A SUFFICIENT CONDITION FOR A COMPLEX POLYNOMIAL TO HAVE ONLY SIMPLE ZEROS AND AN ANALOG OF HUTCHINSON'S THEOREM FOR REAL POLYNOMIALS

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Abstract. We find the constant b_∞ ($b_\infty \approx 4.81058280$) such that if a complex polynomial or entire function $f(z) = \sum_{k=0}^{\omega} a_k z^k$, $\omega \in \{2,3,4,\dots\} \cup \{\infty\}$, with nonzero coefficients satisfy the conditions $\left| \frac{a_k^2}{a_{k-1} a_{k+1}} \right| > b_\infty$ for all $k = 1, 2, \dots, \omega - 1$, then all the zeros of f are simple. We show that the constant b_∞ in the statement above is the smallest possible. We also obtain an analog of Hutchinson's theorem for polynomials or entire functions with real nonzero coefficients.

1. Introduction

In this short note, we obtain a simple sufficient condition for a complex polynomial to have only simple zeros in terms of its coefficients. To formulate our results, we define the second quotients of coefficients for a polynomial.

Let us consider a complex polynomial (or entire function) $f(z) = \sum_{k=0}^{\omega} a_k z^k$, where $a_k \in \mathbb{C} \setminus \{0\}$ and $\omega \in \{2,3,4,\dots\} \cup \{\infty\}$. We define the second quotients of the Taylor coefficients of f by the formula

$$q_n(f) = \frac{a_{n-1}^2}{a_{n-2} a_n}, \quad n \geqslant 2. \tag{1}$$

It is easy to check that

$$a_n = \frac{a_1^n}{a_0^{n-1} q_2^{n-1} q_3^{n-2} \cdot \ldots \cdot q_{n-1}^2 q_n}, \quad n \geqslant 2. \tag{2}$$

One can see that the second quotients of Taylor coefficients are independent parameters that define a function up to multiplication by a constant and changing z to λz.

In 1926, J. I. Hutchinson found quite a simple sufficient condition for an entire function with positive coefficients to have only real simple zeros.

THEOREM A. (J. I. Hutchinson, [3]) *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ *for all k, be an entire function. Then the inequalities* $q_n(f) \geqslant 4$ *for all* $n \geqslant 2$ *hold if and only if the following two conditions are fulfilled:*

(i) The zeros of f *are all real, simple and negative, and*

(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, $m < n$, *formed by taking any number of consecutive terms of* f, *are all real and non-positive.*

For some extensions of Hutchinson's results see, for example, [1], where, in particular, the following theorem is proved.

THEOREM B. (T. Craven and G. Csordas, [1]) *Let* $N \in \mathbb{N}$ *and* $(\gamma_k)_{k=0}^{N}$, $\gamma_0 = 1$, *be a sequence of positive real numbers. Suppose that the inequalities* $\frac{\gamma_n^2}{\gamma_{n-1}\gamma_{n+1}} \geqslant \alpha^2$ *hold for all* $n = 1, 2, \ldots, N-1$, *where* $\alpha = \max(2, \frac{\sqrt{2}}{2}(1 + \sqrt{1 + \gamma_1}))$. *Then the polynomial* $Q(x) = \sum_{n=0}^{N} \gamma_n \cdot \frac{x(x-1)\cdot\ldots\cdot(x-n+1)}{n!}$ *has only real, simple, negative zeros.*

There are a number of works which deal with statements of the following kind: there exists a constant $d > 1$ such that if a real polynomial P satisfies the condition $q_k(P) > d$ for all k, then we can state something about the location of the zeros of P. For example, in [2] the author proved that if for some constant $d > 0$ a real polynomial P satisfies the condition $q_k(P) > d$ for all k, then all the zeros of P lie in a special sector depending on d. In [5] the smallest possible constant $d > 0$ was found such that if a real polynomial P satisfies the condition $q_k(P) > d$ for all k, then P is stable (all the zeros of P lie in the left half-plane). In this paper, we study analogous questions for complex polynomials and entire functions.

Hutchinson's theorem inspired our investigations. The goal of this work is to find sufficient conditions for complex polynomials or entire functions with non-zero coefficients to have only simple zeros. More precisely, we answer the following question: what is the smallest possible constant $c > 0$ such that for every complex polynomial P with nonzero coefficients if the inequalities $q_n(P) > c$ hold for all $n \geqslant 2$, then all the zeros of P are simple.

For $x > 1$ let us consider the function $\phi(x) = 1 - 2\sum_{k=1}^{\infty} x^{\frac{-k^2}{2}}$. We observe that ϕ is an increasing function on $(0, \infty)$, $\lim_{x \to 1+0} \phi(x) = -\infty$ and $\lim_{x \to +\infty} \phi(x) = 1$. So, the equation

$$1 - 2\sum_{k=1}^{\infty} x^{\frac{-k^2}{2}} = 0 \qquad (3)$$

has a unique positive root, which we denote by b_∞. One can check that $b_\infty \approx 4.81058280$.

For $n \in \mathbb{N}$ we also define b_{2n} as the unique positive root of the equation

$$1 - 2\sum_{k=1}^{n} x^{\frac{-k^2}{2}} = 0. \qquad (4)$$

One can see that $(b_{2n})_{n=1}^{\infty}$ is an increasing sequence, $\lim_{n \to \infty} b_{2n} = b_\infty$, and

$$b_2 = 4, \quad b_4 \approx 4.79753651. \qquad (5)$$

The constant b_∞ firstly appeared in the paper [4] where some analogs of Hutchinson's result were obtained for sign-independently hyperbolic polynomials.

THEOREM C. (I. Karpenko and A. Vishnyakova, [4]) *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with positive coefficients. Suppose that $q_k(f) \geqslant b_\infty$ for all $k \geqslant 2$. Then for every $n \in \mathbb{N}$, the n-th section $S_n(z) := \sum_{k=0}^{n} a_k z^k$ is a sign-independently hyperbolic polynomial, meaning that it remains real-rooted after arbitrary sign changes of its coefficients.*

Our first result is the following theorem.

THEOREM 1.1. *(i) Let $n \in \mathbb{N}$ be a given integer, and $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k$, $a_k \in \mathbb{C} \setminus \{0\}$ for all k, be a polynomial. Suppose that the inequalities $|q_k(P_{2n})| > b_{2n}$ hold for all $k = 2, 3, \ldots, 2n$. Then all the zeros of P_{2n} are simple. Moreover, the moduli of all zeros of P_{2n} are pairwise different.*

(ii) Let $n \in \mathbb{N}$ be a given integer, and $P_{2n+1}(z) = \sum_{k=0}^{2n+1} a_k z^k$, $a_k \in \mathbb{C} \setminus \{0\}$ for all k, be a polynomial. Suppose that the inequalities $|q_k(P_{2n+1})| \geqslant b_{2n+2}$ hold for all $k = 2, 3, \ldots, 2n+1$. Then all the zeros of P_{2n+1} are simple. Moreover, the moduli of all zeros of P_{2n+1} are pairwise different.

(iii) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \in \mathbb{C} \setminus \{0\}$ for all k, be an entire function. Suppose that the inequalities $|q_k(f)| \geqslant b_\infty$ hold for all $k \geqslant 2$. Then all the zeros of f are simple. Moreover, the moduli of all zeros of f are pairwise different.

Since the sequence $(b_{2n})_{n=1}^{\infty}$ is monotonic and tends to b_∞, we get the following corollary.

COROLLARY 1.1. *Let $n \geqslant 2$ be a given integer, and P be a complex polynomial with nonzero coefficients of degree n. If the inequalities $|q_k(P)| \geqslant b_\infty$ hold for all $k = 2, 3, \ldots, n$, then all the zeros of P are simple. Moreover, the moduli of all zeros of P are pairwise different.*

The following statement shows the sharpness of Theorem 1.1 for entire functions and polynomials of even degrees, and asymptotical sharpness of Theorem 1.1 for polynomials of odd degrees.

THEOREM 1.2. *(i) For every $n \in \mathbb{N}$ there exists a complex polynomial $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k$, $a_k \in \mathbb{C} \setminus \{0\}$ for all k, such that the equalities $|q_k(P_{2n})| = b_{2n}$ hold for all $k = 2, 3, \ldots, 2n$, and P_{2n} has a multiple root.*

(ii) For every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a complex polynomial $P_{2n+1,\varepsilon}$ with nonzero coefficients, $\deg P_{2n+1,\varepsilon} = 2n+1$, such that $|q_k(P_{2n+1,\varepsilon})| > b_{2n} - \varepsilon$ for all $k = 2, 3, \ldots, 2n+1$, and $P_{2n+1,\varepsilon}$ has a multiple root.

(iii) For every $\varepsilon > 0$ there exists an entire function $f_\varepsilon(z) = \sum_{k=0}^{\infty} a_k(\varepsilon) z^k$, $a_k(\varepsilon) \in \mathbb{C} \setminus \{0\}$ for all k, such that the inequalities $|q_k(f_\varepsilon)| > b_\infty - \varepsilon$ hold for all $k \geqslant 2$, and f has a multiple root.

In the following statement we find the sharp constant for polynomials of the third degree.

THEOREM 1.3. *(i) Let P_3 be a complex polynomial with nonzero coefficients, $\deg P_3 = 3$, and suppose that $|q_k(P_3)| > \sqrt{9 + 6\sqrt{3}}$ for $k = 2, 3$. Then all the zeros*

of P_3 are simple. Moreover, the moduli of all zeros of P_3 are pairwise different. Note that $\sqrt{9+6\sqrt{3}} \approx 4.4036695$, and $4 = q_2 < \sqrt{9+6\sqrt{3}} < q_4 \approx 4.79753651$.

(ii) There exists a complex polynomial Q_3 with nonzero coefficients, $\deg Q_3 = 3$, such that $|q_k(Q_3)| = \sqrt{9+6\sqrt{3}}$ for $k = 2,3$, and Q_3 has a multiple root.

Using Theorem 1.1 and Theorem 1.3 (i) we obtain the following analog of Hutchinson's theorem for real polynomials.

THEOREM 1.4. *(i) Let $n \in \mathbb{N}$ be a given integer, and $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k$, $a_k \in \mathbb{R} \setminus \{0\}$ for all k, be a real polynomial. Suppose that the inequalities $|q_k(P_{2n})| \geqslant b_{2n}$ hold for all $k = 2,3,\ldots,2n$. Then all the zeros of P_{2n} are real.*

(ii) For every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a real polynomial $P_{2n,\varepsilon}$ with nonzero coefficients, $\deg P_{2n,\varepsilon} = 2n$, such that $|q_k(P_{2n,\varepsilon})| > b_{2n} - \varepsilon$ for all $k = 2,3,\ldots,2n$, and $P_{2n,\varepsilon}$ has nonreal roots.

(iii) Let $n \in \mathbb{N}$ be a given integer, and $P_{2n+1}(z) = \sum_{k=0}^{2n+1} a_k z^k$, $a_k \in \mathbb{R} \setminus \{0\}$ for all k, be a real polynomial. Suppose that the inequalities $|q_k(P_{2n+1})| \geqslant b_{2n+2}$ hold for all $k = 2,3,\ldots,2n+1$. Then all the zeros of P_{2n+1} are real.

(iv) For every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a real polynomial $P_{2n+1,\varepsilon}$ with nonzero coefficients, $\deg P_{2n+1,\varepsilon} = 2n+1$, such that $|q_k(P_{2n+1,\varepsilon})| > b_{2n} - \varepsilon$ for all $k = 2,3,\ldots,2n+1$, and $P_{2n+1,\varepsilon}$ has nonreal roots.

(v) Let P_3 be a real polynomial with nonzero coefficients, $\deg P_3 = 3$, and suppose that $|q_k(P_3)| \geqslant \sqrt{9+6\sqrt{3}}$ for $k = 2,3$. Then all the zeros of P_3 are real.

(vi) For every $\varepsilon > 0$ there exists a real polynomial $P_{3,\varepsilon}$ with nonzero coefficients, $\deg P_{3,\varepsilon} = 3$, such that $|q_k(P_{3,\varepsilon})| > \sqrt{9+6\sqrt{3}} - \varepsilon$ for $k = 2,3$, and $P_{3,\varepsilon}$ has nonreal roots.

We see that, unlike Hutchinson's result, the sharp constant for the realrootedness of a real polynomial depends on the degree of the polynomial.

2. Proof of Theorem 1.1

At first we consider the case of polynomials of even degrees. Let $n \in \mathbb{N}$, $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k$, where $a_k \in \mathbb{C} \setminus \{0\}$, and suppose that the inequalities $|q_k(P_{2n})| > b_{2n}$ hold for all $k = 2,3,\ldots,2n$.

Without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider the function $Q_{2n}(z) = a_0^{-1} P_{2n}(a_0 a_1^{-1} z)$ instead of P_{2n}, due to the fact that such a rescaling of P_{2n} preserves its property of having all simple zeros and preserves the second quotients: $q_k(Q_{2n}) = q_k(P_{2n})$ for all k. During the proof we use the notation q_k instead of $q_k(P_{2n})$. So, we can write

$$P_{2n}(z) = 1 + z + \frac{z^2}{q_2} + \frac{z^3}{q_2^2 q_3} + \ldots + \frac{z^{2n}}{q_2^{2n-1} q_3^{2n-2} \cdot \ldots \cdot q_{2n-1}^2 q_{2n}}.$$

We choose an arbitrary $k = 1,2,3,\ldots,2n-1$ and fix this k. Denote by

$$R_1 := \sqrt{|q_2|}, \quad R_k := |q_2 q_3 \ldots q_k \sqrt{q_{k+1}}|, \quad k = 2,3,\ldots,2n-1. \tag{6}$$

We have

$$P_{2n}(z) = \sum_{j=0}^{k-1} \frac{z^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} + \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k}$$

$$+ \sum_{j=k+1}^{2n} \frac{z^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} =: S_{1,k}(z) + S_{2,k}(z) + S_{3,k}(z).$$

We want to prove the inequality

$$\min_{|z|=R_k} |S_{2,k}(z)| > \max_{|z|=R_k} (|S_{1,k}(z)| + |S_{3,k}(z)|). \tag{7}$$

We obtain for every z, $|z| = R_k$,

$$|S_{2,k}(z)| = |q_2 q_3^2 \cdot \ldots \cdot q_k^{k-1} q_{k+1}^{k/2}|. \tag{8}$$

Now we estimate from above $|S_{1,k}(z)|$ for $|z| = R_k$. We have

$$|S_{1,k}(z)| \leqslant \sum_{j=0}^{k-1} \frac{|z^j|}{|q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j|}$$

$$= \sum_{j=0}^{k-1} \left| \frac{q_2^j q_3^j \cdot \ldots \cdot q_k^j q_{k+1}^{j/2}}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} \right|$$

$$= |q_2 q_3^2 \cdot \ldots \cdot q_{k-1}^{k-2} q_k^{k-1} q_{k+1}^{(k-1)/2}| + |q_2 q_3^2 \cdot \ldots \cdot q_{k-2}^{k-3} q_{k-1}^{k-2} q_k^{k-2} q_{k+1}^{(k-2)/2}|$$

$$+ |q_2 q_3^2 \cdot \ldots \cdot q_{k-2}^{k-3} q_{k-1}^{k-3} q_k^{k-3} q_{k+1}^{(k-3)/2}| + \ldots + |q_2 q_3 \cdot \ldots \cdot q_k \sqrt{q_{k+1}}| + 1$$

(we rewrite the sum from the end to the beginning). Thus, we get

$$|S_{1,k}(z)| \leqslant |q_2 q_3^2 \cdot \ldots \cdot q_{k-1}^{k-2} q_k^{k-1} q_{k+1}^{k/2}| \cdot \left(\left| \frac{1}{q_{k+1}^{1/2}} \right| + \left| \frac{1}{q_k q_{k+1}} \right| + \left| \frac{1}{q_{k-1} q_k^2 q_{k+1}^{3/2}} \right| \right.$$

$$+ \ldots + \left| \frac{1}{q_{k-j+1} q_{k-j+2}^2 \cdot \ldots \cdot q_{k-1}^{j-1} q_k^j q_{k+1}^{(j+1)/2}} \right|$$

$$+ \ldots + \left| \frac{1}{q_3 q_4^2 \cdot \ldots \cdot q_k^{k-2} q_{k+1}^{(k-1)/2}} \right| + \left. \left| \frac{1}{q_2 q_3^2 q_4^3 \cdot \ldots \cdot q_k^{k-1} q_{k+1}^{k/2}} \right| \right).$$

Using our assumption $|q_k(P_{2n})| > b_{2n}$ for $k = 2, 3, \ldots, 2n$, we obtain

$$|S_{1,k}(z)| < \left| q_2 q_3^2 \cdot \ldots \cdot q_{k-1}^{k-2} q_k^{k-1} q_{k+1}^{k/2} \right| \cdot \sum_{j=1}^{k} b_{2n}^{\frac{-j^2}{2}}. \tag{9}$$

Now we estimate $|S_{3,k}(z)|$ from above for $|z| = R_k$. We have

$$|S_{3,k}(z)| \leqslant \sum_{j=k+1}^{2n} \left| \frac{q_2^j q_3^j \cdot \ldots \cdot q_k^j q_{k+1}^{j/2}}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} \right|$$

$$= |q_2 q_3^2 \cdot \ldots \cdot q_k^{k-1} q_{k+1}^{(k-1)/2}| + \left| \frac{q_2 q_3^2 \cdot \ldots \cdot q_k^{k-1} q_{k+1}^{(k-2)/2}}{q_{k+2}} \right|$$

$$+ \ldots + \left| \frac{q_2 q_3^2 \cdot \ldots \cdot q_k^{k-1} q_{k+1}^{(2k-j)/2}}{q_{k+2}^{j-k-1} \cdot \ldots \cdot q_{j-1}^2 q_j} \right| + \ldots + \left| \frac{q_2 q_3^2 \cdot \ldots \cdot q_k^{k-1} q_{k+1}^{(2k-2n)/2}}{q_{k+2}^{2n-k-1} \cdot \ldots \cdot q_{2n-1}^2 q_{2n}} \right|$$

$$= \left| q_2 q_3^2 \cdot \ldots \cdot q_{k-1}^{k-2} q_k^{k-1} q_{k+1}^{k/2} \right| \left(\left| \frac{1}{q_{k+1}^{1/2}} \right| + \left| \frac{1}{q_{k+1} q_{k+2}} \right| + \ldots \right.$$

$$+ \left| \frac{1}{q_{k+1}^{(j-k)/2} q_{k+2}^{j-k-1} \cdot \ldots \cdot q_{j-1}^2 q_j} \right| + \ldots + \left. \left| \frac{1}{q_{k+1}^{(2n-k)/2} q_{k+2}^{2n-k-1} \cdot \ldots \cdot q_{2n-1}^2 q_{2n}} \right| \right).$$

Using our assumption $|q_k(P_{2n})| > b_{2n}$ for $k = 2, 3, \ldots, 2n$, we obtain

$$|S_{3,k}(z)| < \left| q_2 q_3^2 \cdot \ldots \cdot q_{k-1}^{k-2} q_k^{k-1} q_{k+1}^{k/2} \right| \cdot \sum_{j=1}^{2n-k} b_{2n}^{\frac{-j^2}{2}}. \tag{10}$$

Thus, by virtue of (8), (9) and (10), the desired inequality (7) follows from

$$1 - \sum_{j=1}^{k} b_{2n}^{\frac{-j^2}{2}} - \sum_{j=1}^{2n-k} b_{2n}^{\frac{-j^2}{2}} \geqslant 0. \tag{11}$$

Since the summands in both sums in the above inequality are strictly decreasing in j we have

$$1 - \sum_{j=1}^{k} b_{2n}^{\frac{-j^2}{2}} - \sum_{j=1}^{2n-k} b_{2n}^{\frac{-j^2}{2}} \geqslant 1 - 2 \sum_{j=1}^{n} b_{2n}^{\frac{-j^2}{2}} = 0 \tag{12}$$

by the definition (4) of the constant b_{2n}. We have proved that for every $k = 1, 2, 3, \ldots,$ $2n - 1$ the inequality (7) is valid. Thus, by virtue of Rouché's theorem, we obtain that for every $k = 1, 2, 3, \ldots, 2n - 1$ the polynomial P_{2n} has exactly k zeros in the circle $\{z : |z| < R_k\}$. Whence, the polynomial P_{2n} has one zero in the circle $\{z : |z| < R_1\}$, one zero in the annulus $\{z : R_1 \leqslant |z| < R_2\}$, one zero in the annulus $\{z : R_2 \leqslant |z| < R_3\}$, and so on, one zero in the annulus $\{z : R_{2n-2} \leqslant |z| < R_{2n-1}\}$ and one zero in the set $\{z : |z| \geqslant R_{2n-1}\}$. We have proved that all the zeros of P_{2n} are simple. Moreover, the moduli of all zeros of P_{2n} are pairwise different.

Now we consider the case of polynomials of odd degrees. Let $n \in \mathbb{N}$ and

$$P_{2n+1}(z) = 1 + z + \frac{z^2}{q_2} + \frac{z^3}{q_2^2 q_3} + \ldots + \frac{z^{2n+1}}{q_2^{2n} q_3^{2n-1} \cdot \ldots \cdot q_{2n}^2 q_{2n+1}}$$

be a complex polynomial with nonzero coefficients. Suppose that the inequalities $|q_k(P_{2n+1})| \geqslant b_{2n+2}$ hold for all $k = 2, 3, \ldots, 2n+1$. We choose an arbitrary $k = 1, 2, 3, \ldots, 2n$ and fix this k. We use the same notation R_k as in (6). We have

$$P_{2n+1}(z) = \sum_{j=0}^{k-1} \frac{z^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} + \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k}$$

$$+ \sum_{j=k+1}^{2n+1} \frac{z^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} =: S_{1,k}(z) + S_{2,k}(z) + S_{3,k}(z).$$

We want to prove the inequality

$$\min_{|z|=R_k} |S_{2,k}(z)| > \max_{|z|=R_k} (|S_{1,k}(z)| + |S_{3,k}(z)|). \tag{13}$$

As in the previous case, this inequality follows from

$$1 - \sum_{j=1}^{k} b_{2n+2}^{\frac{-j^2}{2}} - \sum_{j=1}^{2n+1-k} b_{2n+2}^{\frac{-j^2}{2}} > 0. \tag{14}$$

We have

$$1 - \sum_{j=1}^{k} b_{2n+2}^{\frac{-j^2}{2}} - \sum_{j=1}^{2n+1-k} b_{2n+2}^{\frac{-j^2}{2}} > 1 - 2\sum_{j=1}^{n+1} b_{2n+2}^{\frac{-j^2}{2}} = 0 \tag{15}$$

by the definition (4) of the constant b_{2n+2}. Using Rouché's theorem, we obtain that all the zeros of P_{2n+1} are simple. Moreover, the moduli of all zeros of P_{2n+1} are pairwise different.

It remains to consider the case of entire functions. Let $f(z) = 1 + z + \frac{z^2}{q_2} + \frac{z^3}{q_2^2 q_3} + \frac{z^4}{q_2^3 q_3^2 q_4} + \ldots$ be an entire function. Suppose that the inequalities $|q_k(f)| \geqslant b_\infty$ hold for all $k \geqslant 2$. For all $k \in \mathbb{N}$ we have

$$f(z) = \sum_{j=0}^{k-1} \frac{z^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j} + \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_k} + \sum_{j=k+1}^{\infty} \frac{z^j}{q_2^{j-1} q_3^{j-2} \cdot \ldots \cdot q_j}$$

$$=: S_{1,k}(z) + S_{2,k}(z) + S_{3,k}(z).$$

We want to obtain the inequality

$$\min_{|z|=R_k} |S_{2,k}(z)| > \max_{|z|=R_k} (|S_{1,k}(z)| + |S_{3,k}(z)|). \tag{16}$$

This inequality follows from

$$1 - \sum_{j=1}^{k} b_\infty^{\frac{-j^2}{2}} - \sum_{j=1}^{\infty} b_\infty^{\frac{-j^2}{2}} > 0. \tag{17}$$

We have

$$1 - \sum_{j=1}^{k} b_\infty^{\frac{-j^2}{2}} - \sum_{j=1}^{\infty} b_\infty^{\frac{-j^2}{2}} > 1 - 2\sum_{j=1}^{\infty} b_\infty^{\frac{-j^2}{2}} = 0 \tag{18}$$

by the definition (3) of the constant b_∞. Using Rouché's theorem, we obtain that all the zeros of f are simple, moreover, the moduli of all zeros of f are pairwise different.

Theorem 1.1 is proved. $\square$

3. Proof of Theorem 1.2

At first we prove Theorem 1.2 (i). For every $n \in \mathbb{N}$ and $c > 0$ we consider the following polynomial

$$P_{2n,c}(z) = \sum_{k=0}^{2n} c^{k(2n-k)/2} z^k - 2c^{n^2/2} z^n. \tag{19}$$

Note that, for all $k = 0, 1, \ldots, 2n$ the modulus of the k-th coefficient of $P_{2n,c}$ is equal to $c^{k(2n-k)/2}$, so that

$$|q_k(P_{2n,c})| = \frac{c^{(k-1)(2n-k+1)}}{c^{(k-2)(2n-k+2)/2} \cdot c^{k(2n-k)/2}} = c \tag{20}$$

for $k = 2, 3, \ldots, 2n$.

We observe that $P_{2n,c}(z) = z^{2n} \cdot P_{2n,c}(\frac{1}{z})$, so $P_{2n,c}$ is a self-reciprocal polynomial. We have

$$P'_{2n,c}(z) = 2nz^{2n-1} \cdot P_{2n,c}\left(\frac{1}{z}\right) - z^{2n} P'_{2n,c}\left(\frac{1}{z}\right) \frac{1}{z^2}.$$

Thus, if $P_{2n,c}(1) = 0$, we get

$$P'_{2n,c}(1) = -P'_{2n,c}(1).$$

It means that if $P_{2n,c}(1) = 0$ then $P'_{2n,c}(1) = 0$, so 1 is a multiple root for this polynomial. Now we consider the equation

$$P_{2n,c}(1) = \sum_{k=0}^{2n} c^{k(2n-k)/2} - 2c^{n^2/2} = 0.$$

We rewrite it in the form $2\sum_{k=0}^{n-1} c^{k(2n-k)/2} - c^{n^2/2} = 0$. After dividing by $-c^{n^2/2}$ we get

$$1 - 2\sum_{k=0}^{n-1} c^{-(n-k)^2/2} = 0,$$

or, changing the index in the sum: $n - k = j$,

$$1 - 2\sum_{j=1}^{n} c^{-j^2/2} = 0.$$

The unique positive root of this equation is b_{2n} (see (4)), so we get that the polynomial $P_{2n,b_{2n}}$ has a multiple root and $|q_k(P_{2n,b_{2n}})| = b_{2n}$ for all $k = 2, 3, \ldots, 2n$.

Let us fix an arbitrary $n \in \mathbb{N}$ and $\varepsilon > 0$. To prove Theorem 1.2 (ii) we use the polynomial $P_{2n,b_{2n}}$. Let $Q_{2n+1,d}(z) = P_{2n,b_{2n}}(z) \cdot (1 + \frac{z}{d})$, where $d > 0$. We have $\deg Q_{2n+1,d} = 2n + 1$, and $Q_{2n+1,d}$ has a multiple root at the point 1 since $P_{2n,b_{2n}}$ has a multiple root at that point. Let $P_{2n,b_{2n}}(z) = \sum_{k=0}^{2n} a_k z^k$, then

$$Q_{2n+1,d}(z) = a_0 + \left(\frac{a_0}{d} + a_1\right)z + \left(\frac{a_1}{d} + a_2\right)z^2 + \ldots + \left(\frac{a_{2n-1}}{d} + a_{2n}\right)z^{2n} + \frac{a_{2n}}{d}z^{2n+1}.$$

Thus, for all $k = 3, 4, \ldots, 2n$ we have

$$q_k(Q_{2n+1,d}) = \frac{(\frac{a_{k-2}}{d} + a_{k-1})^2}{(\frac{a_{k-1}}{d} + a_k)(\frac{a_{k-3}}{d} + a_{k-2})} \to q_k(P_{2n,b_{2n}}), \quad d \to \infty.$$

For $k = 2$ and $k = 2n + 1$ we have

$$q_2(Q_{2n+1,d}) = \frac{(\frac{a_0}{d} + a_1)^2}{a_0(\frac{a_1}{d} + a_2)} \to q_2(P_{2n,b_{2n}}), \quad d \to \infty$$

and

$$q_{2n+1}(Q_{2n+1,d}) = \frac{(\frac{a_{2n-1}}{d} + a_{2n})^2}{\frac{a_{2n}}{d}(\frac{a_{2n-2}}{d} + a_{2n-1})} \to \infty, \quad d \to \infty.$$

So, for d being large enough we obtain $|q_k(Q_{2n+1,d})| > b_{2n} - \varepsilon$ for all $k = 2, 3, \ldots, 2n + 1$.

It remains to prove Theorem 1.2 (iii). Let us fix an arbitrary $\varepsilon > 0$. Since $\lim_{n \to \infty} b_{2n} = b_\infty$, there exists $n_0 \in \mathbb{N}$ such that $b_{2n_0} > b_\infty - \varepsilon/3$. We consider an entire function of the form

$$f_\varepsilon(z) = P_{2n_0, b_{2n_0}}(z) \prod_{j=1}^{\infty} \left(1 + \frac{z}{d_j}\right),$$

where the polynomial $P_{2n_0, b_{2n_0}}$ is defined by (19), and positive constants $(d_j)_{j=1}^{\infty}$, such that $\sum_{j=1}^{\infty} \frac{1}{d_j} < \infty$, will be chosen inductively. We see that f_ε has a multiple zero at the point 1.

We know that $|q_k(P_{2n_0, b_{2n_0}})| > b_\infty - \varepsilon/3$ for all $k = 2, 3, \ldots, 2n_0$. As we have proved above, for $d_1 > 0$ being large enough for the polynomial $T_1(z) := P_{2n_0, b_{2n_0}}(z)(1 + \frac{z}{d_1})$ we have $|q_k(T_1)| > b_\infty - \varepsilon/3 - \varepsilon/4$ for all $k = 2, 3, \ldots, 2n_0 + 1$. We additionally suppose that $d_1 > 2$. We fix such d_1, and now choose d_2. For all $d_2 > 0$ being large enough for the polynomial $T_2(z) := T_1(z)(1 + \frac{z}{d_2})$ we have $|q_k(T_2)| > b_\infty - \varepsilon/3 - \varepsilon/4 - \varepsilon/8$ for all $k = 2, 3, \ldots, 2n_0 + 2$. We additionally suppose that $d_2 > 2^2$. Reasoning analogously, we construct a sequence of positive constants $(d_j)_{j=1}^{\infty}$, such that for every j the polynomial $T_j(z) := P_{2n_0, b_{2n_0}}(z) \prod_{l=1}^{j} \left(1 + \frac{z}{d_l}\right)$ has the property $|q_k(T_j)| > b_\infty - \varepsilon/3 - \varepsilon/4 - \varepsilon/8 - \ldots - \varepsilon/2^{j+1}$ for all $k = 2, 3, \ldots, 2n_0 + j$, and $d_j > 2^j$. Thus, $\sum_{j=1}^{\infty} \frac{1}{d_j} < \infty$, and f_ε is an entire function. We also observe that for every natural $k \geqslant 2$

$$|q_k(f_\varepsilon)| \geqslant b_\infty - \varepsilon/3 - \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} = b_\infty - \varepsilon/3 - \varepsilon/2 > b_\infty - \varepsilon.$$

Theorem 1.2 is proved. $\square$

4. Proof of Theorem 1.3

We consider a complex polynomial of degree 3 with nonzero coefficients

$$P_{3,a,b}(z) = 1 + z + \frac{z^2}{a} + \frac{z^3}{a^2 b}, \quad a, b \in \mathbb{C} \setminus \{0\},$$

so that $q_2(P_{3,a,b}) = a$, $q_3(P_{3,a,b}) = b$. The polynomial $P_{3,a,b}$ has multiple roots if and only if its discriminant is equal to zero. We recall that, if $Q(z) = \alpha z^3 + \beta z^2 + \gamma z + \delta \in \mathbb{C}[z]$, $\alpha \neq 0$, then its discriminant is equal to $D(Q) = -4\beta^3\delta + \beta^2\gamma^2 - 4\alpha\gamma^3 + 18\alpha\beta\gamma\delta - 27\alpha^2\delta^2$. So we have

$$D(P_{3,a,b}) = -\frac{4}{a^3} + \frac{1}{a^2} - \frac{4}{a^2b} + \frac{18}{a^3b} - \frac{27}{a^4b^2}$$

$$= -\frac{1}{a^4b^2}\left(4ab^2 - a^2b^2 + 4a^2b - 18ab + 27\right).$$

Thus, $P_{3,a,b}$ has multiple roots if and only if

$$4ab^2 - a^2b^2 + 4a^2b - 18ab + 27 = 0. \tag{21}$$

Denote by $S := \{(a,b) : a,b \in \mathbb{C} \setminus \{0\}, 4ab^2 - a^2b^2 + 4a^2b - 18ab + 27 = 0\}$, and

$$c := \sup_{(a,b)\in S} (\min(|a|,|b|).$$

By Theorem 1.1 (ii) we have $c \leqslant b_4$. By the definition of c, if $|a| > c$ and $|b| > c$ then all the zeros of $P_{3,a,b}$ are simple. We want to prove that $c = \sqrt{9 + 6\sqrt{3}}$.

We rewrite (21) in the form

$$(4a - a^2)b^2 + (4a^2 - 18a)b + 27 = 0. \tag{22}$$

By our assumption $a \neq 0$, consider now the case $a = 4$. Then we have $b = \frac{27}{8}$ and $\min(|a|,|b|) = \frac{27}{8} < \sqrt{9 + 6\sqrt{3}}$.

Let $(a_0, b_0) \in S$, $a_0 \neq 4$, and $|a_0| \neq |b_0|$. Without loss of generality we suppose that $|a_0| < |b_0|$ since (21) is symmetric with respect to a, b. Let $a_0 = re^{i\alpha}$, $r > 0$, $\alpha \in \mathbb{R}$. For $\varepsilon > 0$ being small enough we denote by $a_\varepsilon = (r+\varepsilon)e^{i\alpha}$, such that $a_\varepsilon \neq 4$. We have $|a_\varepsilon| > |a_0|$. Then we substitute a_ε in the equation (22) and find the solution b_ε, such that $(a_\varepsilon, b_\varepsilon) \in S$. By the continuity reasoning $\lim_{\varepsilon \to 0} b_\varepsilon = b_0$, so for $\varepsilon > 0$ being very small we have $|b_\varepsilon| > |a_0|$. Thus, $\min(|a_0|,|b_0|) = |a_0| < \min(|a_\varepsilon|,|b_\varepsilon|)$. Thus, we conclude that

$$c = \sup_{(a,b)\in S,\ |a|=|b|} (\min(|a|,|b|).$$

Now let $a \in \mathbb{C} \setminus \{0\}$, $b = ae^{i\gamma}$, $\gamma \in \mathbb{R}$. We substitute a, b into (21) and get

$$4a^3e^{2i\gamma} - a^4e^{2i\gamma} + 4a^3e^{i\gamma} - 18a^2e^{i\gamma} + 27 = 0.$$

We are searching for the root of the last equation with the maximal possible modulus. We rewrite the equation in the form

$$a^4e^{2i\gamma} - 4a^3(e^{2i\gamma} + e^{i\gamma}) + 18a^2e^{i\gamma} - 27 = 0,$$

or

$$a^4e^{2i\gamma} - 8a^3e^{\frac{3i\gamma}{2}}\cos\frac{\gamma}{2} + 18a^2e^{i\gamma} - 27 = 0.$$

We denote by x such complex number that $a = xe^{\frac{-i\gamma}{2}}$, and note that $|a| = |x|$, we also denote by $\lambda = \cos\frac{\gamma}{2}$, $\lambda \in [0,1]$. After substituting in the last equation we get

$$x^4 - 8\lambda x^3 + 18x^2 - 27 = 0, \tag{23}$$

and we are searching for the root of the last equation with the maximal possible modulus.

At first let us estimate the maximal real positive root. We have for $\lambda \in [0,1]$

$$x^4 - 8\lambda x^3 + 18x^2 - 27 \geqslant x^4 - 8x^3 + 18x^2 - 27 = (x+1)(x-3)^3,$$

so for all $\lambda \in [0,1]$ the maximal positive root of the equation (23) is less than or equal to 3.

Now we estimate the minimal real negative root. We have for $x = -y$, $y > 0$, and $\lambda \in [0,1]$

$$y^4 + 8\lambda y^3 + 18y^2 - 27 \geqslant y^4 + 18y^2 - 27$$
$$= (y^2 + 6\sqrt{3} - 9)(y + \sqrt{9 + 6\sqrt{3}})(y - \sqrt{9 + 6\sqrt{3}}),$$

so for all $\lambda \in [0,1]$ the maximal modulus of the negative root of the equation (23) is less than or equal to $\sqrt{9 + 6\sqrt{3}}$.

Now we consider non-real roots of the equation (23). We will use the classical Ferrari method to solve (23). For $w \in \mathbb{C}$ we rewrite the equation (23) in the form

$$(x^2 - 4\lambda x + w)^2 - \left((16\lambda^2 + 2w - 18)x^2 - 8\lambda wx + (w^2 + 27)\right) = 0. \tag{24}$$

We want to find such $w \in \mathbb{C}$, that the discriminant of the quadratic expression in the brackets will be zero. We have the equation

$$\frac{D}{4} = 16\lambda^2 w^2 - (w^2 + 27)(16\lambda^2 + 2w - 18)$$
$$= 16\lambda^2 w^2 - 16\lambda^2 w^2 - 2w^3 + 18w^2 - 27 \cdot 16\lambda^2 - 54w + 27 \cdot 18 = 0,$$

or

$$w^3 - 9w^2 + 27w - 27 - 27(8 - 8\lambda^2) = 0.$$

We get the equation

$$(w - 3)^3 = 27 \cdot 8(1 - \lambda^2) \geqslant 0.$$

Let us put $w = 3 + 6\sqrt[3]{1 - \lambda^2}$ and denote by $t = \sqrt[3]{1 - \lambda^2}$, $t \in [0,1]$, so that $\lambda^2 = 1 - t^3$, $w = 3 + 6t$. For the second quadratic expression from (24) we have

$$(16\lambda^2 + 2w - 18)x^2 - \lambda wx + (w^2 + 27)$$
$$= (16 - 16t^3 + 6 + 12t - 18)x^2 - 8\sqrt{1 - t^3}(3 + 6t)x + (9 + 36t + 36t^2 + 27)$$
$$= (4 - 16t^3 + 12t)x^2 - 8\sqrt{1 - t^3}(3 + 6t)x + (36 + 36t + 36t^2)$$

$$= \frac{4}{1-t} \cdot \left((1-t)^2(2t+1)^2x^2 - 6(1-t)(1+2t)\sqrt{1-t^3}x + 9(1-t^3) \right)$$

$$= \frac{4}{1-t} \cdot \left((1-t)(2t+1)x - 3\sqrt{1-t^3} \right)^2.$$

We substitute this in (24) and get

$$(x^2 - 4\sqrt{1-t^3}x + 3 + 6t)^2 - \left(\frac{2}{\sqrt{1-t}} \right)^2 \cdot \left((1-t)(2t+1)x - 3\sqrt{1-t^3} \right)^2 = 0.$$

We write the last equation in the form

$$\left(x^2 - 4\sqrt{1-t^3}x + 3 + 6t - 2\sqrt{1-t}(2t+1)x + 6\sqrt{1+t+t^2} \right)$$

$$\cdot \left(x^2 - 4\sqrt{1-t^3}x + 3 + 6t + 2\sqrt{1-t}(2t+1)x - 6\sqrt{1+t+t^2} \right) = 0,$$

or

$$\left(x^2 - (4\sqrt{1-t^3} - 2\sqrt{1-t}(2t+1))x + (3 + 6t + 6\sqrt{1+t+t^2}) \right)$$

$$\cdot \left(x^2 - (4\sqrt{1-t^3} + 2\sqrt{1-t}(2t+1))x + (3 + 6t - 6\sqrt{1+t+t^2}) \right) = 0.$$

So, all 4 roots of the equation (23) are the roots of two quadratic equations above, where $t = \sqrt[3]{1-\lambda^2}$, $t \in [0,1]$. Both of these quadratic equations have real coefficients. Since $3 + 6t - 6\sqrt{1+t+t^2} < 0$, the roots of the second equation are real, and we have proved above that the moduli of the roots are less than or equal to $\sqrt{9+6\sqrt{3}}$. If the roots of the first equation are real, then we have proved that the moduli of the roots are less than or equal to $\sqrt{9+6\sqrt{3}}$. If the roots of the first equation are complex and conjugate, then their moduli are equal to $\sqrt{3 + 6t + 6\sqrt{1+t+t^2}}$. It remains to check that for all $t \in [0,1]$ we have

$$\sqrt{3 + 6t + 6\sqrt{1+t+t^2}} \leqslant \sqrt{9+6\sqrt{3}},$$

that is obviously valid. So we have proved that

$$c = \sup_{(a,b)\in S} (\min(|a|,|b|)) = \sqrt{9+6\sqrt{3}}.$$

Theorem 1.3 (i) is proved.   □

To prove Theorem 1.3 (ii) we consider the polynomial

$$Q_3(z) = 1 + z + \frac{z^2}{\sqrt{9+6\sqrt{3}}} - \frac{z^3}{(\sqrt{9+6\sqrt{3}})^3},$$

so that $\deg Q_3 = 3$, $q_2(Q_3) = \sqrt{9+6\sqrt{3}}$, $q_3(Q_3) = -\sqrt{9+6\sqrt{3}}$, whence $|q_2(Q_3)| = |q_3(Q_3)| = \sqrt{9+6\sqrt{3}}$. We recall that a complex polynomial $P_{3,a,b} = 1 + z + \frac{z^2}{a} + \frac{z^3}{a^2b}$ has multiple roots if and only if $4ab^2 - a^2b^2 + 4a^2b - 18ab + 27 = 0$ (see (21)). For

the polynomial Q_3 we have $b = -a$ and the condition for having multiple roots takes the form

$$4a^3 - a^4 - 4a^3 + 18a^2 + 27 = 0 \Leftrightarrow a^4 - 18a^2 - 27 = 0.$$

It is easy to check that $a = \sqrt{9 + 6\sqrt{3}}$ is a root of this equation, so for such a the polynomial Q_3 has multiple roots. These multiple roots can be found explicitly, but the expression is rather cumbersome.

Theorem 1.3 is proved. $\square$

5. Proof of Theorem 1.4

Let $n \in \mathbb{N}$ be a given integer, and $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k$, $a_k \in \mathbb{R} \setminus \{0\}$ for all k, be a real polynomial. Suppose that the inequalities $|q_k(P_{2n})| \geqslant b_{2n}$ hold for all $k = 2, 3, \ldots, 2n$. For an arbitrary λ, $0 < \lambda < 1$, we consider a real polynomial $P_{2n,\lambda}(z) = \sum_{k=0}^{2n} a_k \cdot \lambda^{k^2} z^k$. We have for $k = 2, 3, \ldots, 2n$

$$q_k(P_{2n,\lambda}) = \frac{a_{k-1}^2 \cdot \lambda^{2(k-1)^2}}{a_{k-2} \cdot \lambda^{(k-2)^2} \cdot a_k \cdot \lambda^{k^2}} = \frac{q_k(P_{2n})}{\lambda^2} > b_{2n}.$$

By Theorem 1.1 (i) the moduli of all zeros of $P_{2n,\lambda}$ are pairwise different. So, $P_{2n,\lambda}$ can not have complex conjugate zeros, whence all the zeros of $P_{2n,\lambda}$ are real. Since $\lim_{\lambda \to 1} P_{2n,\lambda}(z) = P_{2n}(z)$, and this limit is uniform on the compacts in $\mathbb{C}$, using the Hurwitz's theorem we obtain that all the zeros of P_{2n} are real.

Theorem 1.4 (i) is proved. Using analogous reasoning we prove Theorem 1.4 (iii) and Theorem 1.4 (v).

Statements (ii), (iv) and (vi) in Theorem 1.4 can be proved using an analogous reasoning. To prove Theorem 1.4 (ii), for example, we fix an arbitrary $n \in \mathbb{N}$ and $\varepsilon > 0$, and consider a polynomial $P_{2n,b_{2n}}(z) = \sum_{k=0}^{2n} b_{2n}^{k(2n-k)/2} z^k - 2b_{2n}^{n^2/2} z^n$ (see (19)). We recall that $|q_k(P_{2n,b_{2n}})| = b_{2n}$ for all $k = 2, 3, \ldots, 2n$ (see (20)), and that $P_{2n,b_{2n}}$ has a double zero at the point 1. Since the sequence of coefficients of $P_{2n,b_{2n}}$ has two sign changes (all coefficients, except the n-th, are positive, and the n-th coefficient is negative), we conclude, using Descartes' rule of signs, that $P_{2n,b_{2n}}$ has not more than two positive zeros. So the polynomial $P_{2n,b_{2n}}$ has exactly two positive zeros counting multiplicities. Whence,

$$P_{2n,b_{2n}} \geqslant 0 \quad \text{for all} \quad x \geqslant 0.$$

Now for a small $\delta > 0$ we consider a polynomial $Q_{2n,\delta}(x) = P_{2n,b_{2n}}(x) + \delta x^n$. Then we get $Q_{2n,\delta}(x) > 0$ for all $x \geqslant 0$. We observe that $q_k(Q_{2n,\delta}) = q_k(P_{2n,b_{2n}})$ for all $k = 2, 3, \ldots, n-1, n+3, n+4, \ldots, 2n$, so that $|q_k(Q_{2n,\delta})| = b_{2n}$ for all $k = 2, 3, \ldots, n-1, n+3, n+4, \ldots, 2n$. We also see that $|q_n(Q_{2n,\delta})| > |q_n(P_{2n,b_{2n}})| = b_{2n}$ and $|q_{n+2}(Q_{2n,\delta})| > |q_{n+2}(P_{2n,b_{2n}})| = b_{2n}$. Since $\lim_{\delta \to 0} |q_{n+1}(Q_{2n,\delta})| = |q_{n+1}(P_{2n,b_{2n}})| = b_{2n}$, we obtain that $|q_{n+1}(Q_{2n,\delta})| > b_{2n} - \varepsilon$ for δ being small enough. Thus, $|q_k(Q_{2n,\delta})| > b_{2n} - \varepsilon$ for all $k = 2, 3, \ldots, 2n$ for δ being small enough. It remains to show that the polynomial $Q_{2n,\delta}$ has nonreal zeros. Suppose that all the zeros of $Q_{2n,\delta}$ are real. Then, since $Q_{2n,\delta}(x) > 0$ for all $x \geqslant 0$, we get that all the zeros of $Q_{2n,\delta}$ are negative. Then all the

coefficients of $Q_{2n,\delta}$ have the same signs, but we know that all coefficients, except the n-th, are positive, and the n-th coefficient is negative for δ being small enough. Thus, we have proved that $Q_{2n,\delta}$ has nonreal roots.

Theorem 1.4 is proved. □

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