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ON THE IRREGULARITY OF GRAPHS BASED ON THE ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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This paper is dedicated to the memory of Professor Ali Reza Ashrafi, a mentor and a good friend, who passed away on January 9, 2023

(*Communicated by J. Jakšetić*)

Abstract. For a graph G of order n, size m and degree sequence $\mathscr{D}(G) = (d_1, d_2, \dots, d_n)$, a new measure of irregularity

$$\mathrm{I}_{AG}(G) = 1 - n^n(d_1 + r)(d_2 + r)\cdots(d_n + r)/(2m + rn)^n,$$

$r \in \mathbb{R}_{\geqslant 0}$, is introduced. It is shown that if G has maximum I_{AG}-irregularity among all connected graphs of order n and size m, then (i) $\Delta(G) = n-1$; (ii) for each $u, v \in V(G)$ with the property $d_G(u) \leqslant d_G(v)$, it holds that $N(G,u) \subseteq N[G,v]$, where $N(G,w)$ and $N[G,w]$ are the neighbourhood and the closed neighbourhood of w in G, respectively; (iii) G is a threshold graph. Further, it is proven that if a graph H has a minimum value of I_{AG}-irregularity among all irregular graphs of the same order and size, then $\Delta(H) - \delta(H) = 1$. Finally, the graphs with minimum and maximum I_{AG}-irregularity in the classes of trees, unicyclic and bicyclic graphs are characterized.

1. Introduction

Let G be a simple graph, with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The quantities n and m are called the order and size of G, respectively. The degree of vertex w in G, $d_G(w)$, is the number of vertices adjacent to w. When from the context it is clear, which graph is considered, the index of G in $d_G(w)$ will be omitted. $N(G,v)$ denotes the set of vertices adjacent to v and $N[G,v] = N(G,v) \cup \{v\}$. The maximum degree of G, denoted by $\Delta(G)$, and the minimum degree of G, denoted by $\delta(G)$, are the maximum and minimum of its vertices' degrees. $\mathscr{D}(G) = (d_G(v_1), d_G(v_2), \dots, d_G(v_n))$, with $d_G(v_1) \geqslant d_G(v_2) \geqslant \cdots \geqslant d_G(v_n)$ is called the *degree sequence* of G. The graph G is said to be *regular of degree* k, when $\mathscr{D}(G) = (k, k, \dots, k)$. Otherwise, the graph is irregular.

A *pendant* vertex is a vertex of degree one. For a connected graph G of order n and size m, the *cyclomatic number* of G is defined as $c = m - n + 1$. Graphs with

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cyclomatic numbers $c = 1$ and 2 are said to be unicyclic and bicyclic, respectively. A *clique* is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent. An *independent set* is a set of vertices in a graph, no two of which are adjacent. A connected graph G with the property that the vertex set $V(G)$ can be partitioned into two subsets A and B, such that A is independent and B is a clique, is called a *split graph*. If G has n vertices and the subset A has exactly k elements, then the graph G is denoted by $S(n,k)$. If each vertex in A is adjacent to each vertex in B, then it is called *complete split graph* and we denote it by $KS(n,k)$. A *cograph* is a graph with no induced path on four vertices, P_4. A graph is a *threshold graph* if and only if it is both a cograph and a split graph [8]. Some alternative notations of threshold graph can be found in [5].

Let $\mathbb{H}$ be a subclass of a class of all non-isomorphic graphs $\mathbb{G}$. The real function $\mathscr{I} : \mathbb{H} \longrightarrow \mathbb{R}_{\geqslant 0}$ is called an *irregularity measure* on $\mathbb{H}$, if for all $G \in \mathbb{H}$, $\mathscr{I}(G) = 0$ if and only if G is regular. We refer the interested readers to consult the recently published book of Ali et al. [4] for more information on this topic.

To the best of our knowledge, the first irregularity measure of graphs was introduced by Collatz and Sinogowitz [9]. For a simple graph G of order n and size m with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, they proved that $\lambda_1 \geqslant 2m/n$ with equality if and only if G is regular. Therefore, this property leads to an irregularity measure $\mathrm{CS}(G) = \lambda_1 - 2m/n$. Collatz and Sinogowitz proved that the star and path on n vertices have the maximum and minimum values of CS among all $n-$vertex trees, and checked that for all graphs with at most five vertices $\mathrm{CS}(G) \leqslant \sqrt{n-1} - 2 + 2/n$, with equality if and only if G is isomorphic to the $n-$vertex star graph S_n. They also conjectured that the $n-$vertex star graph S_n has maximum value of CS among all $n-$vertex graphs. Counterexamples for this conjecture were presented by Cvetković and Rowlinson [10].

Bell [7] introduced the *vertex degree variance* of the graph G, $\mathrm{Var}(G)$, as $\mathrm{Var}(G) = \frac{1}{n}\sum_{v\in V(G)}\left(d(v) - \frac{2m}{n}\right)^2$. He determined the most irregular graphs with respect to irregularity measures CS and Var for various classes of graphs.

The *edge imbalance* of an edge $e = xy$ of a graph G is defined as $|d(x) - d(y)|$. The sum of imbalances over all edges of G is called the *irregularity* of G. It was introduced by Albertson [3], who also determined the maximum irregularity of various classes of graphs and proved that the irregularity of an arbitrary $n-$vertex graph has a tight bound of $4n^3/27$. In [2] it is shown that for general graphs with n vertices the upper bound $\lfloor n/3 \rfloor \lceil 2n/3 \rceil (\lceil 2n/3 \rceil - 1)$ is sharp. Albertson also proved that the irregularity of any graph is an even positive integer.

Motivated to overcome some disadvantages of the (Albertson) irregularity, Abdo et al. [1] introduced the *total irregularity* of G as $irr_t(G) = \sum_{\{u,v\}\subseteq V(G)} |d_G(u) - d_G(v)|$. They determined all graphs with maximal total irregularity and showed that among all $n-$vertex trees the star graph S_n has the maximal total irregularity. A comparison of irregularity and total irregularity was studied in [11]. In [6] it was proven that if G is an irregular $n-$vertex graph, then $irr_t(G) \geqslant \begin{cases} n-1, & \text{if } \ 2 \nmid n \\ 2n-4, & \text{if } \ 2 \mid n \end{cases}$. There, also all graphs for which the equality is satisfied were determined. In [16] tight upper and lower bounds on the total irregularity of an $n-$edge connected graph with a cyclomatic number c and

p pendant vertices were established.

Estrada [12] introduced a new measure of irregularity for graphs without isolated vertices, $\rho(G) = (\sum_{uv\in E(G)}(d(u)^{-1/2} - d(v)^{-1/2})^2/(n-2\sqrt{n-1})$, which in a context of networks was named as a *degree heterogeneity*. He proved that $0 \leqslant \rho(G) \leqslant 1$ with equality on the left side if and only if G is regular. The equality on the right side occurs if and only if G is a star graph. A *scale-free network* is a network whose degree distribution follows a power law. Many real-world networks have been reported to be scale-free. The irregularity of a scale-free network is very close to the irregularity of the star graph, considered by Estrada as a graph, which is most appropriate to be the graph with maximal irregularity. Estrada analyzed several real-world networks to support his expectation and intuition. In [13], he studied 17 real-world networks representing food webs in a variety of ecological environments. A graphical method for representing every graph in a degree heterogeneity space was introduced. Estrada [14] proved a few analytical results showing the relation of degree heterogeneity index to the number of pendant nodes, and to some irregularity indices proposed in the literature.

Nikiforov [18] presented the *degree deviation of graphs*, which is another irregularity measure of graphs based on vertex degrees. This irregularity measure is defined as $S(G) = \sum_{v\in V(G)}\left|d(v) - \frac{2m}{n}\right|$. Nikiforov proved that $\mathrm{Var}(G)/(2\sqrt{2m}) \leqslant \mathrm{CS}(G) \leqslant \sqrt{S(G)}$, $S(G)^2/(2n^2\sqrt{2m}) \leqslant \mathrm{CS}(G) \leqslant \sqrt[4]{n^2\mathrm{Var}(G)}$, $S(G)^2/n^2 \leqslant \mathrm{Var}(G) \leqslant S(G)$, $\lambda_n(G) + \lambda_n(G^c) \leqslant -1 - S(G)^2/2n^3$ and $\lambda_k(G) + \lambda_{n-k+2}(G^c) \leqslant -1 - 2\sqrt{2S(G)}$, where $2 \leqslant k \leqslant n$ and G^c denotes the complement of G. Two of the present authors [15] proved that if

$$k = \begin{cases} \dfrac{n}{3} & \text{if} \quad 3 \mid n \\ \dfrac{n-1}{3} & \text{if} \quad 3 \mid n-1 \\ \dfrac{n-2}{3} \quad \dfrac{n+1}{3} & \text{if} \quad 3 \mid n-2 \end{cases},$$

then $\mathrm{KS}(n,k)$ has the maximum degree deviation among all $n-$ vertex graphs.

The aim of this paper is to propose a new irregularity measure, which is also an invariant regarding a given degree sequence. It is based on the arithmetic-geometric mean inequality. By modifying the value of the parameter r, different graphs with maximal irregularity can be obtained. We conjecture that the star graph will be one among them. Also, we expect that we can tune r such that the discrimination ability of this measure will increase. The justification of this will come when the stronger versions of both conjectures at the end of the paper will be resolved (for any possible r). Also, additional theoretical and practical support will be needed. This is however beyond the scope of this work and could be considered in the future. We introduce this new irregularity measure in the next section, where also some of its properties will be shown. Those properties lead to characterizations of extremal graphs for a few classes of graphs, which will be done in the last section.

2. I_{AG}-irregularity

Let G be a simple graph of order n, size m and with a degree sequence $\mathscr{D}(G) = (d_1, d_2, \ldots, d_n)$. By the classical result of Euler, $\sum_{i=1}^{n} d_i = 2m$. Suppose that r is a non-negative real number. By the arithmetic-geometric mean inequality,

$$\frac{(d_1+r)+(d_2+r)+\cdots+(d_n+r)}{n} \geqslant \sqrt[n]{(d_1+r)(d_2+r)\cdots(d_n+r)}$$

with inequality if and only if $d_1 = d_2 = \cdots = d_n$. Thus,

$$\frac{n^n(d_1+r)(d_2+r)\cdots(d_n+r)}{(2m+rn)^n} \leqslant 1,$$

with equality if and only if G is regular. From the last inequality, we can derive the following quantity:

$$\mathrm{I}_{AG}^{r}(G) = 1 - \frac{n^n(d_1+r)(d_2+r)\cdots(d_n+r)}{(2m+rn)^n},$$

where, $\mathrm{I}_{AG}^{r}(G) = 0$ if and only if G is regular, otherwise $0 < \mathrm{I}_{AG}^{r}(G) < 1$. Therefore, I_{AG}^{r} is a measure of irregularity. If we are not considering a particular value of r, then r will be omitted as a superscript in the notation.

2.1. Some properties of the graphs with extremal I_{AG}-irregularity

THEOREM 1. *Let G be a graph with maximal I_{AG}-irregularity among all connected graphs of order n and size m. Then, $\Delta(G) = n-1$.*

Proof. Suppose $\Delta(G) \leqslant n-2$, and $d(v) = \Delta(G) = \Delta$, for some $v \in V(G)$. Let $N(G,v) = \{x_1, x_2, \ldots, x_\Delta\}$. Since G is connected there exists i, $1 \leqslant i \leqslant \Delta$, such that $A = N(G,x_i) \backslash N(G,v) \neq \emptyset$. Let $a \in A$ be a vertex adjacent to x_i. Consider the graph G_1 obtained by deleting the edge $x_i a$ and adding the edge va, i.e., $G_1 = G - \{x_i a | a \in A\} + \{va | a \in A\}$. Then G_1 is a graph of order n and size m. It holds that

$$\begin{aligned} \frac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_1)} &= \frac{\left(n^n \prod_{v \in V(G)} (d_G(v)+r)\right)/(2m+rn)^n}{\left(n^n \prod_{v \in V(G_1)} (d_{G_1}(v)+r)\right)/(2m+rn)^n} \\ &= \frac{(\Delta+r)(d_G(x_i)+r)}{(\Delta+|A|+r)(d_G(x_i)-|A|+r)}. \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} (\Delta+r)(d_G(x_i)+r) - (\Delta+|A|+r)(d_G(x_i)-|A|+r) &= |A|(|A|+\Delta-d_G(x_i)) \\ &\geqslant |A|(|A|+\Delta-\Delta) > 0. \end{aligned}$$

Applying (1) it can be deduced that $\dfrac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_1)} > 1$. Thus, $1-\mathrm{I}_{AG}(G) > 1-\mathrm{I}_{AG}(G_1)$ and so $\mathrm{I}_{AG}(G) < \mathrm{I}_{AG}(G_1)$, a contradiction. Therefore, $\Delta(G) = n-1$. $\square$

Next, we characterize the difference between the minimum and maximum degree of the graphs with minimal I_{AG}-irregularity.

THEOREM 2. *Let G be a graph with minimal I_{AG}-irregularity among all connected irregular graphs of order n and size m. Then, $\Delta(G)-\delta(G)=1$.*

Proof. Assume that $\Delta(G)-\delta(G) \geqslant 2$. Choose two vertices u and v of G such that $d(u)=\Delta(G)=\Delta$ and $d(v)=\delta(G)=\delta$. Under these conditions there exists a vertex $w \in N(G,u) \setminus N(G,v)$. Let G_2 be the graph obtained from G by deleting the edge wu and adding the edge wv, i.e., $G_2 = G - wu + wv$. Due to the condition $\Delta(G)-\delta(G) \geqslant 2$, we can choose the vertex w in such a way that G_2 is connected. Note that G_2 has order n and size m. Further, we have

$$\frac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_2)} = \frac{(\Delta+r)(\delta+r)}{(\Delta-1+r)(\delta+1+r)}. \tag{2}$$

On the other hand, $(\Delta+r)(\delta+r)-(\Delta-1+r)(\delta+1+r)=\delta+1-\Delta<0$. Hence, by (2), $\dfrac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_2)}<1$, which implies that $\mathrm{I}_{AG}(G_2)<\mathrm{I}_{AG}(G)$, which is a contradiction by minimality of G. We now apply our assumption that G is irregular to deduce that $\Delta(G)-\delta(G)=1$. $\square$

By a similar argument, one can see that the previous theorem is valid for non-connected graphs.

The next result charaterizes the neighbourhood of the vertices of the graphs with maximal I_{AG}-irregularity.

THEOREM 3. *Let G be a graph with the maximum value of the I_{AG}-irregularity among all connected graphs of order n and size m. If $u,v \in V(G)$ and $d_G(u) \leqslant d_G(v)$, then $N(G,u) \subseteq N[G,v]$.*

Proof. Let u and v be two vertices of G such that $d_G(u) \leqslant d_G(v)$ and $N(G,u) \setminus N[G,v] \neq \emptyset$. Choose the vertex $x \in N(G,u) \setminus N[G,v] \neq \emptyset$. Let G_3 be the graph obtained by deleting the edge xu and adding the edge xv, i.e., $G_3 = G - xu + xv$. By definition of G_3,

$$\frac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_3)} = \frac{(d_G(u)+r)(d_G(v)+r)}{(d_G(u)-1+r)(d_G(v)+1+r)}. \tag{3}$$

On the other hand,

$$(d_G(u)+r)(d_G(v)+r)-(d_G(u)-1+r)(d_G(v)+1+r)=d_G(v)+1-d_G(u)>0.$$

Hence by (3), $\dfrac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_3)}>1$ and so $\mathrm{I}_{AG}(G)<\mathrm{I}_{AG}(G_3)$, contradicts by maximality of G. Therefore, for all vertices u and v of G with condition $d_G(u) \leqslant d_G(v)$, we have $N(G,u) \subseteq N[G,v]$, as desired. $\square$

By a similar argument as in the previous theorem, it can be easily seen that Theorem 3 is also valid for non-connected graphs.

Finally, we are presenting the closest characterization of graphs of fixed order and size with maximal I_{AG}-irregularity so far.

THEOREM 4. *If the graph G has maximal I_{AG}-irregularity among all connected graphs of order n and size m, then G is a threshold graph.*

Proof. Recall that a graph is a threshold graph if and only if it is both a split graph and a cograph.

Firstly, we show that G is a split graph. Suppose S is the largest subset of $V(G)$ such that $G[S]$ is a complete graph and $\min\{d_G(x)|x \in S\} \geqslant \max\{d_G(x)|x \in V(G)\setminus S\}$. We claim that $V(G)\setminus S$ is an independent set. If not, the subgraph $G[V(G)\setminus S]$ has at least one edge. Suppose $v \in V(G)\setminus S$ such that $N(G,v)\cap(V(G)\setminus S) \neq \emptyset$, and $d_G(v) \geqslant \max\{d_G(x)|x \in V(G)\setminus S \,\mathrm{and}\, N(G,x)\cap(V(G)\setminus S) \neq \emptyset\}$. By Theorem 3 and maximality of G, we can see that for each vertex $u \in V(G)\setminus S$ which satisfies the condition $N(G,u)\cap(V(G)\setminus S) = \emptyset$ we have $d_G(u) < d_G(v)$. Note that in the other case, $N(G,v) \subseteq N(G,u)$ which contradicts $N(G,u)\cap(V(G)\setminus S) = \emptyset$. Thus for every $y \in V(G)\setminus S$, we have $d_G(y) \leqslant d_G(v)$. By definition of S, $S\setminus N(G,v) \neq \emptyset$. Suppose $a \in S\setminus N(G,v)$ and $b \in N(G,v)\cap(V(G)\setminus S)$. Define $G_4 = G - vb + va$. By Theorem 1 and this assumption that G has maximum value of I_{AG}, we can see that $ba \in E(G)$. This proves that G_4 is a connected graph of order n and size m and by definition of G_4,

$$\frac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_4)} = \frac{(d_G(a)+r)(d_G(b)+r)}{(d_G(a)+1+r)(d_G(b)-1+r)}. \tag{4}$$

On the other hand,

$$(d_G(a)+r)(d_G(b)+r)-(d_G(a)+1+r)(d_G(b)-1+r) = d_G(a)+1-d_G(b) > 0.$$

Thus by (4), $\dfrac{1-\mathrm{I}_{AG}(G)}{1-\mathrm{I}_{AG}(G_4)} > 1$ and so $\mathrm{I}_{AG}(G) < \mathrm{I}_{AG}(G_4)$, which contradicts by maximality of G. This proves that $V(G)\setminus S$ is independent and G is a split graph.

Next, we show that G does not contain induced paths on 4 vertices, which will imply that G is also a cograph. Assume that there is a path P on 4 vertices in G. Observe that P must contain at least two vertices from the clique, otherwise, it cannot exist. On the other hand, P cannot contain three or four vertices from the clique, because those vertices will induce at least one triangle. Thus, P must contain two vertices, u_1 and u_2, from the clique and two vertices, v_1 and v_2, from the independent set. The vertices u_1 and u_2 must be adjacent in P, otherwise, we have a triangle, due to the edge u_1u_2 (u_1 and u_2 are adjacent since they belong to the clique). So the remaining configuration of P is when its end-vertices are v_1 and v_2. We may assume that $P = v_1u_1u_2v_2$ and that $d(v_1) \geqslant d(v_2)$. Then, by Theorem 3, v_1 is also adjacent to u_2, and we have again triangle. Thus, we can conclude that G does not have induced paths on 4 vertices, i.e., G is also a cograph. $\square$

From the above results, the characterization of the graphs with minimal and maximal I_{AG}-irregularity in the classes of trees, unicyclic and bicyclic graphs follows, and it will be presented in the next section. In addition, we will characterize the c-cyclic graphs with minimal I_{AG}-irregularity.

3. Trees and c-cyclic graphs with extremal I_{AG}-irregularity

The aim of this section is to study the extremal c-cyclic graphs under I_{AG}-irregularity.

COROLLARY 1. *If T is an $n-$vertex tree, then*

$$1-n^n(r+2)^{n-2}(1+r)^2(rn+2n-2)^{-n} \leqslant \mathrm{I}_{AG}(T) \tag{5}$$

$$\leqslant 1-n^n(1+r)^{n-1}(n-1+r)(rn+2n-2)^{-n}. \tag{6}$$

The equality in (5) is satisfied if and only if $T \cong P_n$, and the equality in (6) is satisfied if and only if $T \cong S_n$.

Proof. Let T be a tree of order n. It is easy to see that $\Delta(T) = n-1$ if and only if $T \cong S_n$. Moreover, $\Delta(T) = 2$ if and only if $T \cong P_n$. Apply Theorem 2 to deduce that the minimum value of I_{AG} among all trees of order n are attained if and only if T is a path. The star graph S_n is the only threshold graph of order n and size $n-1$, and we know from Theorem 4 that threshold graphs maximize the I_{AG}. On the other hand,

$$\mathrm{I}_{AG}(S_n) = 1-n^n(r+2)^{n-2}(1+r)^2(rn+2n-2)^{-n},$$
$$\mathrm{I}_{AG}(P_n) = 1-n^n(1+r)^{n-1}(n-1+r)(rn+2n-2)^{-n},$$

proving the result. □

Suppose S_n^e is a unicyclic graph obtained from the star S_n by adding one edge e. Note that all such unicyclic graphs are isomorphic and so the graph structure is independent of the choice of the edge e.

COROLLARY 2. *Let U be a unicyclic graph of order $n \geqslant 4$. Then,*

$$0 \leqslant \mathrm{I}_{AG}(U) \leqslant 1-n^n(1+r)^{n-3}(r+2)^2(n-1+r)((r+2)n)^{-n}.$$

The equality on the left is satisfied if and only if $U \cong C_n$, and the equality on the right is attained if and only if $U \cong S_n^e$.

Proof. Since $\sum_{v\in V(U)} d(v) = 2n$, if U has a vertex of degree $n-1$, then it must have two vertices of degree 2 and $n-3$ vertices of degree 1. This shows that $\Delta(U) = n-1$ if and only if $U \cong S_n^e$. Since $U \cong S_n^e$ is the unique threshold graph amnong uncyclic connected graphs, from Theorem 4 it follows that it maximizes the I_{AG}-irregularity. It can be easily calculated that $\mathrm{I}_{AG}(S_n^e) = 1-n^n(1+r)^{n-3}(r+2)^2(n-1+r)((r+2)n)^{-n}$. On the other hand, the cycle C_n is the unique regular unicyclic graph, and $\mathrm{I}_{AG}(C_n) = 0$. □

Denote by S_n^{2e} the graph constructed from S_n by adding two edges, so that its degree sequence is

$$\mathscr{D}(S_n^{2e}) = (n-1,3,2,2,\overbrace{1,\ldots,1}^{n-4}).$$

We also assume that $\mathscr{B}_n(2,3)$ is the family of all connected bicyclic graphs with degree sequence

$$(3,3,\overbrace{2,\ldots,2}^{n-2}).$$

COROLLARY 3. *Let B be a bicyclic $n-$vertex graph. Then,*

$$\mathrm{I}_{AG}(B) \leqslant 1 - n^n (1+r)^{n-4} (r+2)^2 (3+r)(n-1+r)(rn+2n+2)^{-n}$$

with equality if and only if $B \cong S_n^{2e}$. Moreover,

$$\mathrm{I}_{AG}(B) \geqslant 1 - n^n (r+2)^{n-2} (3+r)^2 (rn+2n+2)^{-n}$$

with equality if and only if $B \in \mathscr{B}_n(2,3)$.

Proof. It can be seen that if $\Delta(B) = n-1$ and for all vertices $u,v \in V(B)$ with condition $d_B(u) \leqslant d_B(v)$, we have $N(B,u) \subseteq N[B,v]$, then $B \cong S_n^{2e}$. Furthermore, there is no regular bicyclic graph and $\Delta(B) - \delta(B) = 1$ if and only if $B \in \mathscr{B}_n(2,3)$. After simplifying the calculations, we can obtain that

$$\mathrm{I}_{AG}(S_n^{2e}) = 1 - n^n (1+r)^{n-4} (r+2)^2 (3+r)(n-1+r)(rn+2n+2)^{-n},$$

and if $B \in \mathscr{B}_n(2,3)$, then

$$\mathrm{I}_{AG}(B) = 1 - n^n (r+2)^{n-2} (3+r)^2 (rn+2n+2)^{-n}. \quad \square$$

Connected graphs of fixed order and size share the same cyclomatic number, and therefore, the problem of determining extremal graphs with given order and size is equivalent to the problem of determining extremal graphs with given cyclomatic number c and order (or size).

Let $\mathscr{C}_n(\Delta-1,\Delta)$ be the family of all connected c-cyclic graphs with degree sequence

$$(\overbrace{\Delta,\ldots,\Delta}^{n_\Delta},\overbrace{\Delta-1,\ldots,\Delta-1}^{n-n_\Delta}).$$

COROLLARY 4. *Among all c-cyclic graphs, $c \geqslant 1$, of order n the uniquely determined graph with minimal I_{AG}-irregularity is a graph from $\mathscr{C}_n(\Delta-1,\Delta)$.*

Proof. By Theorem 2, for an irregular graph G with minimal I_{AG} it holds that $\Delta(G)-\delta(G)=1$, and thus, $G\in\mathscr{C}_n(\Delta-1,\Delta)$. Recall, that if $\Delta(G)=\delta(G)$, G is regular, and $I_{AG}=0$. For given n and c the number of edges m is also determined ($m=n+c-1$). Together with $n_\Delta\Delta+(n-n_\Delta)(\Delta-1)=2m$, it follows that

$$\Delta=\left\lceil\frac{3n+2c-n_\Delta-2}{n}\right\rceil,$$

and

$$n_\Delta=3n+2c-n\Delta-2.$$

Hence, for given c and n the graphs in $\mathscr{C}_n(\Delta-1,\Delta)$ are uniquely determined. □

We end this paper with two conjectures.

CONJECTURE 1. Let G be a graph with maximal I^0_{AG}-irregularity among all connected graphs of order n. Then, $G\cong S_n$.

We believe that the answer to this conjecture is affirmative. It is known that graphs with maximal Collatz-Sinogowitz irregularity are also threshold graphs, but, different from star graphs [10]. A further comparison study could help to find how consistent the I_{AG}-irregularity and Collatz-Sinogowitz's irregularity are.

CONJECTURE 2. Among all c-cyclic graphs, $c\geqslant 1$, of order n, the uniquely determined graph with maximal I^0_{AG}- irregularity is a threshold graph whose independent set has at most one vertex of degree larger than one.

Conflict of interest. The authors declare that there is no conflict of interest regarding the publication of this paper.

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