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SHARP, DOUBLE INEQUALITIES BOUNDING THE FUNCTION $(1+x)^{1/x}$ AND A REFINEMENT OF CARLEMAN'S INEQUALITY

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Abstract. In the expansion

$$(1+x)^{1/x} = e\cdot\sum_{j=0}^{\infty}(-1)^j B_j \cdot\left(\frac{x}{x+2}\right)^j, \quad \text{for } -1<x\neq 0,$$

the sequence B_n is monotonically decreasing, bounded as $\frac{7}{10} < \lim_{n\to\infty} B_n < B_n < \frac{8}{10}$, for $n \geqslant 4$, and is given recursively as

$$B_0 = 1 \quad \text{and} \quad B_{2m} = B_{2m+1} = \frac{1}{m}\sum_{j=1}^{m}\frac{4j+1}{4j+2}B_{2m-2j}, \quad \text{for} \quad m \geqslant 1.$$

For any integers $m,n \geqslant 1$, the double inequality

$$e\cdot\sum_{j=0}^{2m-1}(-1)^j\frac{B_j}{(2n+1)^j} < \left(1+\frac{1}{n}\right)^n < e\cdot\sum_{j=0}^{2m}(-1)^j\frac{B_j}{(2n+1)^j}$$

holds, together with improved Carleman's inequality

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}x_i\right)^{1/n} < e\cdot\sum_{n=1}^{\infty}\left(1-\sum_{j=1}^{2m}(-1)^{j+1}\frac{B_j}{(2n+1)^j}\right)x_n,$$

true for every sequence $x_n \geqslant 0$ such that $0<\sum_{n=1}^{\infty}x_n<\infty$.

1. Introduction

In 1922 the Swedish mathematician Carleman [1] presented the inequality

$$x_1+(x_1x_2)^{1/2}+(x_1x_2x_3)^{1/3}+\cdots+(x_1x_2x_3\cdots x_n)^{1/n}+\cdots < e\left(x_1+x_2+x_3+\cdots\right), \quad (1)$$

valid for $x_n \geqslant 0$ with $0<x_1+x_2+x_3+\ldots<\infty$. It is now called Carleman's inequality.

First important generalization of (1) was done in 1925 by the English mathematician Hardy [8]. Later, in 1926 the Hungarian mathematician Pólya [12], in his proof of (1), derived the crucial improvement

$$\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}x_i\right)^{1/n} \leqslant \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n x_n, \quad (2)$$

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true for $x_n \geqslant 0$ such that $0 < \sum_{n=1}^{\infty} x_n < \infty$. Until recently, many authors provided various generalizations/improvements of (1), see e.g. [7].

Over the last twenty years, many articles have addressed strengthening (1) by using (2). This was done mostly on the basis of several estimates of the sequence $(1+1/n)^n$, which continued with the search for an accurate estimate of $(1+x)^{1/x}$, for $0<x\leqslant 1$, or equivalently, an estimate of $(1+\frac{1}{x})^x$, for $x \geqslant 1$. For example, in 1999, Yang [16], using the left estimate of his inequality $e/(2x+2) < e-(1+1/x)^x < e/(2x+1)$, valid for $x>0$, improved (1) by the estimate $\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n} x_i\right)^{1/n} < e\cdot\sum_{n=1}^{\infty}\left(1-1/(2n)\right)x_n$, true for $x_n \geqslant 0$ such that $0<\sum_{n=1}^{\infty} x_n < \infty$. In fact, most authors sought as accurate estimates of the function $(1+x)^{1/x}$ as possible, see e.g. [2, 3, 4, 5, 9, 10, 11, 13, 14, 16, 17].

In 2002 H.-W. Chen [2, Theorem 1] provided the expansion

$$\left(1+\frac{1}{x}\right)^x = e\cdot\left(1-\sum_{j=1}^{\infty}\frac{b_j^{**}}{(1+x)^j}\right) \qquad (x>0), \tag{3}$$

where $b_n^{**}>0$, for $n\in\mathbb{N}$, $b_1^{**}=1/2$, and $b_n^{**}=\frac{1}{n}\left(\frac{1}{n+1}-\sum_{j=1}^{n-1}\frac{b_j^{**}}{n+1-j}\right)$, for $n\geqslant 2$.

Just recently [6], the authors presented their study of the function $x\mapsto(1+x)^{1/x}$, for $x>0$. They found the expansion

$$(1+x)^{1/x} = e\sum_{j=0}^{\infty}(-1)^j b_j x^j \qquad (-1<x\leqslant 1), \tag{4}$$

where the sequence b_n is monotonically decreasing, converging to 0, and is defined recursively as

$$b_0 = 1 \quad \text{and} \quad b_n = \frac{1}{n}\sum_{j=1}^{n}\frac{j}{j+1}b_{n-j}, \text{ for } n\geqslant 1. \tag{5}$$

Unfortunately, the convergence in (4) is extremely poor in the left immediate neighborhood of the point $x=1$ as it is seen in Figure 1, where the graph of the function $x\mapsto(1+x)^{1/x}$, together with the graphs of its approximations $\sigma_n(x) := e\sum_{j=0}^{n}(-1)^j b_j x^j$ and $\sigma_{n+1}(x)$, for $n\in\{2,10\}$, are plotted. This deficiency motivates our contribution.

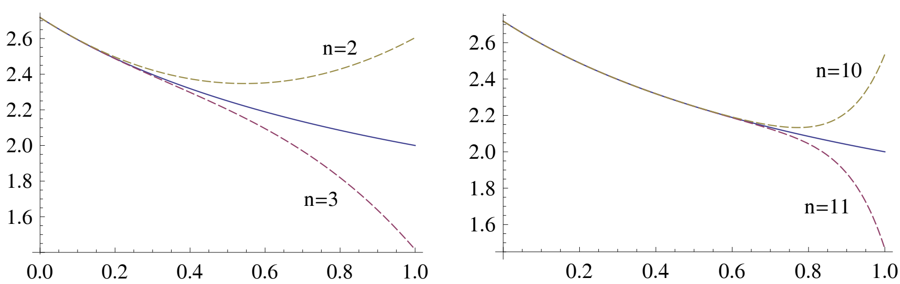

Figure 1: *Illustration of the double inequality* $\sigma_{n+1}(x) < (1+x)^{1/x} < \sigma_n(x)$, *for* $\sigma_n(x) := e\sum_{j=0}^{n}(-1)^j b_j x^j$, $x\in(0,1]$ *and* $n\in\{2,10\}$.

2. Discussion

We have

$$(1+x)^{1/x} = \exp\left(\frac{1}{x}\ln(1+x)\right), \quad \text{for } x > -1,$$

where, according to Maclaurin's expansion,

$$\ln(1+t) = \sum_{j=0}^{\infty}(-1)^j \frac{t^j}{j+1}, \quad \text{for } t \in (-1,1] \setminus \{0\},$$

we have the expansion

$$\ln\left(\frac{1+t}{1-t}\right) = \ln(1+t) - \ln(1-t) = 2\sum_{j=1}^{\infty}\frac{t^{2j-1}}{2j-1}, \quad \text{for } |t| < 1. \tag{6}$$

Using in (6) the substitution

$$\frac{1+t}{1-t} = 1+x, \quad \text{i.e} \quad t = \frac{x}{2+x}, \quad \text{i.e} \quad x = \frac{2t}{1-t} =: x(t)\,, \tag{7}$$

for $x \in (-1,\infty) \setminus \{0\}$, or equivalently, for $t \in (-1,1) \setminus \{0\}$, we obtain

$$\begin{aligned} \frac{1}{x}\ln(1+x) &= \frac{1-t}{2t}\ln\left(\frac{1+t}{1-t}\right) = \frac{1-t}{t}\sum_{j=1}^{\infty}\frac{t^{2j-1}}{2j-1} \\ &= (1-t)\sum_{i=0}^{\infty}\frac{t^{2i}}{2i+1}\,. \end{aligned} \tag{8}$$

Hence, for $x \in (-1,\infty) \setminus \{0\}$, i.e. for $|t| < 1$,

$$\begin{aligned} (1+x)^{1/x} = \left(\frac{1+t}{1-t}\right)^{\frac{1-t}{2t}} &= \exp\left(\sum_{i=0}^{\infty}\frac{t^{2i}}{2i+1} - \sum_{i=0}^{\infty}\frac{t^{2i+1}}{2i+1}\right) \\ &= \exp\left(\sum_{j=0}^{\infty} a_j t^j\right), \end{aligned} \tag{9}$$

where, for $j \geqslant 0$,

$$a_j = (-1)^j c_j \quad \text{and} \quad c_j = \begin{cases} \dfrac{1}{j+1}, & j \text{ even} \\[2mm] \dfrac{1}{j}, & j \text{ odd.} \end{cases} \tag{10}$$

We shall use the following lemma, demonstrated quite elementarily.

LEMMA 1. *If an analytic function* $s(t)$ *has the expansion* $s(t) = \sum_{j=0}^{\infty} a_j t^j$, *for* $|t| < r$ *with some* $r \in \mathbb{R}^+$, *then the function* $f(t) := \exp\big(s(t)\big)$ *has the expansion*[1]

$$f(t) = \sum_{j=0}^{\infty} a_j^* t^j = e^{a_0} \sum_{j=0}^{\infty} b_j^* t^j \qquad (|t| < r), \tag{11}$$

where $a_j^* = e^{a_0} b_j^*$, *for* $j \geqslant 0$, *with*

$$a_0^* = e^{a_0} \quad \textit{and} \quad a_n^* = \frac{1}{n} \sum_{k=0}^{n-1} (n-k)\, a_{n-k} a_k^* \qquad (n \geqslant 1), \tag{12}$$

$$b_0^* = 1 \qquad \textit{and} \quad b_n^* = \frac{1}{n} \sum_{k=0}^{n-1} (n-k)\, a_{n-k} b_k^* = \frac{1}{n} \sum_{j=1}^{n} j \cdot a_j b_{n-j}^* \qquad (n \geqslant 1). \tag{13}$$

Proof. Let all the suppositions of Lemma 1 be satisfied. Then, due to the analyticity, the function $f(t) = \exp\big(s(t)\big)$ has the Taylor series expansion, consequently the nth coefficient a_n^* is given as

$$a_n^* = \frac{f^{(n)}(0)}{n!}, \quad \text{for} \quad n \geqslant 0. \tag{14}$$

Thus

$$a_0^* = f(0) = e^{s(0)} = e^{a_0}. \tag{15}$$

Since $f'(t) = e^{s(t)} s'(t) = f(t) s'(t)$, we have, using (14) and the Leibniz theorem on the nth derivative of a product,

$$\begin{aligned}
\big(f(t)\big)^{(n+1)} &= \big(f'(t)\big)^{(n)} \\
&= \Big(f(t) s'(t)\Big)^{(n)} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t)\, s^{(1+n-k)}(t) \qquad (n \geqslant 0).
\end{aligned}$$

Consequently, considering (14), we obtain, for $n \geqslant 0$,

$$\begin{aligned}
a_{n+1}^* &= \frac{1}{(n+1)!} \sum_{k=0}^{n} \frac{n!}{\cancel{k!} \cdot (n-k)!} \cdot \cancel{(k!)}\, a_k^* \cdot (1+n-k)!\, a_{1+n-k} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} (n+1-k)\, a_k^* a_{n+1-k},
\end{aligned}$$

that is, for $n \geqslant 1$,

$$a_n^* = \frac{1}{n} \sum_{k=0}^{n-1} (n-k) a_{n-k} a_k^*.$$

[1]See the interesting discussions on an expansion of the function $(1+1/x)^x$ given in [3] and [13], the latter as a revisit using the Faà di Bruno formula.

This way we approved (12) and consequently also (13), due to the obvious identity $a_j^* = e^{a_0} b_j^*$ $(j \geqslant 0)$. $\square$

Thanks to (9)–(10) and Lemma 1 we have the expansion

$$\left(\frac{1+t}{1-t}\right)^{\frac{1-t}{2t}} = e \cdot \sum_{j=0}^{\infty} b_j^* t^j \qquad \big(t \in (-1,1) \setminus \{0\}\big), \tag{16}$$

where, according to (9), (10) and (13), the coefficients b_j^* are given as

$$b_0^* = 1 \quad \text{and} \quad b_n^* = \frac{1}{n} \sum_{j=1}^{n} j \cdot (-1)^j c_j b_{n-j}^* . \qquad (n \geqslant 1). \tag{17}$$

Consequently, for the sequence

$$B_n := (-1)^n b_n^* \qquad (n \geqslant 0), \tag{18}$$

we have (see (10)),

$$B_0 = 1, \quad 0 < B_n = \frac{1}{n} \sum_{j=1}^{n} j c_j B_{n-j} \leqslant 1, \quad \text{and} \quad b_n^* = (-1)^n B_n, \quad \text{for } n \geqslant 1. \tag{19}$$

Indeed, referring to (17) and (18), we obtain

$$\cancel{(-1)^n} B_n = b_n^* = \frac{1}{n} \sum_{j=1}^{n} j \cancel{(-1)^j} c_j (-1)^{\cancel{n-j}} B_{n-j}, \quad \text{for } n \geqslant 1.$$

In addition, considering (10), i.e. the obvious inequalities $0 < j c_j \leqslant 1$, and using the induction, we approve the estimate $0 < B_n \leqslant 1$.

Due to (19) we have

$$\begin{aligned} B_0 &= B_1 = 1, \\ B_2 &= B_3 = \tfrac{5}{6} \approx 0.83, \\ B_4 &= B_5 = \tfrac{287}{360} \approx 0.80, \\ B_6 &= B_7 = \tfrac{7085}{9072} \approx 0.78 . \end{aligned} \tag{20}$$

3. Monotonicity of the sequence $n \mapsto B_n$

3.1. Direct discrete approach

We would like to confirm the detected monotonicity of the sequence $n \mapsto B_n$ perceived in (20). Thanks to (10) and (19), we obtain, for $n \geqslant 1$,

$$\begin{aligned} B_{n+1} - B_n &= \frac{1}{n+1}\left(1 \cdot 1 \cdot B_n + \sum_{j=2}^{n+1} j\, c_j\, B_{n+1-j}\right) - B_n \\ &= \frac{1}{n+1}\left(-n B_n + \sum_{j=2}^{n+1} j\, c_j\, B_{n+1-j}\right) \\ &= \frac{1}{n+1}\left(-\sum_{j=1}^{n} j\, c_j\, B_{n-j} + \sum_{j=2}^{n+1} j\, c_j\, B_{n-(j-1)}\right) \\ &= -\frac{1}{n+1}\sum_{i=1}^{n} \underbrace{\Big(i\, c_i - (i+1)\, c_{i+1}\Big)}_{=:\,(-1)^{i+1}\beta_i} B_{n-i}\,. \end{aligned} \tag{21}$$

According to (10), we have, for $i \geqslant 0$,

$$\beta_i := (-1)^{i+1}\Big(i\, c_i - (i+1)\, c_{i+1}\Big) = \begin{cases} \dfrac{1}{i+1}\,, & i \text{ even} \\[2mm] \dfrac{1}{i+2}\,, & i \text{ odd}\,. \end{cases} \tag{22}$$

The sequence β_i is decreasing, satisfying the following relations

$$1 = \beta_0 > \beta_{2j-1} = \beta_{2j} > \beta_{2j+1} = \beta_{2j+2} > 0 \qquad (j \geqslant 1). \tag{23}$$

Now, using (21) and (23), we obtain, for any integer $m \geqslant 1$,

$$\begin{aligned} B_{2m+1} - B_{2m} &= -\frac{1}{2m+1}\sum_{i=1}^{2m} (-1)^{i+1}\beta_i\, B_{2m-i} \\ &= -\frac{1}{2m+1}\sum_{j=1}^{m} \beta_{2j-1}\big(\underline{B_{2(m-j)+1} - B_{2(m-j)}}\big). \end{aligned} \tag{24}$$

Thus, if for some integer $m \geqslant 1$ we have $B_{2k+1} = B_{2k}$ for $k \in \{0, 1, \ldots, m-1\}$, then also $B_{2m+1} = B_{2m}$. Hence, since $B_0 = 1 = B_1$, we conclude

$$B_{2k+1} = B_{2k} \qquad (k \geqslant 0). \tag{25}$$

Consequently, using (25) and (21)–(23), we find, for $m \geqslant 1$,

$$\begin{aligned} B_{2m+2}-B_{2m} &= B_{2m+2}-B_{2m+1} \\ &= -\frac{1}{2m+2}\Big(\beta_1 B_{2m} + \sum_{i=2}^{2m+1}(-1)^{i+1}\beta_i B_{2m+1-i}\Big) \\ &= -\frac{1}{2m+2}\left(\frac{B_{2m}}{3} + \sum_{j=1}^{m}(\beta_{2j+1}-\beta_{2j})B_{2m-2j}\right) \\ &= -\frac{1}{6(m+1)}\left(B_{2m} - 6\sum_{j=1}^{m}\frac{B_{2m-2j}}{(2j+1)(2j+3)}\right). \end{aligned} \tag{26}$$

We can not demonstrate that the expression between the last round parenthesis in (26) is positive, for all positive integers m, although Mathematica [15] find $B_{2m+2}-B_{2m}>0$, for $m \leqslant 500$. Therefore, we try differently. Thanks to (19), (25) and (10), we have

$$\begin{aligned} B_{2m} &= \frac{1}{2m}\sum_{i=1}^{2m} i\,\alpha_i B_{2m-i} \\ &= \frac{1}{2m}\left(\sum_{j=1}^{m}(2j-1)\,\alpha_{2j-1}\underbrace{B_{2m-2j+1}}_{=B_{2m-2j}} + \sum_{j=1}^{m}(2j)\,\alpha_{2j}B_{2m-2j}\right) \\ &= \frac{1}{2m}\sum_{j=1}^{m}\Big((2j-1)\,\alpha_{2j-1}+(2j)\,\alpha_{2j}\Big)B_{2m-2j} \\ &= \frac{1}{m}\sum_{j=1}^{m}\left(1-\frac{1}{4j+2}\right)B_{2m-2j}, \qquad \text{for } m \geqslant 1. \end{aligned} \tag{27}$$

Consequently, using (26), we obtain

$$B_{2m+2}-B_{2m} = -\frac{1}{6(m+1)}\sum_{j=1}^{m}\frac{1}{2j+1}\left(\frac{4j+1}{2m}-\frac{6}{2j+3}\right)B_{2m-2j}. \tag{28}$$

Unfortunately, we fail to prove that the sum in (28) is positive, for all integers $m \geqslant 1$. So, we shall use a complex analysis approach, used in [6].

OPEN PROBLEM 1. *Demonstrate directly/elementarily the monotonicity of the sequence $m \mapsto B_{2m}$.*

3.2. Complex analysis approach to the monotonicity of B_{2m}

The logarithmic function[2] $L(z) := \int\limits_{C:1}^{z} \frac{\mathrm{d}\zeta}{\zeta}$ (C is any piecewise smooth curve connecting 1 and z) is analytic on the simply connected domain $\mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$ and satisfies the equalities $L(z) = \ln(z)$, for $z \in \mathbb{R}^+$ and $z = \exp\big(L(z)\big)$, for $z \in \mathbb{C}^-$. For $z \in \mathbb{C}^-$, we have

$$L(z) = \ln(|z|) + \mathrm{i}\operatorname{Arg}(z), \tag{29}$$

where $\operatorname{Arg}(z) \in (-\pi, \pi]$ is the principal value of the argument of z.

For $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}^-$, the α-power of z we define as $z^\alpha := \exp\big(\alpha L(z)\big)$. For $z \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, this definition of a power coincides with the standard one. Consequently, considering the expansion[3] (16) and the identity $\frac{1+z}{1-z} = \frac{1-|z|^2+2\mathrm{i}\Im(z)}{|1-z|^2}$, the composite function

$$f(z) := \left(\frac{1+z}{1-z}\right)^{\frac{1-z}{2z}} = \exp\left(\frac{1-z}{2z} L\Big(\frac{1+z}{1-z}\Big)\right) \tag{30}$$

is analytic on the domain $\mathbb{C} \setminus (-\infty, -1] \setminus \{1\}$.

We will show that the singularity of $f(z)$ at $z = 1$ is removable. Indeed, according to (29)–(30), for $r \in (0, 1/e)$ and $t \in (-\pi, \pi]$, we estimate

$$\begin{aligned} \left|f(1 + re^{\mathrm{i}t})\right| &\leqslant \exp\left(\tfrac{r}{2(1-r)}\left(\ln\tfrac{2+r}{r} + \pi\right)\right) \\ &< \exp\left(3 \cdot \tfrac{r}{3}\left(\ln\tfrac{3}{r} + \pi\right)\right) < \exp\left(3\left(\tfrac{1}{e} + \pi\right)\right) < 4\pi. \end{aligned}$$

Since $f(z)$ is bounded on the open punctured disk $D'(1, \frac{1}{e}) := \{z \in \mathbb{C} : 0 < |z-1| < \frac{1}{e}\}$, the Laurent expansion of $f(z)$ on $D'(1, 1/e)$ reduces to the Taylor expansion guaranteeing the existence of the finite $\lambda := \lim\limits_{z \to 1} f(z)$. Therefore, using the additional definition $f(1) := \lambda$, the extension $f(z)$ becomes analytic also on the disk $|z-1| < 1/e$.

For the function $f(z)$, being analytic on the simply–connected domain $\mathscr{D} := \mathbb{C} \setminus (-\infty, -1]$, we use the Cauchy's integral formula for derivatives,

$$f^{(n)}(0) = \frac{n!}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z^{n+1}}\,\mathrm{d}z \qquad (n \in \mathbb{N}), \tag{31}$$

where $C \subset \mathscr{D}$ is any piecewise smooth, simple closed curve enclosing the point $z = 0$. In addition, referring to (18), (16) and (30), we have also

$$e(-1)^n B_n = e\,b_n = \frac{f^{(n)}(0)}{n!}. \tag{32}$$

Hence, we obtain

$$B_n = \frac{(-1)^n}{2\pi\mathrm{i}\,e} \oint_C \frac{f(z)}{z^{n+1}}\,\mathrm{d}z \qquad (n \in \mathbb{N}),$$

[2] called the principal branch of the logarithm
[3] $z = 0$ is a removable singular point of $f(z)$

consequently

$$B_{2m} - B_{2m+2} = \frac{1}{2\pi \mathrm{i}\, e} \oint_C \frac{(z^2-1)f(z)}{z^{2m+3}}\, \mathrm{d}z \qquad (m \in \mathbb{N}), \tag{33}$$

Here, in contrast to the function $f(z)$, the function $g(z) := (z^2-1)f(z) = (z-1)(1+z)f(z)$ is bounded on the notched disk $D := \{-1+re^{\mathrm{i}t} : 0 < r \leqslant \frac{1}{2}, -\pi < t < \pi\}$. Indeed, using (30), for $(-1+re^{\mathrm{i}t}) \in D$, we have

$$\begin{aligned} \left| g(-1+re^{\mathrm{i}t}) \right| &\leqslant (2+r)\, r \cdot \exp\left(\frac{r}{2(1-r)} \left(\left| \ln \frac{r}{2-r} \right| + \pi \right) \right) \\ &< 3r \cdot \exp\left(r \left(\ln \tfrac{2}{r} + \pi \right) \right) = 3(2e^{\pi})^r \cdot r^{1-r} < 6e^{\pi}. \end{aligned} \tag{34}$$

Now let, for (small) $\varepsilon \in \left(0, \frac{1}{4}\right]$ and (large) $R > 2$, the curve $C = C(\varepsilon, R)$ be the oriented sum of consistently oriented curves, $C(\varepsilon,R) = C_1(\varepsilon,R) + C_2(\varepsilon,R) + C_3(\varepsilon) + C_2^*(\varepsilon,R)$, where, as is indicated in Figure 2, $C_1(\varepsilon,R)$ is the circular arc with center at $z=0$ and radius R, $C_2(\varepsilon,R)$ and $C_2^*(\varepsilon,R)$ are horizontal segments, and $C_3(\varepsilon) = \{z \in \mathbb{C} : |z+1| = \varepsilon,\ \Re(z) \geqslant 1\}$, the semicircle.

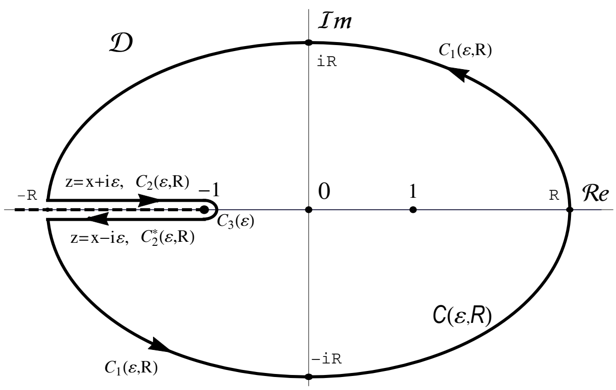

Figure 2: *The piecewise smooth, simple closed curve $C(\varepsilon,R) = C_1(\varepsilon,R) + C_2(\varepsilon,R) + C_3(\varepsilon) + C_2^*(\varepsilon,R)$ in a simply connected domain $\mathscr{D} := \mathbb{C} \setminus (-\infty,-1]$, enclosing the point $z=0$.*

We have

$$\oint_{C(\varepsilon,R)} \frac{(z^2-1)f(z)}{z^{2m+3}}\, \mathrm{d}z = \int_{C_1(\varepsilon,R)} + \int_{C_2(\varepsilon,R)} + \int_{C_3(\varepsilon)} + \int_{C_2^*(\varepsilon,R)} \frac{(z^2-1)f(z)}{z^{2m+3}}\, \mathrm{d}z \tag{35}$$

and

$$\frac{1+z}{1-z} = \frac{1-|z|^2 + 2\mathrm{i} \cdot \Im(z)}{|1-z|^2}, \quad \lim_{\substack{\Re(z)<0 \\ \Im(z)\downarrow 0}} \operatorname{Arg}\left(\frac{1+z}{1-z}\right) = \pi, \quad \lim_{\substack{\Re(z)<0 \\ \Im(z)\uparrow 0}} \operatorname{Arg}\left(\frac{1+z}{1-z}\right) = -\pi, \tag{36}$$

where, using (30) and (36), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left(\int_{C_2(\varepsilon,R)}\frac{(z^2-1)f(z)}{z^{2m+3}}\,\mathrm{d}z+\int_{C_2^*(\varepsilon,R)}\frac{(z^2-1)f(z)}{z^{2m+3}}\,\mathrm{d}z\right)\\
&=\int_{-R}^{-1}\frac{(x^2-1)\exp\left(\frac{1-x}{2x}\left(\ln\left|\frac{1+x}{1-x}\right|+\mathrm{i}\pi\right)\right)\mathrm{d}x}{x^{2m+3}}\\
&\quad+\int_{-1}^{-R}\frac{\exp\left((x^2-1)\frac{1-x}{2x}\left(\ln\left|\frac{1+x}{1-x}\right|-\mathrm{i}\pi\right)\right)\mathrm{d}x}{x^{2m+3}}\\
&=2\mathrm{i}\int_{-R}^{-1}(x^2-1)\left|\frac{1+x}{1-x}\right|^{\frac{1-x}{2x}}\frac{\sin\left(\frac{1-x}{2x}\pi\right)}{x^{2m+3}}\,\mathrm{d}x\\
&=2\mathrm{i}\int_{1}^{R}(t^2-1)\left|\frac{1+t}{1-t}\right|^{\frac{1+t}{2t}}\frac{\cos\left(\frac{\pi}{2t}\right)}{t^{2m+3}}\,\mathrm{d}t\,. \qquad (37)
\end{aligned}$$

Thanks to (29)–(30), for $R \geqslant 3$ and $t \in [-\pi, \pi]$, we estimate

$$\left|f(Re^{\mathrm{i}t})\right| \leqslant \exp\left(\frac{1+R}{2R}\ln\left(\frac{1+R}{R-1}\right)+\pi\right) < \exp\left(1\cdot\ln(2)+\pi\right) = 2e^{\pi}.$$

Therefore, for integer $m \geqslant 1$, $\varepsilon \in (0, 1/4)$ and $R > 3$, we have

$$\left|\int_{C_1(\varepsilon,R)}\frac{(z^2-1)f(z)\,\mathrm{d}z}{z^{2m+3}}\right| < \int_{C_1(\varepsilon,R)}\frac{2R^2\cdot 2e^{\pi}\,|\mathrm{d}z|}{R^{2m+3}} < \frac{4e^{\pi}}{R^{2m+1}}\cdot 2\pi R = \frac{8\pi e^{\pi}}{R^{2m}}.$$

Thus,

$$\lim_{R\uparrow\infty,\,\varepsilon\downarrow 0}\int_{C_1(\varepsilon,R)}\frac{(z^2-1)f(z)\,\mathrm{d}z}{z^{2m+3}} = 0\,. \qquad (38)$$

Similarly, according to (34), we have

$$\lim_{\varepsilon\downarrow 0}\int_{C_3(\varepsilon)}\frac{(z^2-1)f(z)\,\mathrm{d}z}{z^{2m+3}} = 0\,. \qquad (39)$$

Now, considering (35), (37), (38) and (39), we get the equality

$$\lim_{R\uparrow\infty,\,\varepsilon\downarrow 0}\oint_{C(\varepsilon,R)}\frac{(z^2-1)f(z)}{z^{2m+3}}\,\mathrm{d}z = 2\mathrm{i}\int_1^{\infty}(t^2-1)\left|\frac{1+t}{1-t}\right|^{\frac{1+t}{2t}}\frac{\cos\left(\frac{\pi}{2t}\right)}{t^{2m+3}}\,\mathrm{d}t\,.$$

Hence, using (33) we find, for integer $m \geqslant 1$,

$$\begin{aligned}
B_{2m}-B_{2m+2} &= \frac{1}{e\pi}\int_1^{\infty}(t^2-1)\left(\frac{t+1}{t-1}\right)^{\frac{t+1}{2t}}\frac{\cos\left(\frac{\pi}{2t}\right)}{t^{2m+3}}\,\mathrm{d}t\\
&= \frac{1}{e\pi}\int_0^1\underbrace{(1-\tau^2)\left(\frac{1+\tau}{1-\tau}\right)^{\frac{1+\tau}{2}}\tau^{2m-1}\cos\left(\tfrac{\pi}{2}\tau\right)}_{\geqslant 0}\,\mathrm{d}\tau > 0. \qquad (40)
\end{aligned}$$

3.3. Rough bounding the sequence $n \mapsto B_n$

The integral $I(m)$ in (40),

$$
\begin{aligned}
I(m) :&= \int_0^1 (1-\tau^2)\left(\tfrac{1+\tau}{1-\tau}\right)^{\frac{1+\tau}{2}} \tau^{2m-1}\cos\left(\tfrac{\pi}{2}\tau\right)\mathrm{d}\tau \\
&= \int_0^1 (1-\tau)^{\frac{1-\tau}{2}}(1+\tau)^{\frac{3+\tau}{2}}\tau^{2m-1}\cos\left(\tfrac{\pi}{2}\tau\right)\mathrm{d}\tau, \qquad (41)
\end{aligned}
$$

we roughly estimate[4] from below, using (41), as follows

$$
\begin{aligned}
I(m) &> \int_0^1 e^{-1/(2e)}\left(1+\tfrac{3}{2}\tau\right)\cdot\tau^{2m-1}\cdot\left(\tfrac{\pi}{2}-\tfrac{\pi}{2}\tau-\tfrac{1}{6}\left(\tfrac{\pi}{2}-\tfrac{\pi}{2}\tau\right)^3\right)\mathrm{d}\tau \\
&= \frac{\pi\exp\left(-1/(2e)\right)}{192}\cdot\frac{240m^3+936m^2+3(352-5\pi^2)m+12(24-\pi^2)}{m(m+1)(m+2)(2m+1)(2m+3)} \\
&> \frac{\pi\exp\left(-1/(2e)\right)}{192}\cdot\frac{240m^3+936m^2+606m+168}{m(m+1)(m+2)(2m+1)(2m+3)} \\
&> \frac{\pi\exp\left(-1/(2e)\right)}{192}\cdot\frac{42\cdot\cancel{(m+2)(2m+1)(2m+3)}}{m(m+1)\cancel{(m+2)(2m+1)(2m+3)}} \\
&= \frac{21\pi\exp\left(-1/(2e)\right)}{96\,m(m+1)} > \frac{0.181\,\pi}{m(m+1)} \qquad (m \geqslant 1).
\end{aligned}
$$

Thus, using (40), we obtain[5]

$$
\begin{aligned}
B_{2m+2n} &< B_{2m} - \frac{21\exp(-1/(2e))}{96e}\sum_{j=0}^{n-1}\frac{1}{(m+j)(m+j+1)} \\
&= B_{2m} - \frac{21\exp(-1/(2e))}{96e}\left(\frac{1}{m}-\frac{1}{m+n}\right) \\
&< B_{2m} - 0.066\left(\frac{1}{m}-\frac{1}{m+n}\right) \qquad (m,n \geqslant 1). \qquad (42)
\end{aligned}
$$

Similarly, using (41), we estimate from above

$$
\begin{aligned}
I(m) &< \int_0^1 1\cdot(1+\tau)^2\cdot\tau^{2m-1}\left(\tfrac{\pi}{2}-\tfrac{\pi}{2}\tau-\tfrac{1}{6}\left(\tfrac{\pi}{2}-\tfrac{\pi}{2}\tau\right)^3+\tfrac{1}{120}\left(\tfrac{\pi}{2}-\tfrac{\pi}{2}\tau\right)^5\right)\mathrm{d}\tau \\
&= \frac{\pi}{3840}\cdot\frac{A(m)+B(m)+C(m)}{m(m+1)(m+2)(m+3)(2m+1)(2m+3)(2m+5)(2m+7)} \\
&< \frac{\pi}{3840}\cdot\frac{1920\cdot\cancel{(m+2)(m+3)(2m+1)(2m+3)(2m+5)(2m+7)}}{m(m+1)\cancel{(m+2)(m+3)(2m+1)(2m+3)(2m+5)(2m+7)}} \\
&= \frac{\pi}{2m(m+1)} = \frac{0.500\,\pi}{m(m+1)},
\end{aligned}
$$

---

[4] using the inequalities $e^{-1/(2e)} \leqslant x^{x/2} \leqslant 1$ and $(1+x)^{3/2} \geqslant 1+\frac{3}{2}x$, both true for $0 \leqslant x \leqslant 1$, and $\frac{\pi}{2} - x - \frac{1}{6}\left(\frac{\pi}{2}-x\right)^3 \leqslant \cos(x) \leqslant \frac{\pi}{2}-x-\frac{1}{6}\left(\frac{\pi}{2}-x\right)^3+\frac{1}{120}\left(\frac{\pi}{2}-x\right)^5$, valid for $0 < x \leqslant \frac{\pi}{2}$

[5] using the induction and the expansion $\frac{1}{(m+1)(m+2)} = \frac{1}{m+1} - \frac{1}{m+2}$

where

$$A(m) := 30720m^6 + 384000m^5 + 1920(986 - \pi^2)m^4 + 1920(2416 - 9\pi^2)m^3$$

$$B(m) := 120(48360 - 448\pi^2 + \pi^4)m^2 + 60(55632 - 1100\pi^2 + 7\pi^4)m$$

$$C(m) := 315(1920 - 80\pi^2 + \pi^4).$$

Consequently, considering (40), we get[6]

$$\begin{aligned} B_{2m+2n} &> B_{2m} - \tfrac{1}{2e}\sum_{j=0}^{n-1}\tfrac{1}{(m+j)(m+j+1)} \\ &= B_{2m} - \tfrac{1}{2e}\left(\tfrac{1}{m} - \tfrac{1}{m+n}\right) \\ &> B_{2m} - 0.184\left(\tfrac{1}{m} - \tfrac{1}{m+n}\right) \qquad (m,n \geqslant 1). \end{aligned} \tag{43}$$

Letting $n \to \infty$ in (42)–(43), we obtain, for any integer $m \geqslant 1$, the double inequality

$$B_{2m} - \frac{23}{125\,m} < \lim_{n\to\infty} B_n < B_{2m} - \frac{8}{125\,m}. \tag{44}$$

For example, using $m = 2$, we estimate

$$\frac{7}{10} < \lim_{n\to\infty} B_n < \frac{8}{10}.$$

Figure 3 shows, for $m \in \{10, 30\}$ the graphs of the functions $n \mapsto B_{2m+2n}$ together with the graphs of the lower and upper bounds, $n \mapsto B_{2m} - \frac{23}{125}\left(\frac{1}{m} - \frac{1}{m+n}\right)$ and $n \mapsto B_{2m} - \frac{8}{125}\left(\frac{1}{m} - \frac{1}{m+n}\right)$, respectively.

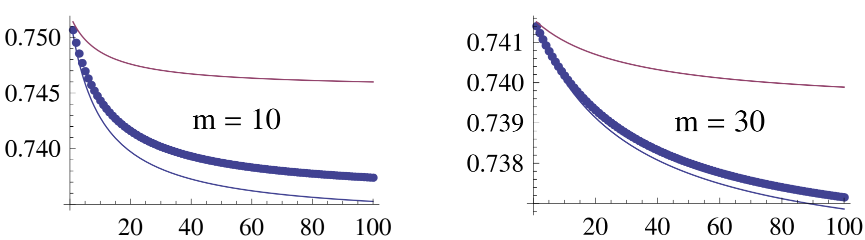

Figure 3: *The graphs of the functions* $n \mapsto B_{2m+2n}$ *and the graphs of the lower and upper bounds,* $n \mapsto B_{2m} - \frac{23}{125}\left(\frac{1}{m} - \frac{1}{m+n}\right)$ *and* $n \mapsto B_{2m} - \frac{8}{125}\left(\frac{1}{m} - \frac{1}{m+n}\right)$.

[6]by induction

4. Series expansion of $(1+x)^{1/x}$

We summarize the results of our previous discussions, the items (7), (9), (16), (18), (25), (27), (40), (42) and (43), as the following theorem.

THEOREM. *For $-1 < x \neq 0$, the expansion*

$$(1+x)^{1/x} = e \cdot \sum_{j=0}^{\infty} (-1)^j B_j \cdot \left(\frac{x}{x+2}\right)^j, \tag{45}$$

holds, having the series absolutely convergent, where the sequence B_n is monotonically decreasing (B_{2m} strictly monotonically decreasing) and given recursively as

$$B_0 = 1 \quad \text{and} \quad B_{2m} = B_{2m+1} = \frac{1}{m}\sum_{j=1}^{m} \frac{4j+1}{4j+2} B_{2m-2j} = \frac{1}{m}\sum_{i=0}^{m-1} \frac{4(m-i)+1}{4(m-i)+2} B_{2i}, \tag{46}$$

for $m \geqslant 1$.

The sequence $n \mapsto B_{2m+2n}$ satisfies, for all $m,n \geqslant 1$, the double inequality

$$B_{2m} - \frac{23}{125}\left(\frac{1}{m} - \frac{1}{m+n}\right) < B_{2m+2n} < B_{2m} - \frac{8}{125}\left(\frac{1}{m} - \frac{1}{m+n}\right), \tag{47}$$

resulting from the identity

$$B_{2m} - B_{2m+2} = \frac{1}{e\pi}\int_0^1 (1-\tau^2)\left(\frac{1+\tau}{1-\tau}\right)^{\frac{1+\tau}{2}} \tau^{2m-1} \cos\left(\tfrac{\pi}{2}\tau\right) \mathrm{d}\tau \qquad (m \geqslant 1). \tag{48}$$

COROLLARY 1. *We have $\frac{7}{10} < \lim_{n\to\infty} B_n < B_n < \frac{8}{10}$, for $n \geqslant 4$.*

COROLLARY 2. *For any integer $m \geqslant 1$ and every real $x > 0$ we have the following relations*

$$(1+x)^{1/x} = \frac{2e}{x+2}\sum_{i=0}^{\infty} B_{2i} \cdot \left(\frac{x}{x+2}\right)^{2i}, \tag{49}$$

$$S_{2m-1}(x) < (1+x)^{1/x} < S_{2m}(x), \tag{50}$$

where

$$S_n(x) := e \cdot \sum_{j=0}^{n} (-1)^j B_j \cdot \left(\frac{x}{x+2}\right)^j, \qquad \text{for } n \geqslant 1.$$

Figure 4 shows the graph of the function $x \mapsto (1+x)^{1/x}$, together with the graphs of its approximations $S_n(x) := e\sum_{j=0}^{n}(-1)^j b_j x^j$ and $S_{n+1}(x)$, for $n \in \{2, 4\}$.

Setting $x = \frac{1}{n}$ in (49), we obtain the next corollary.

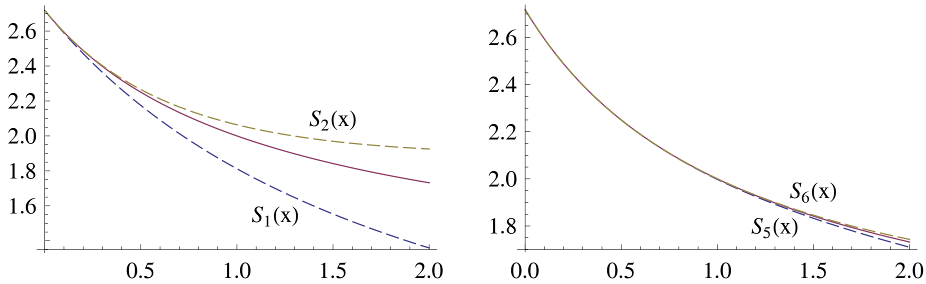

Figure 4: *Illustration of the double inequality* (49)*, for* $m \in \{1,3\}$.

COROLLARY 3. *For any integers* $m,n \geqslant 1$ *there holds the following double inequality*

$$e \cdot \sum_{j=0}^{2m-1} (-1)^j \frac{B_j}{(2n+1)^j} < \left(1+\frac{1}{n}\right)^n < e \cdot \sum_{j=0}^{2m} (-1)^j \frac{B_j}{(2n+1)^j}. \tag{51}$$

Considering Pólya's improvement (2), we get from (51) the following corollary.

COROLLARY 4. (Carleman's inequality improvement) *For any integer* $m \geqslant 1$ *and for every sequence* $x_n \geqslant 0$ *such that* $0 < \sum_{n=1}^{\infty} x_n < \infty$*, we have the following improvement of Carleman's inequality*

$$\sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} x_i\right)^{1/n} < e \cdot \left(1 - \sum_{j=1}^{2m} (-1)^{j+1} \frac{B_j}{(2n+1)^j}\right) x_n = e \cdot \sum_{n=1}^{\infty} \Big(1 - \Delta(m,n)\Big) x_n, \tag{52}$$

where

$$0 < \Delta(m,n) := \sum_{i=1}^{m} \frac{1}{(2n+1)^{2i-1}} \left(B_{2i-1} - \frac{B_{2i}}{2n+1}\right) < 1 \qquad (m,n \geqslant 1).$$

OPEN PROBLEM 2. *Demonstrate that the estimate* (52) *improves the inequality* [6, (3.4)]*, i.e.* $\Delta(m,n) > \delta(m,n) := \sum_{j=1}^{2m} (-1)^{j+1} \frac{b_j}{n^j}$*, for* $m,n \geqslant 1$.

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