l_{p,q}(\Lambda)$. By using the relation (9) we get

$$\|a\|_{l_{p,q}} \asymp \left(\sum_{n=0}^{\infty} \left(2^{\frac{n}{p}} \left(\frac{1}{2^n} \sum_{r=1}^{2^n} a_r^* \right) \right)^q \right)^{\frac{1}{q}}$$

$$\leqslant \left(\sum_{n=0}^{\infty} \left(2^{\frac{n}{p}} \left(\frac{1}{2^n} \sum_{r=1}^{2^n} (a_r^*)^2 \right)^{\frac{1}{2}} \right)^q \right)^{\frac{1}{q}}$$

$$\leqslant \left(\sum_{n=0}^{\infty} \left(2^{\frac{n}{p}} \left(\frac{1}{2^n} \sum_{(m_1,m_2)\in G_n} (a_{m_1 m_2}^{*_1 *_2})^2 \right)^{\frac{1}{2}} \right)^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{n=0}^{\infty} \left(2^{n(\frac{1}{p}-\frac{1}{2})} \left(\sum_{r=0}^{n} u_r^2 \right)^{\frac{1}{2}} \right)^q \right)^{\frac{1}{q}},$$

where

$$u_r = \left(\sum_{(m_1,m_2)\in\Lambda_r} (a_{m_1 m_2}^{*_1 *_2})^2 \right)^{\frac{1}{2}}.$$

Finally, by using Hardy's inequality (Lemma 2) we arrive at

$$\|a\|_{l_{p,q}} \leqslant c\|a\|_{l_{p,q}(\Lambda)}. \quad \square$$

Let (A_0, A_1) be compatible pair of Banach spaces (see, e.g. [1]). Let

$$K(t,a;A_0,A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \;\; a \in A_0 + A_1, \quad t > 0,$$

be a Peetre functional.

We have

$$(A_0,A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^{\infty} (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

for $0 < q < \infty$ and $0 < \theta < 1$, as well as

$$(A_0,A_1)_{\theta,\infty} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\}$$

when $q = \infty$.

The following interpolation theorem holds for the spaces $l_{p,q}(\Lambda)$.

THEOREM 1. *Let* $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \leqslant \infty$ *and* $1 < \theta < \infty$. *Then we have*

$$\left(l_{p_0,q_0}(\Lambda),\, l_{p_1,q_1}(\Lambda)\right)_{\theta,q} \hookrightarrow l_{p,q}(\Lambda), \tag{13}$$

and

$$\left(n_{p_0,q_0}(G),\, n_{p_1,q_1}(G)\right)_{\theta,q} \hookrightarrow n_{p,q}(G), \tag{14}$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof. Let $a = \{a_{m_1m_2}\}_{(m_1,m_2)\in\mathbb{N}^2} \in \left(l_{p_0,\infty}(\Lambda),\ l_{p_1,\infty}(\Lambda)\right)_{\theta,q}$. Let $a = u+v$ be a representation, where $u \in l_{p_0,\infty}(\Lambda)$ and $v \in l_{p_1,\infty}(\Lambda)$.

Let $n \in \mathbb{N}$. Since

$$(u+v)^{*_1*_2}_{k_1k_2} \leqslant u^{*_1*_2}_{\left[\frac{k_1}{2}\right]\left[\frac{k_2}{2}\right]} + v^{*_1*_2}_{\left[\frac{k_1}{2}\right]\left[\frac{k_2}{2}\right]}$$

for $n > 1$, we get

$$\left(\frac{1}{2^n}\sum_{(r_1,r_2)\in\Lambda_n}\left(a^{*_1*_2}_{r_1r_2}\right)^2\right)^{\frac12} \leqslant \left(\frac{2}{2^{n-2}}\sum_{(r_1,r_2)\in\Lambda_{n-2}}\left(u^{*_1*_2}_{r_1r_2}\right)^2\right)^{\frac12}$$
$$+\left(\frac{2}{2^{n-2}}\sum_{(r_1,r_2)\in\Lambda_{n-2}}\left(v^{*_1*_2}_{r_1r_2}\right)^2\right)^{\frac12}$$
$$\leqslant 2^{\frac12}2^{\left(-\frac{n-2}{p_0}\right)}\left[\|u\|_{l_{p_0,\infty}(\Lambda)} + 2^{(n-2)\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}\|v\|_{l_{p_1,\infty}(\Lambda)}\right],$$

and

$$\left(\frac{1}{2^n}\sum_{(r_1,r_2)\in\Lambda_n}\left(a^{*_1*_2}_{r_1r_2}\right)^2\right)^{\frac12} \leqslant 2(u^{*_1*_2}_{11} + v^{*_1*_2}_{11})$$

for $0 \leqslant n \leqslant 1$.

Since $a = u+v$ is arbitrary representation, we have

$$2^{\frac{n}{p_0}}\left(\frac{1}{2^n}\sum_{(r_1,r_2)\in\Lambda_n}(a^{*_1*_2}_{r_1r_2})^2\right)^{\frac12} \leqslant cK\left(2^{(n-2)(\frac{1}{p_0}-\frac{1}{p_1})}, a;\ l_{p_0,\infty}(\Lambda), l_{p_1,\infty}(\Lambda)\right).$$

It yields

$$\|a\|_{l_{p,q}(\Lambda)} \leqslant c\left(\sum_{k=1}^{\infty}\left(2^{k(\frac1p-\frac{1}{p_0})}K\left(2^{k(\frac{1}{p_0}-\frac{1}{p_1})}, a;\ l_{p_0,\infty}(\Lambda), l_{p_1,\infty}(\Lambda)\right)\right)^q\right)^{\frac1q}$$
$$\asymp \left(\int_1^\infty \left(t^{-\theta}K\left(t, a;\ l_{p_0,\infty}(\Lambda), l_{p_1,\infty}(\Lambda)\right)\right)^q \frac{dt}{t}\right)^{\frac1q} \leqslant \|a\|_{\left(l_{p_0,\infty}(\Lambda),\ l_{p_1,\infty}(\Lambda)\right)_{\theta,q}}.$$

Combining this with the embedding $l_{p_i,q_i}(\Lambda) \hookrightarrow l_{p_i,\infty}(\Lambda)$, $i = 0,1$, we obtain (13). The relation (14) is proved similarly. □

3. Hardy-Littlewood-Stein inequalities

LEMMA 5. *Let* $0 < p < 2$. *Then we have*

$$\left(\sum_{(k_1,k_2)\in\mathbb{N}^2}\left(k_1^{\frac{1}{p'}}k_2^{\frac{1}{p'}}a_{k_1k_2}^{*_1*_2}\right)^2\frac{1}{k_1}\frac{1}{k_1}\right)^{1/2}\leqslant\left(\sum_{(k_1,k_2)\in\mathbb{N}^2}(k_1k_2)^{p-2}\left(a_{k_1k_2}^{*_1*_2}\right)^p\right)^{1/p}.$$

Proof. A direct computation gives

$$\left(\sum_{(k_1,k_2)\in\mathbb{N}^2}\left(k_1^{\frac{1}{p'}}k_2^{\frac{1}{p'}}a_{k_1k_2}^{*_1*_2}\right)^2\frac{1}{k_1}\frac{1}{k_1}\right)^{1/2}$$

$$=\left(\sum_{(k_1,k_2)\in\mathbb{N}^2}\left(k_1^{\frac{1}{p'}}k_2^{\frac{1}{p'}}a_{k_1k_2}^{*_1*_2}\right)^p\left(k_1^{\frac{1}{p'}}k_2^{\frac{1}{p'}}a_{k_1k_2}^{*_1*_2}\right)^{2-p}\frac{1}{k_1}\frac{1}{k_1}\right)^{1/2}$$

$$\leqslant\left(\sup_{k_i\in\mathbb{N}}k_1^{\frac{1}{p'}}k_2^{\frac{1}{p'}}a_{k_1k_2}^{*_1*_2}\right)^{1-\frac{p}{2}}\left(\sum_{(k_1,k_2)\in\mathbb{N}^2}(k_1k_2)^{p-2}(a_{k_1k_2}^{*_1*_2})^p\right)^{1/2}. \tag{15}$$

On the other hand, we have $(k_1k_2)^{\frac{1}{p'}}\asymp\left(\sum_{r_1=1}^{k_1}\sum_{r_1=1}^{k_1}(r_1r_2)^{p-2}\right)^{1/p}$. Thus, we get

$$\sup_{k_i\in\mathbb{N}}k_1^{\frac{1}{p'}}k_2^{\frac{1}{p'}}a_{k_1k_2}^{*_1*_2}\lesssim\left(\sum_{(r_1,r_2)\in\mathbb{N}^2}(r_1r_2)^{p-2}(a_{r_1r_2}^{*_1*_2})^p\right)^{1/p}.$$

Substituting this in (15) we obtain the desired result. □

THEOREM 2. *Let* $\Phi=\{\varphi_k\}_{k=1}^\infty$ *and* $\Psi=\{\psi_k\}_{k=1}^\infty$ *be bounded orthonormal systems in* $L_2[0,1]$. *Let* $1\leqslant q\leqslant\infty$. *If* $1<p<2$, $f\in L_{p,q}[0,1]^2$ *and* $f\sim\sum_{k_1=1}^\infty\sum_{k_2=1}^\infty a_{k_1k_2}\varphi_{k_1}(x_1)\psi_{k_2}(x_2)$, *then* $a=\{a_{k_1k_2}\}\in l_{p',q}(\Lambda)$ *and*

$$\|a\|_{l_{p',q}(\Lambda)}\leqslant c\|f\|_{L_{p,q}[0,1]^2}. \tag{16}$$

If $2<p<\infty$, $f\sim\sum_{k_1=1}^\infty\sum_{k_2=1}^\infty a_{k_1k_2}\varphi_{k_1}(x_1)\psi_{k_2}(x_2)$ *and* $a\in l_{p',q}(\Lambda)$, *then* $f\in L_{p,q}[0,1]^2$ *and*

$$\|f\|_{L_{p,q}[0,1]^2}\leqslant c\|a\|_{l_{p',q}(\Lambda)}. \tag{17}$$

Proof. Assuming $1<p\leqslant 2$, first we show

$$\|a\|_{l_{p',\infty}(\Lambda)}\leqslant c_p\|f\|_{L_p} \tag{18}$$

Let $f \in L_p[0,1]^2$, $f \sim \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} a_{k_1k_2}\varphi_{k_1k_2}$, and $n \in \mathbb{N}$. A direct computation gives

$$2^{\frac{n}{p'}} a_n(\Lambda) = 2^{\frac{n}{p'}} \left(\frac{1}{2^n} \sum_{(m_1,m_2)\in\Lambda_n} \left(a^{*_1*_2}_{m_1m_2}\right)^2 \right)^{\frac{1}{2}}$$

$$\leqslant 2 \left(\sum_{(k_1,k_2)\in\Lambda_n} \left(k_1^{\frac{1}{p'}} k_2^{\frac{1}{p'}} a^{*_1*_2}_{k_1k_2} \right)^2 \frac{1}{k_1}\frac{1}{k_1} \right)^{1/2}$$

$$\leqslant 2 \left(\sum_{(k_1,k_2)\in\mathbb{N}^2} \left(k_1^{\frac{1}{p'}} k_2^{\frac{1}{p'}} a^{*_1*_2}_{k_1k_2} \right)^2 \frac{1}{k_1}\frac{1}{k_1} \right)^{1/2}.$$

By using Lemma 5, and using the fact that n is arbitrary parameter, we get

$$\|a\|_{l_{p',\infty}(\Lambda)} \lesssim \left(\sum_{(k_1,k_2)\in\mathbb{N}^2} (k_1k_2)^{p-2} \left(a^{*_1*_2}_{k_1k_2}\right)^p \frac{1}{k_1}\frac{1}{k_1} \right)^{1/p}.$$

Thus, by using the inequality (6), we obtain (18).

Now we consider the case $1 < p < 2$. Then there exist $p_0, p_1, \theta \in (0,1)$ such that

$$1 < p_0 < p < p_1 < 2, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Form (18) it follows that

$$\|a\|_{l_{p_0',\infty}(\Lambda)} \leqslant c_{p_0} \|f\|_{L_{p_0}[0,1]^2},$$

and

$$\|a\|_{l_{p_1',\infty}(\Lambda)} \leqslant c_{p_1} \|f\|_{L_{p_1}[0,1]^2}.$$

So, according to the real interpolation method, we get

$$\|a\|_{\left(l_{p_0',\infty}(\Lambda), l_{p_1',\infty}(\Lambda)\right)_{\theta,q}} \leqslant c(c_{p_0})^{1-\theta}(c_{p_1})^{\theta} \|f\|_{\left(L_{p_0}[0,1]^2, L_{p_1}[0,1]^2\right)_{\theta,q}}.$$

Thus, taking into account Theorem 1 and the fact that (see [1])

$$\left(L_{p_0}[0,1]^2, L_{p_1}[0,1]^2\right)_{\theta,q} = L_{p,q}[0,1]^2,$$

we obtain (16).

Let $2 < p < \infty$, $f \sim \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} a_{k_1 k_2} \varphi_{k_1}(x_1) \psi_{k_2}(x_2)$ and $a \in l_{p',q}$. By using the dual representation of the norm of the Lorentz space and Parseval's equality, we get

$$\|f\|_{L_{p,q}} \asymp \sup_{\|g\|_{L_{p'q'}}=1} \int_0^1 \int_0^1 f(x_1,x_2)\overline{g(x_1,x_2)}dx_1dx_2 = \sup_{\|g\|_{L_{p',q'}}=1} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} a_{k_1k_2}\overline{b_{k_1k_2}}$$

$$\leqslant \sup_{\|g\|_{L_{p',q'}}=1} \sum_{k=1}^{\infty} \sum_{k_2=1}^{\infty} a_{k_1k_2}^{*1*2} b_{k_1k_2}^{*1*2} = \sup_{\|g\|_{L_{p',q'}}=1} \sum_{n=1}^{\infty} \sum_{(k_1,k_2)\in\Lambda_n} a_{k_1k_2}^{*1*2} b_{k_1k_2}^{*1*2}$$

$$\leqslant \sup_{\|g\|_{L_{p',q'}}=1} \sum_{n=1}^{\infty} \left(\sum_{(k_1,k_2)\in\Lambda_n} \left(a_{k_1k_2}^{*1*2}\right)^2 \right)^{\frac{1}{2}} \left(\sum_{(k_1,k_2)\in\Lambda_n} \left(b_{k_1k_2}^{*1*2}\right)^2 \right)^{\frac{1}{2}}$$

$$\leqslant \sup_{\|g\|_{L_{p',q'}}=1} \left(\sum_{n=1}^{\infty} \left(2^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \left(\sum_{(k_1,k_2)\in\Lambda_n} \left(b_{k_1k_2}^{*1*2}\right)^2 \right)^{\frac{1}{2}} \right)^{q'} \right)^{\frac{1}{q'}}$$

$$\times \left(\sum_{n=1}^{\infty} \left(2^{n\left(\frac{1}{p'}-\frac{1}{2}\right)} \left(\sum_{(k_1,k_2)\in\Lambda_n} \left(a_{k_1k_2}^{*1*2}\right)^2 \right)^{\frac{1}{2}} \right)^{q} \right)^{\frac{1}{q}}.$$

Since $2 < p < \infty$, we have $1 < p' < 2$. This allows us to apply inequality (16), that is,

$$\sum_{n=1}^{\infty} \left(2^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \left(\sum_{(k_1,k_2)\in\Lambda_n} \left(b_{k_1k_2}^{*1*2}\right)^2 \right)^{\frac{1}{2}} \right)^{q'} \leqslant c\|g\|_{L_{p',q'}}^{q'}.$$

Thus, we arrive at

$$\|f\|_{L_{p,q}}^q \leqslant c \sum_{n=1}^{\infty} \left(2^{\frac{n}{p'}} \left(\frac{1}{2^n} \sum_{(k_1,k_2)\in\Lambda_n} \left(a_{k_1k_2}^{*1*2}\right)^2 \right)^{\frac{1}{2}} \right)^q. \quad \square$$

REMARK 1. From Lemma 3, it follows that inequalities (16) and (17) imply inequalities (4) and (5), respectively. Let us show that the converse argument is not valid.

Let n be an arbitrary natural number. We consider the function

$$f(x_1,x_2) = \sum_{(k_1,k_2)\in G_n} \varphi_{k_1}(x_1)\psi_{k_2}(x_2)$$

Inequalities (16) and (17) imply the estimates

$$2^{\frac{n}{p'}} n^{\frac{1}{2}} \leqslant c\|f\|_{L_{p,q}}, \quad 1 < p < 2,$$

and

$$\|f\|_{L_{p,q}} \leqslant c\, 2^{\frac{n}{p'}} n^{\frac{1}{2}}, \quad 2 < p < \infty.$$

Meanwhile, inequalities (4) and (5) lead us to estimates of a different order

$$2^{\frac{n}{p'}} n^{\frac{1}{p'}} \leqslant \|f\|_{L_{p,q}}, \quad 1 < p < 2,$$

and

$$\|f\|_{L_{p,q}} \leqslant c\, 2^{\frac{n}{p'}} n^{\frac{1}{p'}}, \quad 2 < p < \infty,$$

which are essentially less accurate.

4. Inequality for Fourier series in regular systems

A complete orthonormal system $\Phi = \{\phi_k(x)\}_{k=1}^{\infty}$ is called regular (see, e.g. [9]) if there exists a constant B such that:

1) For any segment e from $[0,1]$ and any $k \in \mathbb{N}$ we have

$$\left| \int_e \phi_k(x)dx \right| \leqslant B \min(|e|, 1/k).$$

2) For any segment w from $\mathbb{N}$ and any $t \in [0,1]$ we have

$$\left(\sum_{k \in w} \phi_k(\cdot) \right)^{*} (t) \leqslant B \min(|w|, 1/t),$$

where $\left(\sum_{k\in w} \phi_k(\cdot)\right)^* (t)$ is the non-increasing rearrangement of the function $\sum_{k\in w} \phi_k(x)$. For example, trigonometric, Walsh and Price systems are regular.

We recall the following statement which is a special case of [9, Theorem 9].

LEMMA 6. *Let $2 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. Let Φ and Ψ be regular systems. Let $f \sim \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} a_{k_1k_2} \varphi_{k_1}(x_1) \psi_{k_2}(x_2)$. If for some $A > 0$*

$$\sup_{n_1, n_2 \in \mathbb{N}} n_1^{\frac{1}{p'}} n_2^{\frac{1}{p'}} |a_{n_1 n_2}| \leqslant A,$$

$$\sup_{n_1 \in \mathbb{N}} n_1^{\frac{1}{p'}} \left(\sum_{k_2=1}^{\infty} k_2^{p-1} |a_{n_1 k_2+1} - a_{n_1 k_2}|^p \right)^{\frac{1}{p}} \leqslant A,$$

$$\sup_{n_2 \in \mathbb{N}} n_2^{\frac{1}{p'}} \left(\sum_{k_1=1}^{\infty} k_1^{p-1} |a_{k_1+1 n_2} - a_{k_1 n_2}|^p \right)^{\frac{1}{p}} \leqslant A,$$

$$\left(\sum_{k_2=1}^{\infty} \sum_{k_1=1}^{\infty} (k_1k_2)^{p-1} |a_{k_1+1k_2+1} - a_{k_1+1k_2} - a_{k_1k_2+1} + a_{k_1k_2}|^p \right)^{\frac{1}{p}} \leqslant A,$$

then we have $f \in L_p[0,1]^2$ and

$$\|f\|_{L_p[0,1]^2} \leqslant c_p A,$$

where the constant c_p depends only on p.

Note that, in addition to [9], Lemma 6 was studied in a more general case in the papers [5] and [6]. We refer to these papers and references therein for further discussions in this direction.

THEOREM 3. *Let* $1 < p < \infty$ *and* $1 < q \leqslant \infty$. *Let* Φ *and* Ψ *be regular systems. For any* $f \sim \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} a_{k_1k_2}\varphi_{k_1}(x_1)\psi_{k_2}(x_2)$, *we have*

$$\|a\|_{n_{p'q}(G)} \leqslant c\|f\|_{L_{p,q}[0,1]^2}.$$

Proof. Let us first show the weak inequality

$$\|a\|_{n_{p',\infty}(G)} \leqslant c\|f\|_{L_p}.$$

Let $f \sim \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} a_{k_1k_2}\varphi_{k_1}(x_1)\psi_{k_2}(x_2)$ and G_n be the step hyperbolic cross with $n \in \mathbb{N}$. Consider

$$\left|\sum_{(m_1,m_2)\in G_n} a_{m_1m_2}\right| = \left|\int_{[0,1]^2} f(x_1,x_2)D_{G_n}(x_1,x_2)dx_1dx_2\right|.$$

Using the Hölder inequality, we have

$$\left|\sum_{(m_1,m_2)\in G_n} a_{m_1m_2}\right| \leqslant \|f\|_{L_p}\|D_{G_n}\|_{L_{p'}},$$

where $D_{G_n}(x_1,x_2) = \sum_{(k_1,k_2)\in G_n}\varphi_{k_1}(x_1)\psi_{k_2}(x_2)$. Furthermore, using Lemma 6, we obtain

$$\|D_{G_n}\|_{L_{p'}} \lesssim 2^{\frac{n}{p}} n^{\frac{1}{p'}}.$$

It yields

$$\|a\|_{n_{p',\infty}(G)} = \sup_n \left(\frac{2^n}{n}\right)^{\frac{1}{p'}} \bar{a}(G_n)$$

$$\leqslant \sup_n \frac{1}{n^{\frac{1}{p'}} 2^{\frac{n}{p}}}\left|\sum_{(m_1,m_2)\in G_n} a_{m_1m_2}\right| \leqslant \|f\|_{L_p}.$$

Thus, we arrive at

$$\|a\|_{n_{p',\infty}(G)} \leqslant c\|f\|_{L_p}$$

for $1 < p < \infty$.

Similarly as in the proof of Theorem 2, by using Theorem 1 and the interpolation properties of the Lebesgue spaces we establish the inequality

$$\|a\|_{n_{p',q}(G)} \leqslant c\|f\|_{L_{p,q}}$$

for $1 < p < \infty$ and $0 < q \leqslant \infty$. □

REMARK 2. For $1 < p < 2$ both inequalities (16) and (18) are valid. These inequalities are independent of each other. Indeed, let us consider

$$f(x_1,x_2) = \sum_{(k_1,k_2)\in G_n} \varphi_{k_1}(x_1)\psi_{k_2}(x_2)$$

and

$$g(x_1,x_2) = \sum_{(k_1,k_2)\in G_n} (-1)^{k_1+k_2}\varphi_{k_1}(x_1)\psi_{k_2}(x_2).$$

Inequality (16) implies the estimates

$$2^{\frac{n}{p'}} n^{\frac{1}{2}} \leqslant c_{p,q}\|f\|_{L_{p,q}},$$

and

$$2^{\frac{n}{p'}} n^{\frac{1}{2}} \leqslant c_{p,q}\|g\|_{L_{p,q}}.$$

However, inequality (18) yields

$$2^{\frac{n}{p'}} n^{\frac{1}{p}} \leqslant c_{p,q}\|f\|_{L_{p,q}},$$

and

$$1 \leqslant c_{p,q}\|g\|_{L_{p,q}}.$$

Thus, for the function f inequality (18) is sharper than inequality (16), but for the function g inequality (16) is sharper.

Acknowledgements. This research was funded by the Science Committee of the Ministry of Education and Science of the Rep. of Kazakhstan (Grant No. AP09057887). This work was also partially supported by the NU FDCRGP 20122022FD4105 and 20122022CRP1601.

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(Received August 28, 2020)

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