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AN INTERTWINED CAUCHY—SCHWARZ–TYPE INEQUALITY BASED ON A LAGRANGE–TYPE IDENTITY

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Abstract. Based on an apparently new Lagrange-type identity, a Cauchy–Schwarz-type inequality is proved. The mentioned identity is obtained by using certain “macro” variables; it is hoped that such a method can be used to prove or produce other identities and inequalities.

1. Result

Let a_1,a_2,a_3,b_1,b_2,b_3 be any real numbers. The well-known Lagrange identity (see e.g. [3])

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)=(a_1b_1+a_2b_2+a_3b_3)^2+\sum_{1\leqslant i<j\leqslant 3}(a_ib_j-a_jb_i)^2$$

immediately yields the Cauchy–Schwarz inequality (see e.g. [7])

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^3)\geqslant(a_1b_1+a_2b_2+a_3b_3)^2.$$

In this note we shall prove the following, apparently new Cauchy–Schwarz-type inequality, based on an apparently new Lagrange-type identity.

PROPOSITION 1.

$$\begin{aligned}(a_1^2+&b_2^2+b_3^2)(a_2^2+b_3^2+b_1^2)(a_3^2+b_1^2+b_2^2)\\ &\geqslant(a_1b_1+a_2b_2+a_3b_3)^2\,(b_1^2+b_2^2+b_3^2)\\ &\quad+\tfrac{1}{2}\left[b_1^2(a_2b_3-a_3b_2)^2+b_2^2(a_3b_1-a_1b_3)^2+b_3^2(a_1b_2-a_2b_1)^2\right].\end{aligned}\tag{1}$$

Note that – in distinction with the left-hand side of the Cauchy–Schwarz inequality, with the a_i's and b_i's separated in the two factors there – the a_i's and b_i's are intertwined in the three factors on the left-hand side of inequality (1).

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2. Proof

Proof of Proposition 1. Let $\tilde{d}$ denote the difference between the left- and right-hand sides of inequality (1), which can then be rewritten as $\tilde{d} \geqslant 0$. Note that $\tilde{d}$ is a polynomial (of degree 6 in 6 variables). Therefore, in principle, inequality (1) can be verified completely algorithmically, using one of the suitable known tools. One of these tools is the quantifier elimination by cylindrical algebraic decomposition (see e.g. [2]), based on the Tarski theory [8]; for instance, in Mathematica this theory is implemented via `Reduce[]` and related commands. Alternatively, one may try some of the various Positivstellensätze of real algebraic geometry (see e.g. [5, 1, 4]), which can provide a so-called certificate of positivity to a polynomial that is indeed positive on a set defined by a system of polynomial inequalities (over $\mathbb{R}$). However, our polynomial $\tilde{d}$ turns out to be too complicated for these tools to succeed without substantial human intervention.

To prove Proposition 1, many rounds of rewriting of $\tilde{d}$ were done – manually, each round verified with Mathematica. Complete details of this multi-step rewriting can be seen in the 6-page Mathematica notebook 1stRewriting.nb and its pdf image 1stRewriting.pdf, found in the zip file MathematicaVerfication.zip, which can be downloaded at https://works.bepress.com/iosif-pinelis/22/. After that, to verify inequality (1) in the rewritten form, the mentioned Mathematica command `Reduce[]` took about 23 min, which is a very long time for a contemporary computer (with a 3.5 GHz CPU). One may therefore surmise that a description of the execution of this command would possibly take hundreds or thousands of pages when transcribed into regular mathematical writing.

Fortunately, a few more rounds of rewriting, presented in the Mathematica notebook 2ndRewriting.nb and its pdf image 2ndRewriting.pdf in the mentioned zip file MathematicaVerfication.zip, yield a key identity, which allows one to prove inequality (1) rather quickly and easily.

To state this identity, note first that, without loss of generality (wlog), all the a_i's and b_i's are nonzero. For $i = 1, 2, 3$, introduce the new, "macro" variables

$$x_i := a_1 a_2 a_3 / a_i, \quad y_i := b_1 b_2 b_3 / b_i, \quad p_i := (x_i - y_i) y_i, \quad z_i := y_i^2 \geqslant 0,$$

and then

$$c_1 := p_2^2 + p_2 p_3 + p_3^2, \quad c_2 := p_1^2 + p_1 p_3 + p_3^2, \quad c_3 := p_2^2 + p_2 p_1 + p_1^2. \tag{2}$$

Note that $x_1 x_2 x_3 = (a_1 a_2 a_3)^2 > 0$, $y_1 y_2 y_3 = (b_1 b_2 b_3)^2 > 0$, $c_1 \geqslant 0$, $c_2 \geqslant 0$, and $c_3 \geqslant 0$. Moreover,

$$(p_1 + z_1)(p_2 + z_2)(p_3 + z_3) \geqslant 0. \tag{3}$$

The mentioned crucial identity is

$$y_1 y_2 y_3 \tilde{d} = d := p_1 p_2 p_3 + c_1 z_1 + c_2 z_2 + c_3 z_3. \tag{4}$$

As it is clear now, this identity was difficult to obtain. However, it is quite straightforward (but tedious) to verify it. Such a verification is best done using one of a number of available computer algebra programs. E.g., it takes Mathematica only about 0.15 sec to check identity (4); for details, see the Mathematica notebook checkingTheIdentity.nb and/or its pdf image checkingTheIdentity.pdf in the same zip file, MathematicaVerfication.zip.

Since $y_1y_2y_3 > 0$, $\tilde{d}$ equals d in sign. So, it suffices to show that $d \geqslant 0$ – for any real p_i's, the c_i's as in (2), and any nonnegative z_i's satisfying (3).

Note here that without loss of generality $p_1p_2p_3 < 0$ – otherwise, the desired inequality $d \geqslant 0$ immediately follows because the c_i's and z_i's are nonnegative. So, we may assume that the p_i's are are all nonzero and hence the c_i's are all strictly positive.

Take any nonzero real p_i's and any nonnegative z_i's such that (3) holds. Let us then fix those z_1 and z_2, and let z_3 be decreasing as long as z_3 remains nonnegative and (3) holds; clearly, this process can stop only when the value of z_3 becomes either 0 or $-p_3$, and in the latter case we must have $-p_3 > 0$. Moreover, since $c_i > 0$ for all i, the value of d will not increase after this process is complete.

We can then proceed similarly by decreasing z_2 (instead of z_3), and then by decreasing z_1.

Let now (z_1,z_2,z_3) be any minimizer of d, subject to the stated conditions on the z_i's. Then it follows from the above reasoning that $z_i \in \{0,-p_i\}$ for each $i = 1,2,3$; moreover, if at that $z_i = -p_i$ for some i, then we must have $-p_i > 0$. So, by the symmetry with respect to permutations of the indices, it is enough to consider the following four cases:

(i) $z_1 = -p_1 > 0$, $z_2 = -p_2 > 0$, $z_3 = -p_3 > 0$;

(ii) $z_1 = -p_1 > 0$, $z_2 = -p_2 > 0$, $z_3 = 0$;

(iii) $z_1 = -p_1 > 0$, $z_2 = 0$, $z_3 = 0$;

(iv) $z_1 = 0$, $z_2 = 0$, $z_3 = 0$.

In case (i), $\min_{z_1,z_2,z_3} d = -(p_1+p_2)(p_1+p_3)(p_2+p_3) > 0$.

In case (ii), $\min_{z_1,z_2,z_3} d = -p_1p_2(p_1+p_2) - p_1p_2p_3 + (-p_1-p_2)p_3^2$, which is a convex quadratic polynomial in p_3, with discriminant $-p_1p_2(4p_1^2+7p_1p_2+4p_2^2) < 0$, whence again $\min_{z_1,z_2,z_3} d > 0$.

In case (iii), $\min_{z_1,z_2,z_3} d = -p_1(p_2^2+p_3^2) > 0$.

In case (iv), condition (3) becomes $p_1p_2p_3 \geqslant 0$, which contradicts the assumption $p_1p_2p_3 < 0$.

Thus, $\min_{z_1,z_2,z_3} d \geqslant 0$ in all feasible cases, and (1) is proved. □

3. Discussion

Note that each of the factors on the left-hand side of inequality (1) is the sum of three terms. It would be interesting (but possibly very difficult) to extend this inequality to an "intertwined" one similarly involving sums of more than three terms.

As was noted in the proof of Proposition 1, wlog all the b_i's are nonzero. Introducing then $k_i := a_i/b_i$, we can rewrite inequality (1) as follows:

$$\begin{aligned}(k_1^2b_1^2+b_2^2+b_3^2)&(k_2^2b_2^2+b_3^2+b_1^2)(k_3^2b_3^2+b_1^2+b_2^2)\\ &\geqslant (k_1b_1^2+k_2b_2^2+k_3b_3^2)^2\,(b_1^2+b_2^2+b_3^2)\\ &\quad+\tfrac12\,b_1^2b_2^2b_3^2\left[(k_1-k_2)^2+(k_2-k_3)^2+(k_1-k_3)^2\right]\end{aligned}\tag{5}$$

for all real b_i's and k_i's.

REMARK 1. The constant factor $\frac12$ in (5) and hence in (1) is optimal – that is, the greatest possible one. Indeed, if the factor $\frac12$ in (5) is replaced by any real constant $C>\frac12$, then for $k_1=k_2=0$ the difference between the left- and right-hand sides of inequality (5) will be $(1-2C)b_1^2b_2^2b_3^2k_3^2+\left(b_1^2+b_2^2\right)\left(b_1^2+b_3^2\right)\left(b_2^2+b_3^2\right)$, which will go to $-\infty$ as $k_3\to\infty$ if $b_1b_2b_3\neq 0$.

One may also note that (1) immediately implies the following simpler but weaker "intertwined" inequality:

$$(a_1^2+b_2^2+b_3^2)(a_2^2+b_3^2+b_1^2)(a_3^2+b_1^2+b_2^2)\geqslant (a_1b_1+a_2b_2+a_3b_3)^2\,(b_1^2+b_2^2+b_3^2).$$

Inequality (1) was conjectured on the MathOverflow site [6] and proved there by the author of the present note.

4. Conclusion

Looking back at the cases (i)–(iv) in the proof of Proposition 1 and at Remark 1, we notice a rather large number of entire varieties of cases of minima and near-minima of d or $\tilde d$. This may at least partially explain the difficulties with using standard methods, such as cylindrical algebraic decomposition and certificates of positivity provided by Positivstellensätze, mentioned in the beginning of the proof of Proposition 1.

The "macro"-variables method, demonstrated in this note, may turn out to be useful in other settings where the other methods are not feasible. It would be of great interest if computers could be taught this method, as they have been taught the mentioned standard methods.

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