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BORSUK'S PARTITION PROBLEM IN ℓ_p^4

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Abstract. In 1933, K. Borsuk made a conjecture that every n-dimensional bounded set can be divided into $n+1$ subsets of smaller diameter. Up to now, the problem is still open for $4 \leqslant n \leqslant 63$. In this paper, we study the generalized Borsuk's partition problem in ℓ_p^4 and prove that all bounded sets X in every ℓ_p^4 can be divided into 2^4 subsets of smaller diameter.

1. Introduction

Let $\mathbb{E}^n$ be the n-dimensional Euclidean space, and in this paper an n-dimensional vector $\mathbf{x} \in \mathbb{E}^n$ is always treated as a column vector. Let K denote an n-dimensional convex body, a compact convex set with non-empty interior $\mathrm{int}(K)$. By $\mathscr{K}^n$ we denote the set of convex bodies in $\mathbb{E}^n$.

Let $d(X)$ denote the diameter of a bounded set X of $\mathbb{E}^n$ defined by

$$d(X) = \sup\{\|\mathbf{x},\mathbf{y}\| : \mathbf{x},\mathbf{y} \in X\},$$

where $\|\mathbf{x},\mathbf{y}\|$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$. Let $b(X)$ be the smallest number of subsets $X_1, X_2, \ldots, X_{b(X)}$ of X such that

$$X = \bigcup_{i=1}^{b(X)} X_i$$

and $d(X_i) < d(X)$ holds for all $i \leqslant b(X)$. In 1933, K. Borsuk [1] proposed the following problem:

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Borsuk's partition problem. Is it true that

$$b(X) \leqslant n+1$$

holds for every bounded set X in $\mathbb{E}^n$?

Usually, the positive statement of this problem is referred as Borsuk's conjecture. K. Borsuk [1] proved that the inequality $b(X) \leqslant 3$ holds for any bounded set $X \subseteq \mathbb{E}^2$. For $n = 3$, Borsuk's conjecture was confirmed by H. G. Eggleston [4] in 1955. In 1945, H. Hadwiger [7] proved that the inequality $b(K) \leqslant n+1$ holds for every n-dimensional convex body K with smooth boundary. However, in 1993, J. Kahn and G. Kalai [10] discovered counterexamples to Borsuk's conjecture in high dimensions. In 2014, T. Jenrich and A. E. Brouwer [9] discovered a 64-dimensional counterexample. Up to now, the problem is still open for $4 \leqslant n \leqslant 63$. For more detailed information about the problem, we refer to [2, 3, 23, 25].

Let $\mathbb{M}_C^n = (\mathbb{R}^n, \|\cdot\|_C)$ denote the Minkowski space with respect to the norm $\|\cdot\|_C$ determined by a centrally symmetric convex body C centered at the origin $\mathbf{o}$. Clearly, C is the unit ball of $\mathbb{M}_C^n$. For a bounded set $X \subseteq \mathbb{M}_C^n$, let $d_C(X)$ denote the diameter of X defined by $d_C(X) = \sup\{\|\mathbf{x},\mathbf{y}\|_C : \mathbf{x},\mathbf{y} \in X\}$, and let $b_C(X)$ denote the smallest number such that X can be divided into $b_C(X)$ subsets each of which has the diameter strictly smaller than $d_C(X)$.

In 1957, B. Grünbaum [6] firstly studied the problem in Minkowski planes $\mathbb{M}_C^2$. It was mentioned in [3] that for every bounded set $X \subseteq \mathbb{M}_C^2$, if the unit ball C of $\mathbb{M}_C^2$ is a not a parallelogram, then the inequality $b_C(X) \leqslant 3$ holds; otherwise, the inequality $b_C(X) \leqslant 4$ holds.

For every convex body $K \in \mathscr{K}^n$, the covering number $\gamma(K)$ is the smallest number of translates of λK $(0 < \lambda < 1)$ such that their union contains K. In 1957, H. Hadwiger [8] raised the following conjecture, which has a close relation with the Borsuk's partition problem.

Hadwiger's covering conjecture. Every convex body K in $\mathbb{E}^n$ can be covered by 2^n translates of λK (or $\mathrm{int}(K)$), where λ is a suitable positive number satisfying $\lambda < 1$.

The two-dimensional case had been solved by F. W. Levi [12]. In 1984, M. Lassak [11] proved this conjecture for all centrally symmetric convex bodies in $\mathbb{E}^3$. However, this conjecture is open for all $n \geqslant 3$ untill now. The best known upper bound in three-dimensional case is $\gamma(K) \leqslant 14$ and is due to A. Prymak [17] recently. In 2020, A. Prymak and V. Shepelska [18] showed that $\gamma(K) \leqslant 96$ for all $K \in \mathscr{K}^4$, $\gamma(K) \leqslant 1091$ for all $K \in \mathscr{K}^5$ and $\gamma(K) \leqslant 15373$ for all $K \in \mathscr{K}^6$. For further results on this conjecture, we refer to [2, 3, 22, 24, 25].

In 1997, C. A. Rogers and C. Zong [19] obtained an upper bound on $\gamma(K)$:

$$\begin{aligned} \gamma(K) &\leqslant \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)}(n\log n + n\log\log n + 5n) \\ &\leqslant \binom{2n}{n}(n\log n + n\log\log n + 5n) = O(4^n\sqrt{n}\log n) \end{aligned} \tag{1}$$

when $n \geqslant 3$ and $K \in \mathscr{K}^n$ with volume $\mathrm{vol}(K)$, where $K-K$ denotes the difference body of K.

In 1965, V. G. Boltyanski and I. T. Gohberg [2] proved that

$$b_C(X) \leqslant \gamma(\widehat{X}) \tag{2}$$

holds for all n-dimensional Minkowski space $\mathbb{M}_C^n$ and all bounded sets X of $\mathbb{M}_C^n$, where $\widehat{X}$ denotes the closed convex hull of X. Based on this fact, they also proposed the following problem:

PROBLEM 1. Is it true that

$$b_C(X) \leqslant 2^n$$

holds for all n-dimensional Minkowski space $\mathbb{M}_C^n$ and all bounded sets X of $\mathbb{M}_C^n$?

In this paper, we concern the space $\ell_p^n := (\mathbb{R}^n, \|\cdot\|_p)$, whose unit ball is denoted by

$$C_{n,p} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_p \leqslant 1\}.$$

Denote by

$$C_n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \leqslant 1\} = [-1,1]^n,$$

the n-dimensional unit cube and $\{-1,1\}^n$ the vertices of C_n.

In 2009, L. Yu and C. Zong [21] studied Problem 1 and obtained that $b_{C_{3,p}}(X) \leqslant 2^3$ holds for all bounded sets X in every ℓ_p^3. In 2021, Y. Lian and S. Wu [13] showed that each set X having diameter 1 in ℓ_p^3 can be represented as the union of 2^3 subsets of X whose diameters are at most 0.925. Later, this value is improved into 0.9, see [26].

According to (2), $b_C(X)$ has an upper bound via Hadwiger's covering number (1), i.e.,

$$b_C(X) \leqslant O(4^n \sqrt{n} \log n)$$

holds for all n-dimensional Minkowski spaces $\mathbb{M}_C^n$ and all bounded sets $X \subseteq \mathbb{M}_C^n$. Particularly, since $\gamma(K) \leqslant 96$ for all $K \in \mathscr{K}^4$, it is deduced that $b_C(X) \leqslant 96$ for all 4-dimensional Minkowski space $\mathbb{M}_C^4$ and all bounded sets X of $\mathbb{M}_C^4$. In this paper, we continue studying the above problem in ℓ_p^4. Our main result is:

THEOREM 1. *For all bounded sets X in every ℓ_p^4, we have*

$$b_{C_{4,p}}(X) \leqslant 2^4.$$

In order to prove this theorem, we rely on the Banach-Mazur distance. The Banach-Mazur distance between two $\mathbf{o}$-symmetric convex bodies K and L is defined as

$$d_{BM}(K,L) = \min\{r > 0 : K \subset gL \subset rK, g \in GL(n,\mathbb{R})\},$$

where $GL(n,\mathbb{R})$ is the set of invertible linear operators.

2. Proof of the Theorem

In order to prove Theorem 1, let us consider three main situations. First of all, we introduce two lemmas which will be useful for the proof of cases $p > 1$.

LEMMA 1. ([20]) *Let n be a positive integer and $1 \leqslant p,\ q \leqslant \infty$.*

(i) *If $1 \leqslant p \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant q \leqslant \infty$, then $d_{BM}(C_{n,p}, C_{n,q}) = n^{\frac{1}{p}-\frac{1}{q}}$.*

(ii) *If $1 \leqslant p < 2 < q \leqslant \infty$, then $\xi n^{\alpha} \leqslant d_{BM}(C_{n,p}, C_{n,q}) \leqslant \eta n^{\alpha}$, where $\alpha = \max\{\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q}\}$, and ξ, η are universal constants. If $n = 2^k$ ($k \in \mathbb{N}$), then $\eta = 1$.*

LEMMA 2. ([13]) *Let $\mathbb{M}_C^n = (\mathbb{R}^n, \|\cdot\|_C)$, if $d_{BM}(C, C_n) < 2$, we have $b_C(X) \leqslant 2^n$ for all bounded set X of $\mathbb{M}_C^n$.*

2.1. $p > 2$

If $p > 2$, by Lemma 1 (i), we have $d_{BM}(C_{4,p}, C_4) = 4^{\frac{1}{p}} < 2$. Combining with Lemma 2, then $b_{C_{4,p}}(X) \leqslant 2^4$ holds for all bounded set X of ℓ_p^4 with $p > 2$.

REMARK 1. Using the same method, by Lemma 1 (i) and Lemma 2, we can prove that $b_{C_{n,p}}(X) \leqslant 2^n$ holds for all bounded set X of ℓ_p^n with $\log_2 n < p \leqslant +\infty$ and $n \geqslant 3$.

2.2. $1 < p \leqslant 2$

If $1 < p \leqslant 2$, by Lemma 1 (ii), we have $d_{BM}(C_{4,p}, C_4) \leqslant 2$. That is to say, there exists a parallelotope $Q = gC_4$ satisfying

$$\frac{1}{2}Q \subseteq C_{4,p} \subseteq Q, \tag{3}$$

where

$$g = \frac{1}{4^{\frac{1}{p}}} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Denote by $w(X, \mathbf{u})$ the Euclidean width of a bounded set X in the direction $\mathbf{u}$. Let $\mathbf{u}_i = g\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th unit vector, i.e. $\mathbf{u}_1 = 4^{-\frac{1}{p}}(1,1,1,-1)^T$, then $w(C_{4,p}, \mathbf{u}_i) = 4^{1-\frac{1}{p}}$ for $i = 1, \ldots, 4$.

For each bounded set X with $d_{C_{4,p}}(X) = 2$ in ℓ_p^4, we have $w(X, \mathbf{u}_i) \leqslant 4^{1-\frac{1}{p}}$ for $i = 1, \ldots, 4$ and the equality holds if and only if there exists $\mathbf{a}_i, \mathbf{b}_i \in X$ such that

$$\mathbf{a}_i - \mathbf{b}_i = 2\mathbf{u}_i. \tag{4}$$

Up to translation, we may assume that $X \subseteq \cap_{i \in [4]}\{\mathbf{x} : |\langle \mathbf{x}, \mathbf{u}_i \rangle| \leqslant w_i\} = Q_X$ with $w_i \leqslant 4^{1-\frac{2}{p}}$. In fact, $Q = \cap_{i \in [4]}\{\mathbf{x} : |\langle \mathbf{x}, \mathbf{u}_i \rangle| \leqslant 4^{1-\frac{2}{p}}\}$. Now we consider two cases:

1. If there exists some $w_i < 4^{1-\frac{2}{p}}$, then we have $X \subseteq Q_X \subsetneq Q$ and $d_{C_{4,p}}(Q_X) < 4$. In this case, one can divided Q_X into 16 smaller copies of $\frac{1}{2}Q_X$ with $d_{C_{4,p}}(\frac{1}{2}Q_X) < 2$. Then X can also be divided into 16 corresponding parts with diameter strictly smaller than 2. Thus, $b_{C_{4,p}}(X) \leqslant 2^4$.

2. If $w_i = 4^{1-\frac{2}{p}}$ for all $i = 1,\dots,4$, let $F_i = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{u}_i \rangle = 4^{1-\frac{2}{p}}\}$ and $F_{-i} = \{\mathbf{x} : \langle \mathbf{x}, -\mathbf{u}_i \rangle = 4^{1-\frac{2}{p}}\}$, then X touches each pair of opposite facets of Q. Assuming that X touches $Q \cap F_i$ at one point $\mathbf{a}_i$ and touches $Q \cap F_i$ at point $\mathbf{b}_i$ satisfying (4). In addition, since $C_{4,p}$ is strictly convex when $1 < p \leqslant 2$, then X cannot touch F_i (as well as F_{-i}) at more than one point. Also, all $\mathbf{a}_i$, $\mathbf{b}_i$ must be in the relative interior of each facet of Q. If not, suppose $\mathbf{a}_1$ is on the relative boundary of one facet of Q. Without of loss generality, let $\mathbf{a}_1 \in Q \cap F_1 \cap F_2$, by $\mathbf{a}_1 - \mathbf{b}_1 = 2\mathbf{u}_1$, then $\mathbf{b}_1 \in (Q \cap F_{-1} \cap F_2)$. Since $d_{C_{4,p}}(X) = 2$ and $\mathbf{a}_1, \mathbf{b}_1 \in (Q \cap F_2)$, there is no point of X on the opposite facet $(Q \cap F_{-2})$, which contradicts to the assumption that X intersects all facets of Q.

   For $1 < p \leqslant 2$, by the strictly convexity of $C_{4,p}$ and (3), the diameter of $\frac{1}{2}Q$ in ℓ_p^4 is only determined by its eight pairs of symmetric vertices:

$$d_{C_{4,p}}\left(\frac{1}{2}Q\right) = 2 = \left\| \frac{1}{2}g\mathbf{v}, \frac{1}{2}g(-\mathbf{v}) \right\|_{C_{4,p}},$$

   where

$$\mathbf{v} \in \{1,-1\}^4 = \Sigma_{i=1}^4 \delta_i \mathbf{e}_i, \;\; \delta_i \in \{1,-1\}.$$

   Now we still divided Q into 16 smaller copies of $\frac{1}{2}Q$, that is, $Q = \bigcup_{i=1}^{16}(\frac{1}{2}Q + \mathbf{y}_i)$ with $\mathbf{y}_i \in \{\frac{1}{2}g\mathbf{v} : \mathbf{v} \in \{1,-1\}^4\}$. Then we also get 16 corresponding subsets $X_i = X \cap (\frac{1}{2}Q + \mathbf{y}_i)$, $i = 1,\dots,16$. For every translating point pair $\left(\frac{1}{2}g\mathbf{v} + \mathbf{y}_i, \frac{1}{2}g(-\mathbf{v}) + \mathbf{y}_i\right)$, $i = 1,\dots,16$, we will show that at least one point of $\left(\frac{1}{2}g\mathbf{v} + \mathbf{y}_i, \frac{1}{2}g(-\mathbf{v}) + \mathbf{y}_i\right)$ lies on the relative boundary of some facet of Q. Without loss of generality, take a point pair $\left(\frac{1}{2}g\mathbf{v}_0, \frac{1}{2}g(-\mathbf{v}_0)\right)$ with $\mathbf{v}_0 = \Sigma_{i=1}^4 \sigma_i \mathbf{e}_i$, $\sigma_i \in \{1,-1\}$. Then

$$\frac{1}{2}g\mathbf{v}_0 + \mathbf{y}_i = \frac{1}{2}g(\Sigma_{i=1}^4(\sigma_i + \delta_i)\mathbf{e}_i), \tag{5}$$

$$\frac{1}{2}g(-\mathbf{v}_0) + \mathbf{y}_i = \frac{1}{2}g(\Sigma_{i=1}^4(\delta_i - \sigma_i)\mathbf{e}_i). \tag{6}$$

   If all $\delta_i = \sigma_i$, $i = 1,\dots,4$, then the point (5) is contained in $\cap_{i=1}^4 F_{\delta_i(i)}$; if $\delta_i = \sigma_i$, $i = 1,\dots,3$ and $\delta_4 \neq \sigma_4$, then the point (5) is contained in $\cap_{i=1}^3 F_{\delta_i(i)} \cap Q$; if $\delta_i = \sigma_i$, $i = 1,2$ and $\delta_j \neq \sigma_j$, $j = 3,4$, then the point (5) is contained in $\cap_{i=1}^2 F_{\delta_i(i)} \cap Q$; if $\delta_1 = \sigma_1$, $\delta_i \neq \sigma_i$, $i = 2,\dots,4$, then the point (6) is contained in $\cap_{i=2}^4 F_{\delta_i(i)} \cap Q$; if all $\delta_i \neq \sigma_i$, $i = 1,\dots,4$, then the point (6) is contained in $\cap_{i=1}^4 F_{\delta_i(i)}$.

By above discussions and the fact that X touches each facet of Q at exactly one relative interior point, we have $d_{C_{4,p}}(X_i) < 2$ for all $i = 1,\ldots,16$. Therefore, $b_{C_{4,p}}(X) \leqslant 2^4$ holds for all bounded set X of ℓ_p^4 with $1 < p \leqslant 2$.

2.3. $p = 1$

By (2), determining the covering number of a convex body is useful for solving the Borsuk's partition problem. Let m be a positive integer and let $\gamma_m(K)$ be the smallest positive number r such that K can be covered by m translates of rK. Clearly, $\gamma_m(K) < 1$ is equivalent to $\gamma(K) \leqslant m$. First of all, the following lemma gives an estimate on the value of $\gamma_{2n}(C_{n,1})$.

LEMMA 3. ([14]) *$\gamma_{2n}(C_{n,1}) \leqslant \frac{n-1}{n}$ holds for all $n \geqslant 2$.*

In order to show the case of $p = 1$, we use the concept of completeness. A bounded set is called complete if it is not properly contained in a set of the same diameter. Clearly, a complete set is convex and compact. In [5], H. G. Eggleston showed that any bounded set $X \subseteq \mathbb{M}_C^n$ can be embedded in a complete set A of the same diameter, the complete set A is called the *completion* of X. Generally, A is not unique. For every bounded set $X \subseteq \mathbb{M}_C^n$, we have $b_C(X) \leqslant b_C(A)$, since $X \subseteq A \cap X \subseteq \cup_{i=1}^{b_C(A)}(A_i \cap X) = \cup_{i=1}^{b_C(A)}(X_i)$ and $d_C(X_i) = d_C(A_i \cap X) \leqslant d_C(A_i) < d_C(A) = d_C(X)$.

In [15] and [16], J. P. Moreno and R. Schneider gave a new characterization of the complete sets in $\mathbb{M}_C^n$ in terms of supporting slabs. A supporting slab of the convex body $K \in \mathscr{K}^n$ is any closed set $\Sigma \supseteq K$ that is bounded by two parallel supporting hyperplanes H, H' of K. The distance between H and H' is called the width of Σ. For any other convex body M, we say that the supporting slab Σ of K is M-regular if the supporting slab of M that is parallel to Σ has the property that at least one of its bounding hyperplanes contains a smooth boundary point of M (a boundary point through which passes only one supporting hyperplane of M). For the case of a polyhedral norm, the space of translation classes of complete sets of given diameter is a finite polytopal complex. The following two lemmas will be useful for our proof.

LEMMA 4. ([15]) *Let $d > 0$. The n-dimensional convex body $K \in \mathscr{K}^n$ is a complete set of diameter d if and only if the following properties hold:*

(a) *Every C-regular supporting slab of K has width $\leqslant d$, C is the unit ball of $\mathbb{M}_C^n$.*

(b) *Every K-regular supporting slab of K has width d.*

LEMMA 5. ([16]) *Let $\Sigma_1,\ldots,\Sigma_k$ be the C-regular supporting slabs of the polytopal unit ball C. Each complete set K with diameter 2 is of the form*

$$K = \bigcap_{i=1}^{k}(\Sigma_i + \mathbf{t}_i)$$

with $\mathbf{t}_i \in \mathbb{R}^n$, $i = 1,\ldots,k$.

For the polytopal unit ball $C_{4,1}$ of ℓ_1^4, its supporting slabs are Σ_1 with outer normal vectors $\pm\mathbf{u}_1=\pm(1,1,1,1)$, Σ_2 with outer normal vectors $\pm\mathbf{u}_2=\pm(-1,-1,1,1)$, Σ_3 with outer normal vectors $\pm\mathbf{u}_3=\pm(1,-1,-1,1)$, Σ_4 with outer normal vectors $\pm\mathbf{u}_4=\pm(-1,1,-1,1)$, Σ_5 with outer normal vectors $\pm\mathbf{u}_5=\pm(1,1,1,-1)$, Σ_6 with outer normal vectors $\pm\mathbf{u}_6=\pm(-1,1,1,1)$, Σ_7 with outer normal vectors $\pm\mathbf{u}_7=\pm(1,-1,1,1)$ and Σ_8 with outer normal vectors $\pm\mathbf{u}_8=\pm(1,1,-1,1)$. Each slab Σ_i is bounded by two parallel hyperplanes $\Phi_i=\{\mathbf{x}:\langle\mathbf{x},\mathbf{u}_i\rangle=1\}$ and $\Phi_{-i}=\{\mathbf{x}:\langle\mathbf{x},-\mathbf{u}_i\rangle=1\}$, $i=1,\ldots,8$.

For every bounded set $X\subseteq\ell_1^4$ with $d_{C_{4,1}}(X)=2$, there always exists a completion D of X. Up to some translation and by Lemma 5, we may assume that

$$\begin{aligned}X\subseteq D&=D(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\\&=\bigcap_{i=5}^{8}\Sigma_i\cap\bigcap_{i=1}^{4}(\Sigma_i+\alpha_i\mathbf{u}_i)\\&=\left(C_{4,1}\cup(\cup_{i=1}^{4}S_{\pm i})\right)\cap\bigcap_{i=1}^{4}(\Sigma_i+\alpha_i\mathbf{u}_i)\\&=\bigcup_{i=5}D_i,\end{aligned}$$

where $D_1=C_{4,1}\cap\bigcap_{i=1}^{4}(\Sigma_i+\alpha_i\mathbf{u}_i)$, $D_j=S_{j-1}\cap\bigcap_{i=1}^{4}(\Sigma_i+\alpha_i\mathbf{u}_i)$ or $D_j=S_{-(j-1)}\cap\bigcap_{i=1}^{4}(\Sigma_i+\alpha_i\mathbf{u}_i)$ with $|\alpha_i|\leqslant\frac{1}{4}$, $j=2,\ldots,5$, and

$$S_i=\operatorname{conv}\left((C_{4,1}\cap\Phi_i)\cup\frac{1}{2}\mathbf{u}_i\right),S_{-i}=-S_i,\ \ i=1,\ldots,4.$$

For $i=1,\ldots,4$, we obtain that $d_{C_{4,1}}(S_i)=d_{C_{4,1}}(S_{-i})=2$ and that there are exactly five vertices of S_i or S_{-i} such that the distance between each pair is 2. Neither S_i nor S_{-i} is a complete set by Lemma 4, since there exist a S_i (S_{-i})-regular supporting slab of S_i (S_{-i}) with width 1. By Lemma 3, $C_{4,1}$ can be covered by 8 smaller copies of $C_{4,1}$. That is to say, we have

$$C_{4,1}\subseteq\bigcup_{i=1}^{4}\left(\left(\frac{3}{4}C_{4,1}+\mathbf{y}_i\right)\bigcup\left(\frac{3}{4}C_{4,1}+\mathbf{y}_{4+i}\right)\right),$$

with $\mathbf{y}_i=\frac{1}{4}\mathbf{e}_i$, $\mathbf{y}_{4+i}=-\mathbf{y}_i$, $i=1,\ldots,4$. By taking a small sutible positive number ε satisfying $\frac{3}{4}+\varepsilon<1$, then one can see that

$$(1+\varepsilon)C_{4,1}\subseteq\bigcup_{i=1}^{4}\left(\left((\frac{3}{4}+\varepsilon)C_{4,1}+\mathbf{y}_i\right)\bigcup\left((\frac{3}{4}+\varepsilon)C_{4,1}+\mathbf{y}_{4+i}\right)\right).$$

In fact, some vertices with neighbour also have been covered from the covering of $(1+\varepsilon)C_{4,1}$, so the remaining part of S_i or S_{-i} has diameter strictly smaller than 2. Therefore, X can be divided into at most 12 parts, each of which has diameter strictly smaller than 2. Consequently, $b_{C_{4,1}}(X)\leqslant 2^4$ holds for all bounded set X of ℓ_1^4.

In conclusion, $b_{C_{4,p}}(X) \leqslant 2^4$ holds for all bounded set X of all ℓ_p^4 with $1 \leqslant p \leqslant \infty$. This completes the proof of the theorem. $\square$

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REFERENCES

[1] K. BORSUK, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), 177–190.

[2] V. G. BOLTYANSKI AND I. T. GOHBERG, *Results and Problems in Combinatorial Geometry*, Cambridge Univ. Press, Cambridge, 1985; Nauka, Moscow, 1965.

[3] V. G. BOLTYANSKI, H. MARTINI, P. S. SOLTAN, *Excursions into Combinatorial Geometry*, Universitext, Springer, Berlin (1997).

[4] H. G. EGGLESTON, *Covering a three-dimensional set with sets of smaller diameter*, J. London Math. Soc. 30 (1955), 11–24.

[5] H. G. EGGLESTON, *Sets of constant width in finite dimensional Banach spaces*, Isr. J. Math. 3 (1965), 163–172.

[6] B. GRÜNBAUM, *Borsuk's partition conjecture in Minkowski planes*, Bull. Res. Council Israel Sect. F, 7F (1957), 25–30.

[7] H. HADWIGER, *Überdeckung einer Menge durch Mengen kleineren Durchmessers*, Comment. Math. Helv. 18 (1945), 73–75.

[8] H. HADWIGER, *Ungelöste Probleme Nr. 20*, Elem. Math. 12 (1957), 121.

[9] T. JENRICH AND A. E. BROUWER, *A 64-dimensional counterexample to Borsuk's conjecture*, Electron. J. Combin. 21 (2014), 4.29.

[10] J. KAHN AND G. KALAI, *A counterexample to Borsuk's conjecture*, Bull. Amer. Math. Soc. (N.S.) 29 (1993), 60–62.

[11] M. LASSAK, *Solution of Hadwiger's covering problem for centrally symmetric convex bodies in E^3*, J. London Math. Soc. 30 (1984), 501–511.

[12] F. W. LEVI, *Ein geometrisches überdeckungsproblem*, Arch. Math. (Basel), 5 (1954), 476–478.

[13] Y. LIAN AND S. WU, *Partition bounded sets into sets having smaller diameters*, Results Math. 76 (2021), 116.

[14] H. MARTINI, S. WU, *Characterizations of ℓ_∞^n and ℓ_1^n, and their stabilities*, J. Math. Anal. Appl. 419 (2014), 688–702.

[15] J. P. MORENO AND R. SCHNEIDER, *Diametrically complete sets in Minkowski spaces*, Isr. J. Math. 191 (2012), 701–720.

[16] J. P. MORENO AND R. SCHNEIDER, *Structure of the space of diametrically complete sets in a Minkowski space*, Discrete Comput. Geom. 48 (2012), 467–486.

[17] A. PRYMAK, *A new bound for Hadwiger's covering problem in E^3*, (2022), arXiv: 2112.10698v2.

[18] A. PRYMAK, V. SHEPELSKA, *On the Hadwiger covering problem in low dimensions*, J. Geom. 111 (2020), 42.

[19] C. A. ROGERS AND C. ZONG, *Covering convex bodies by translates of convex bodies*, Mathematika 44 (1997), 215–218.

[20] N. TOMCZAK-JAEGERMANN, *Banach-Mazur Distances and Finite-dimensional Operator Ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York (1989).

[21] L. YU AND C. ZONG, *On the blocking number and the covering number of a convex body*, Adv. Geom. 9 (2009), 13–29.

[22] C. ZONG, *A quantitative program for Hadwiger's covering conjecture*, Sci. China Math. 53 (2010), 2551–2560.

[23] C. ZONG, *Borsuk's partition conjecture*, Jpn. J. Math. 16 (2021), 185–201.
[24] C. ZONG, *Strange phenomena in convex and discrete geometry*, Springer-Verlag, New York, 1996.
[25] C. ZONG, *The kissing number, blocking number and covering number of a convex body*, Contemp. Math. 453, Amer. Math. Soc., (2008), 529–548.
[26] L. ZHANG, L. MENG AND S. WU, *Banach-Mazur distance from ℓ_p^3 to ℓ_∞^3*, (2022), arXiv: 2207.05499.

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