Mathematical
Inequalities
& Applications
Volume 26, Number 3 (2023), 545–565
doi:10.7153/mia-2023-26-35

CONVEXITY PROPERTIES OF AREA INTEGRAL MEANS OVER THE ANNULI

YUCONG DUAN AND CHUNJIE WANG*

(*Communicated by I. Perić*)

Abstract. For positive numbers t,p,q,c and an analytic function $f(z)$ in an annulus $R_1 < |z| < R_2$, let $M_{t,\varphi,q,c}(f,r)$ be the area integral means of f with respect to the weighted area measure $\varphi'(|z|^q)|z|^{q-2}\,dA(z)$, where $R_1 \leqslant c < R_2$. We show that $M_{t,\varphi,q,c}(f,r)^{\frac{1}{p}}$ is a convex function of r if f and φ satisfy certain conditions. The convexities of $\log M_{t,\varphi,q,c}(f,r)$ in r and $\log r$ can be obtained as special cases.

1. Introduction

Let $0 \leqslant R_1 < R_2 \leqslant \infty$, and let $\mathscr{H}$ denote the space of all functions $f(z)$ analytic in $R_1 < |z| < R_2$ and continuous on $R_1 \leqslant |z| < R_2$. For any $f \in \mathscr{H}$ and $0 < t < \infty$, the classical integral means of f are defined by

$$M_t(f,r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^t\,d\theta, \qquad R_1 \leqslant r < R_2.$$

These integral means play an important role in classical analysis, especially in the theory of Hardy spaces. The well-known Hardy convexity theorem asserts that $M_t(f,r)$, as a function of r, is logarithmically convex. See [4] for example. Logarithmic convexity here means that the function $r \mapsto \log M_t(f,r)$ is convex in $\log r$.

Let q be a positive number, and let φ be a real-valued function defined on (R_1^q, R_2^q) with positive derivative φ'. We consider the measure

$$dA_\varphi(z) = \varphi'(|z|^q)|z|^{q-2}\,dA(z), \tag{1}$$

where dA is the Euclidean area measure on $R_1 \leqslant |z| < R_2$. Note that when $q = 2$, $R_1 = 0$, $R_2 = 1$ and $\varphi'(x) = (1-x)^\alpha$,

$$dA_\varphi(z) = (1-|z|^2)^\alpha\,dA(z),$$

which is frequently used in the theory of Bergman spaces [2, 3]; When $q = 2$, $R_1 = 0$, $R_2 = \infty$ and $\varphi'(x) = e^{-\alpha x}$,

$$dA_\varphi(z) = \mathrm{e}^{-\alpha|z|^2}\,dA(z),$$

Mathematics subject classification (2020): 30H10, 30H20.
Keywords and phrases: Convex, logarithmically convex, area integral means, Bergman space.
* Corresponding author.

which is frequently used in the theory of Fock spaces [17]. Let $A(a,b)$ be the annulus $\{z\in\mathbb{C}: a\leqslant|z|\leqslant b\}$ if $a<b$ or the annulus $\{z\in\mathbb{C}: b\leqslant|z|\leqslant a\}$ if $a>b$. For $f\in\mathscr{H}$, $0<t<\infty$, $R_1\leqslant c<R_2$, we consider the area integral means

$$M_{t,\varphi,q,c}(f,r)=\frac{\displaystyle\int_{A(c,r)}|f(z)|^t\varphi'(|z|^q)|z|^{q-2}\,dA(z)}{\displaystyle\int_{A(c,r)}\varphi'(|z|^q)|z|^{q-2}\,dA(z)},\quad R_1\leqslant r<R_2,$$

where the functions f and φ' are such that the integrals exist.

Note that if $c=R_1=0$ and f is analytic in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, the area integral means were firstly studied by Xiao and Zhu [16]. It was shown in [12, 14] that, if f is analytic in $\mathbb{D}$ and $dA_\varphi(z)=(1-|z|^2)^\alpha\,dA(z)$, just like the classical integral means, $M_{t,\varphi,2,0}(f,r)$ is also logarithmically convex on $(0,1)$ when $-2\leqslant\alpha\leqslant 0$. Furthermore, if $t=2$, then $M_{2,\varphi,2,0}(f,r)$ is logarithmically convex on $(0,1)$ when $-3\leqslant\alpha\leqslant 0$, and this range for α is best possible. Cui, Wang and Zhu [1], Wang and Yang [13] discussed the logarithmic convexity of area integral means over the annuli. If f is an entire function and $dA_\varphi(z)=\mathrm{e}^{-\alpha|z|^2}\,dA(z)$, we get the Gaussian integral means $G_{t,\varphi,q,0}(f,r)$, which were studied by Wang, Xiao [10, 11], Li, Liu [6] and Li, Wang [7]. See [15] for other work in the area.

In [9] Shniad proved that, if f is analytic in $|z|<R$, $r\mapsto(M_4(f,r))^{\frac{1}{4}}$ is convex. Professor Zhu asked whether the result remains true for area integral means of analytic functions. Recently Hu and Wang [5] study the problem for functions analytic in the disk $|z|<R$ and show that $(M_{p,\varphi,2,0}(f,r))^{\frac{1}{p}}$ is a convex function of r if f and φ satisfy certain conditions.

In this paper we generalize the results of [5] to functions analytic on an annulus. Following Hu and Wang [5], we consider Zhu's problem in a more general setting.

Let $q>0$, and let Φ denote the set of real-valued functions φ defined on $[R_1^q,R_2^q)$ which satisfies the following conditions:

(i) $\varphi(x_0)=0$, where $x_0\in[R_1^q,R_2^q)$;

(ii) φ' is positive on (R_1^q,R_2^q);

(iii) $\varphi',\varphi'',\varphi'''$ are all continuous on (R_1^q,R_2^q).

Note that x_0 is the unique zero of φ on $[R_1^q,R_2^q)$ due to the condition (ii) above. We also let $\mathscr{M}$ denote the set of positive functions M defined on $[R_1^q,R_2^q)$ with continuous second derivative M''.

For $\varphi\in\Phi$ and $M\in\mathscr{M}$, define

$$H(x)=\frac{\displaystyle\int_{x_0}^x M(t)\varphi'(t)\,dt}{\displaystyle\int_{x_0}^x\varphi'(t)\,dt},\quad R_1^q\leqslant x<R_2^q. \tag{2}$$

For $p>0$, we want to find conditions under which the function $H(x^q)^{\frac{1}{p}}$ is convex.

Throughout the paper we always assume $0 \leqslant R_1 < R_2 \leqslant \infty$ whenever R_1 and R_2 appear. We use the symbol $=:$ whenever a new notation is being introduced. We will use the notation $A \sim B$ to mean that A and B have the same sign.

2. Preliminaries

In this section we collect several preliminary results that will be needed for the proof of our main results.

For any twice differentiable function f on $(a,b) \subset (0,\infty)$, we define

$$d_f(x) = x\frac{f'(x)}{f(x)}$$

and

$$D_f(x) = \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} - x\left(\frac{f'(x)}{f(x)}\right)^2. \tag{3}$$

It is easy to check that

$$d_{fg}(x) = d_f(x) + d_g(x), \quad d_{f/g}(x) = d_f(x) - d_g(x), \tag{4}$$

$$(d_f(x))' = D_f(x), \tag{5}$$

and

$$xD_f(x) = d_f(x)(1 + d_{f'}(x) - d_f(x)). \tag{6}$$

Lemma 1 can be found in [1] or [13].

LEMMA 1. *Suppose that f is positive and twice differentiable on $(a,b) \subset (0,\infty)$. Then $\log f(x)$ is convex in $\log x$ if and only if $D_f(x) \geqslant 0$ for all $x \in (a,b)$.*

The special case $q = 2$ of Lemma 2 can be found in [8] or [5]. It is clear that the conclusions hold for any $q > 0$. So we omit the details here.

LEMMA 2. *Suppose that $q > 0$, $f(x)$ is twice differentiable on (R_1^q, R_2^q). Then $f(x^q)$ is convex on (R_1, R_2) if and only if $\left(1 - \frac{1}{q}\right) f'(x) + xf''(x)$ is nonnegative on (R_1^q, R_2^q).*

LEMMA 3. *Suppose that $p > 0$, $q > 0$, f is positive and twice differentiable on (R_1^q, R_2^q). Then $f(x^q)^{\frac{1}{p}}$ is convex on (R_1, R_2) if and only if*

$$d_f(x)\left[1 - \frac{1}{q} + d_{f'}(x) + \left(\frac{1}{p} - 1\right) d_f(x)\right] \geqslant 0$$

holds on (R_1^q, R_2^q).

Proof. It follows from direct calculations that

$$\left(f(x)^{\frac{1}{p}}\right)' = \frac{1}{p} f'(x) f(x)^{\frac{1}{p}-1},$$
$$\left(f(x)^{\frac{1}{p}}\right)'' = \frac{1}{p} f''(x) f(x)^{\frac{1}{p}-1} + \frac{1}{p}\left(\frac{1}{p}-1\right) f'(x)^2 f(x)^{\frac{1}{p}-2}.$$

Since $p > 0$, f is positive, we have

$$\left(1-\frac{1}{q}\right)\left(f(x)^{\frac{1}{p}}\right)' + x\left(f(x)^{\frac{1}{p}}\right)''$$
$$\sim d_f(x)\left[1-\frac{1}{q}+d_{f'}(x)+\left(\frac{1}{p}-1\right)d_f(x)\right].$$

The desired result follows from Lemma 2. □

Suppose that $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, f is positive and twice differentiable on $(a,b) \subset (0,\infty)$. We define

$$D_f^{p,q}(x) = \left(1-\frac{1}{q}\right)\frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} + \left(\frac{1}{p}-1\right)x\left(\frac{f'(x)}{f(x)}\right)^2.$$

Then

$$xD_f^{p,q}(x) = d_f(x)\left[1-\frac{1}{q}+d_{f'}(x)+\left(\frac{1}{p}-1\right)d_f(x)\right].$$

Lemma 3 tells us that $f(x^q)^{\frac{1}{p}}$ is convex if and only if $D_f^{p,q}(x) \geqslant 0$. It is easy to verify that f is convex if and only if $D_f^{1,1}(x) \geqslant 0$; $\log f(x)$ is convex in $\log x$ if and only if $D_f^{\infty,\infty}(x) = D_f(x) \geqslant 0$; $\log f(x)$ is convex in x if and only if $D_f^{\infty,1}(x) \geqslant 0$; $f(x)^{\frac{1}{p}}$ is convex in $\log x$ if and only if $D_f^{p,\infty}(x) \geqslant 0$. This implies that Lemma 3 is valid if p or q is ∞.

LEMMA 4. *Suppose that $q \neq 0$, f is positive and twice differentiable on $(a,b) \subset (0,\infty)$. Then*

(i) $\log f(x^q)$ is convex in $\log x$ if and only if $\log f(x)$ is convex in $\log x$,

(ii) $\log f(x^q)$ is convex if $q \in (-\infty,0) \cup [1,\infty)$ and $\log f(x)$ is convex.

Proof. Consider the function $g(x) = f(x^q)$ and write $y = x^q$. It is easy to check that

$$d_g(x) = q\frac{x^q f'(x^q)}{f(x^q)} = q d_f(y),$$
$$d_{g'}(x) = q-1+q\frac{x^q f''(x^q)}{f'(x^q)} = q-1+q d_{f'}(y).$$

Hence we have

$$xD_g(x) = q^2 d_f(y)[1 + d_{f'}(y) - d_f(y)] = q^2 y D_f(y),$$
$$xD_g^{\infty,1}(x) = q^2 d_f(y)\left[1 - \frac{1}{q} + d_{f'}(y) - d_f(y)\right] = q^2 y D_f^{\infty,q}(y).$$

This completes the proof of the lemma. □

LEMMA 5. *Suppose that* $p > 0$, $q > 0$, f, g *are positive and twice differentiable functions on* (R_1^q, R_2^q). *Then* $(f(x^q)/g(x^q))^{\frac{1}{p}}$ *is convex on* (R_1, R_2) *if and only if*

$$\left(\frac{1}{p} - 1\right) d_f^2 + d_f \left(d_{f'} - \frac{2}{p} d_g + 1 - \frac{1}{q}\right) - d_g \left(d_{g'} - \frac{p+1}{p} d_g + 1 - \frac{1}{q}\right) \geqslant 0$$

holds on (R_1^q, R_2^q).

Proof. It follows from direct calculations that

$$\left(\frac{f}{g}\right)' = \frac{f}{g}\left(\frac{f'}{f} - \frac{g'}{g}\right),$$
$$\left(\frac{f}{g}\right)'' = \frac{f''}{g} - \frac{2f'g'}{g^2} + \frac{2f(g')^2}{g^3} - \frac{fg''}{g^2}$$
$$= \frac{f}{g}\left(\frac{f''}{f} - \frac{2f'g'}{fg} + \frac{2(g')^2}{g^2} - \frac{g''}{g}\right).$$

Therefore

$$d_{(f/g)'} = \frac{d_f d_{f'} - 2d_f d_g + 2d_g^2 - d_g d_{g'}}{d_f - d_g}.$$

We use (4) to obtain

$$d_{f/g}\left[1 - \frac{1}{q} + d_{(f/g)'} + \left(\frac{1}{p} - 1\right) d_{f/g}\right]$$
$$= (d_f - d_g)\left[1 - \frac{1}{q} + \frac{d_f d_{f'} - 2d_f d_g + 2d_g^2 - d_g d_{g'}}{d_f - d_g} + \left(\frac{1}{p} - 1\right)(d_f - d_g)\right]$$
$$= \left(\frac{1}{p} - 1\right) d_f^2 + d_f \left(d_{f'} - \frac{2}{p} d_g + 1 - \frac{1}{q}\right) - d_g \left(d_{g'} - \frac{p+1}{p} d_g + 1 - \frac{1}{q}\right).$$

The desired result follows from Lemma 3. □

We remark that, when $p = \infty$, the conclusion of Lemma 5 can be stated as follows: $\log(f(x^q)/g(x^q))$ is convex if and only if

$$-d_f^2 + d_f \left(d_{f'} + 1 - \frac{1}{q}\right) - d_g \left(d_{g'} - d_g + 1 - \frac{1}{q}\right) \geqslant 0.$$

In particular, when $p=q=\infty$, the conclusion of Lemma 5 can be stated as follows: $\log(f(x)/g(x))$ is convex in $\log x$ if and only if

$$-d_f^2+d_f\left(1+d_{f'}\right)-d_g\left(1+d_{g'}-d_g\right)\geqslant 0.$$

For $\varphi\in\Phi$, using (3) one can see that xD_φ and $xD_{\varphi'}$ are continuous on $(R_1^q,x_0)\cup(x_0,R_2^q)$. Since $\varphi(x_0)=0$, we have

$$\varphi(x)=\int_{x_0}^{x}\varphi'(t)dt,\quad R_1^q<x<R_2^q.$$

Note that $d_\varphi(x)>0$ on (x_0,R_2^q) and $d_\varphi(x)<0$ on (R_1^q,x_0). Moreover, we have the following property.

LEMMA 6. *Let $a<0$, $b\in\mathbb{R}$, $q>0$ and $\varphi\in\Phi$. Suppose that*

$$\frac{1}{d_\varphi}\left[xD_{\varphi'}-\left(d_{\varphi'}+b\right)\left(1+d_{\varphi'}-d_\varphi\right)\right]\geqslant 0$$

holds on (R_1^q,R_2^q). Then $d_{\varphi'}+ad_\varphi+b$ has at most one zero on (R_1^q,R_2^q), say x^*, so that $d_{\varphi'}+ad_\varphi+b<0$ on (x_0,x^*) and $d_{\varphi'}+ad_\varphi+b>0$ on $(R_1^q,x_0)\cup(x^*,R_2^q)$.

Proof. Consider the function $\kappa(x)=(d_{\varphi'}+ad_\varphi+b)/d_\varphi$. Observe that $\kappa(x)$ is continuous on (R_1^q,R_2^q) and

$$\begin{aligned}\kappa'(x)&=\frac{d_\varphi D_{\varphi'}-\left(d_{\varphi'}+b\right)D_\varphi}{d_\varphi^2}\\&=\frac{d_\varphi\left[xD_{\varphi'}-\left(d_{\varphi'}+b\right)\left(1+d_{\varphi'}-d_\varphi\right)\right]}{xd_\varphi^2}\\&\geqslant 0\end{aligned}$$

by the assumption. Hence $\kappa(x)$ has at most one zero on (R_1^q,R_2^q), say x^*.

If $x<x^*$, then $\kappa(x)\leqslant\kappa(x^*)=0$. Therefore $d_{\varphi'}+ad_\varphi+b\geqslant 0$ on (R_1^q,x_0) and $d_{\varphi'}+ad_\varphi+b\leqslant 0$ on (x_0,x^*). If $x>x^*$, then $\kappa(x)\geqslant\kappa(x^*)=0$. Therefore $d_{\varphi'}+ad_\varphi+b\geqslant 0$ since $d_\varphi\geqslant 0$. The desired result follows. □

LEMMA 7. *Let $p\geqslant 1$, $q>0$. Suppose $\{h_k(x)\}$ is a sequence of positive and twice differentiable functions on $(a,b)\subset(0,\infty)$ such that the function $H(x)=\sum_{k=0}^{\infty}h_k(x)$ is also twice differentiable on (a,b). If for each k the function $h_k(x^q)^{\frac{1}{p}}$ is convex, then $H(x^q)^{\frac{1}{p}}$ is also convex.*

Proof. By Hölder's inequality,

$$\left(H'(x)\right)^2=\left(\sum_{k=0}^{\infty}h_k'(x)\right)^2\leqslant\sum_{k=0}^{\infty}h_k(x)\sum_{k=0}^{\infty}\frac{h_k'(x)^2}{h_k(x)}=H(x)\sum_{k=0}^{\infty}\frac{h_k'(x)^2}{h_k(x)}.$$

Therefore

$$
\begin{aligned}
D_H^{p,q}(x) &= \frac{d_H(x)}{x}\left[1-\frac{1}{q}+d_{H'}(x)+\left(\frac{1}{p}-1\right)d_H(x)\right]\\
&= \left(1-\frac{1}{q}\right)\frac{H'(x)}{H(x)}+x\frac{H''(x)}{H(x)}+\left(\frac{1}{p}-1\right)x\left(\frac{H'(x)}{H(x)}\right)^2\\
&\geqslant \left(1-\frac{1}{q}\right)\frac{H'(x)}{H(x)}+x\frac{H''(x)}{H(x)}+\left(\frac{1}{p}-1\right)x\frac{\sum_{k=0}^{\infty}h_k'(x)^2/h_k(x)}{H(x)}\\
&= \frac{1}{H(x)}\sum_{k=0}^{\infty}\left[\left(1-\frac{1}{q}\right)h_k'(x)+xh_k''(x)+\left(\frac{1}{p}-1\right)x\frac{h_k'(x)^2}{h_k(x)}\right]\\
&= \frac{1}{H(x)}\sum_{k=0}^{\infty}h_k(x)D_{h_k}^{p,q}(x).
\end{aligned}
$$

By the assumption and Lemma 3, one gets $D_H^{p,q}(x)\geqslant 0$. The desired result follows. □

3. Convexity of the function $H(x^q)^{\frac{1}{p}}$

In this section we consider the convexity of the function $H(x^q)^{\frac{1}{p}}$, where $H(x)$ is defined by (2).

We establish the following theorems, which give an answer to the problem proposed in Section 1.

THEOREM 1. *Suppose that $p>1$, $q>0$, $\varphi\in\Phi$ and $M\in\mathscr{M}$. Then $H(x^q)^{\frac{1}{p}}$ is convex for $x\in(R_1,R_2)$ if M and φ satisfy the following conditions:*

(i) $M'>0$ and $M(x^q)^{\frac{1}{p}}$ is convex,

(ii) The inequality

$$
\frac{1}{d_\varphi}\left[xD_{\varphi'}-\left(1-\frac{1}{q}+d_{\varphi'}\right)\left(1+d_{\varphi'}-d_\varphi\right)\right]\geqslant 0 \tag{7}
$$

holds for $x\in(R_1^q,R_2^q)$.

Proof. For $\varphi\in\Phi$ and $M\in\mathscr{M}$, let

$$
\varphi(x)=\int_{x_0}^{x}\varphi'(t)\,dt, \tag{8}
$$

and

$$
h(x)=\int_{x_0}^{x}M(t)\varphi'(t)dt, \tag{9}
$$

where $x_0,x\in[R_1^q,R_2^q)$. Then $h'(x)=M(x)\varphi'(x)$. It follows from (4) and (5) that

$$
d_{h'}=d_M+d_{\varphi'}, \tag{10}
$$

and

$$D_{h'} = D_M + D_{\varphi'}. \tag{11}$$

Note that $M(x)$ is an increasing function of x. If $x > x_0$, since

$$h(x) = \int_{x_0}^{x} M(t)\varphi'(t)dt \leqslant M(x)\int_{x_0}^{x} \varphi'(t)dt = M(x)\varphi(x),$$

we have $d_h \geqslant d_\varphi$. If $x < x_0$, since

$$h(x) = -\int_{x}^{x_0} M(t)\varphi'(t)dt \leqslant -M(x)\int_{x}^{x_0} \varphi'(t)dt = M(x)\varphi(x),$$

we also have $d_h \geqslant d_\varphi$.

If $x > x_0$, noticing that $d_\varphi \geqslant 0, d_M \geqslant 0$, we obatin

$$\begin{aligned}&\left(d_{h'} - \frac{2}{p}d_\varphi + 1 - \frac{1}{q}\right)^2 - 4\left(1-\frac{1}{p}\right)d_\varphi\left(d_{\varphi'} - \frac{p+1}{p}d_\varphi + 1 - \frac{1}{q}\right)\\ =&\left(d_{h'} - 2d_\varphi + 1 - \frac{1}{q}\right)^2 + 4\left(1-\frac{1}{p}\right)d_\varphi d_M \geqslant 0. \end{aligned}\tag{12}$$

If $x < x_0$, then $d_\varphi \leqslant 0$, by Lemma 6

$$1 - \frac{1}{q} + d_{\varphi'} - 2d_\varphi \geqslant 0 \tag{13}$$

and

$$1 - \frac{1}{q} + d_{\varphi'} - \frac{p+1}{p}d_\varphi \geqslant 0. \tag{14}$$

We also obtain

$$\left(d_{h'} - \frac{2}{p}d_\varphi + 1 - \frac{1}{q}\right)^2 - 4\left(1-\frac{1}{p}\right)d_\varphi\left(d_{\varphi'} - \frac{p+1}{p}d_\varphi + 1 - \frac{1}{q}\right) \geqslant 0. \tag{15}$$

For convenience, we write

$$S = \sqrt{\left(d_{h'} - \frac{2}{p}d_\varphi + 1 - \frac{1}{q}\right)^2 - 4\left(1-\frac{1}{p}\right)d_\varphi\left(d_{\varphi'} - \frac{p+1}{p}d_\varphi + 1 - \frac{1}{q}\right)}.$$

By Lemma 5, we just need to prove that $\Delta(x) \geqslant 0$ for $x \in (R_1^q, R_2^q)$, where

$$\begin{aligned}\Delta(x) &= \left(\frac{1}{p}-1\right)d_h^2 + d_h\left(d_{h'} - \frac{2}{p}d_\varphi + 1 - \frac{1}{q}\right) - d_\varphi\left(d_{\varphi'} - \frac{p+1}{p}d_\varphi + 1 - \frac{1}{q}\right)\\ &\sim -\left(z_1 - 2\left(1-\frac{1}{p}\right)(d_h - d_\varphi)\right)\left(z_2 - 2\left(1-\frac{1}{p}\right)(d_h - d_\varphi)\right),\end{aligned}$$

and

$$z_1 = d_{h'} - 2d_\varphi + 1 - \frac{1}{q} - S,$$
$$z_2 = d_{h'} - 2d_\varphi + 1 - \frac{1}{q} + S. \tag{16}$$

We proceed to show that $\Delta(x) \geqslant 0$ on (x_0, R_2^q) and (R_1^q, x_0), respectively.

Case 1. Suppose that $x \in (x_0, R_2^q)$. Note that $d_\varphi \geqslant 0$, $z_1 \leqslant 0$,

$$z_2 \geqslant 0, \tag{17}$$

and

$$2\left(1 - \frac{1}{p}\right) d_\varphi + z_2 \geqslant 0. \tag{18}$$

Since $d_h \geqslant d_\varphi$, we have

$$z_1 - 2\left(1 - \frac{1}{p}\right)(d_h - d_\varphi) \leqslant z_1 \leqslant 0.$$

Therefore

$$\begin{aligned}\Delta(x) &\sim z_2 - 2\left(1 - \frac{1}{p}\right)(d_h - d_\varphi)\\ &\sim h - \frac{2\left(1 - \frac{1}{p}\right) x h'}{2\left(1 - \frac{1}{p}\right) d_\varphi + z_2} =: \delta(x).\end{aligned}$$

It follows from direct computations that

$$\begin{aligned}\delta'(x) = h' &- \frac{2\left(1 - \frac{1}{p}\right) h'(1 + d_{h'})}{2\left(1 - \frac{1}{p}\right) d_\varphi + z_2}\\ &+ \frac{2\left(1 - \frac{1}{p}\right) x h' \left(2\left(1 - \frac{1}{p}\right) D_\varphi + z_2'\right)}{\left(2\left(1 - \frac{1}{p}\right) d_\varphi + z_2\right)^2}.\end{aligned}$$

By Lemma 3, the assumption (i) implies that

$$x D_M = d_M(1 + d_{M'} - d_M) \geqslant d_M \left(\frac{1}{q} - \frac{d_M}{p}\right). \tag{19}$$

Combining (17), (18), (19) with the assumption (ii) we deduce that

$$
\begin{aligned}
& xS\left(2\left(1-\frac{1}{p}\right)D_\varphi+z_2'\right)\\
=& xD_M\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_2\right)+x\left(D_{\varphi'}-\frac{2}{p}D_\varphi\right)z_2\\
& -2x\left(1-\frac{1}{p}\right)D_\varphi\left(d_{\varphi'}-2d_\varphi+1-\frac{1}{q}\right)\\
\geqslant& d_M\left(\frac{1}{q}-\frac{d_M}{p}\right)\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_2\right)\\
& +(1+d_{\varphi'}-d_\varphi)\left(d_{\varphi'}-\frac{2}{p}d_\varphi+1-\frac{1}{q}\right)z_2\\
& -2\left(1-\frac{1}{p}\right)d_\varphi\left(d_{\varphi'}-2d_\varphi+1-\frac{1}{q}\right)(1+d_{\varphi'}-d_\varphi)\\
=& d_M\left(\frac{1}{q}-\frac{d_M}{p}\right)\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_2\right)\\
& +(1+d_{\varphi'}-d_\varphi)\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_2\right)(S-d_M).
\end{aligned}
$$

Since $h'>0,\ z_2\geqslant 0$, we have

$$
\begin{aligned}
\delta'(x)\geqslant& h'\left[1-\frac{2\left(1-\frac{1}{p}\right)(1+d_{h'})}{2\left(1-\frac{1}{p}\right)d_\varphi+z_2}-\frac{2\left(1-\frac{1}{p}\right)d_M\left(1-\frac{1}{q}+d_{\varphi'}-d_\varphi+\frac{d_M}{p}\right)}{S\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_2\right)}\right.\\
&\left.+\frac{2\left(1-\frac{1}{p}\right)(1+d_{\varphi'}-d_\varphi)}{2\left(1-\frac{1}{p}\right)d_\varphi+z_2}\right]\\
=& h'\left[1-\frac{2\left(1-\frac{1}{p}\right)(d_M+d_\varphi)}{2\left(1-\frac{1}{p}\right)d_\varphi+z_2}-\frac{2\left(1-\frac{1}{p}\right)d_M\left(1-\frac{1}{q}+d_{\varphi'}-d_\varphi+\frac{d_M}{p}\right)}{S\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_2\right)}\right]\\
\sim& \left(z_2-2\left(1-\frac{1}{p}\right)d_M\right)S-2\left(1-\frac{1}{p}\right)d_M\left(1-\frac{1}{q}+d_{\varphi'}-d_\varphi+\frac{d_M}{p}\right)\\
=& \frac{1}{2}\left(S+1-\frac{1}{q}+d_{\varphi'}-2d_\varphi+\left(\frac{2}{p}-1\right)d_M\right)^2\\
\geqslant& 0.
\end{aligned}
$$

Hence $\delta(x)\geqslant\delta(x_0)=0$ and $\Delta(x)\geqslant 0$ on (x_0,R_2^q).

Case 2. Suppose that $x \in (R_1^q, x_0)$. Then $d_\varphi \leqslant 0$, $d_h \leqslant 0$. Using (13) and (14) one can get

$$z_1 \geqslant 0, \tag{20}$$

$$2\left(1-\frac{1}{p}\right)d_\varphi + z_1 \leqslant 0, \tag{21}$$

and

$$z_2 - 2\left(1-\frac{1}{p}\right)(d_h - d_\varphi) \geqslant z_2 + 2\left(1-\frac{1}{p}\right)d_\varphi \geqslant 0.$$

Therefore

$$\begin{aligned}\Delta(x) \sim & 2\left(1-\frac{1}{p}\right)(d_h - d_\varphi) - z_1 \\ \sim & -h + \frac{2\left(1-\frac{1}{p}\right)xh'}{2\left(1-\frac{1}{p}\right)d_\varphi + z_1} =: \delta(x).\end{aligned}$$

It follows from direct computations that

$$\delta'(x) = -h' + \frac{2\left(1-\frac{1}{p}\right)h'(1+d_{h'})}{2\left(1-\frac{1}{p}\right)d_\varphi + z_1} - \frac{2\left(1-\frac{1}{p}\right)xh'\left(2\left(1-\frac{1}{p}\right)D_\varphi + z_1'\right)}{\left(2\left(1-\frac{1}{p}\right)d_\varphi + z_1\right)^2}.$$

Combining (20), (21), (19) with the assumption (ii) we deduce that

$$\begin{aligned}& xS\left(2\left(1-\frac{1}{p}\right)D_\varphi + z_1'\right) \\ = & -xD_M\left(2\left(1-\frac{1}{p}\right)d_\varphi + z_1\right) - x\left(D_{\varphi'} - \frac{2}{p}D_\varphi\right)z_1 \\ & + 2x\left(1-\frac{1}{p}\right)D_\varphi\left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q}\right) \\ \geqslant & -d_M\left(\frac{1}{q} - \frac{d_M}{p}\right)\left(2\left(1-\frac{1}{p}\right)d_\varphi + z_1\right) \\ & - \left(d_{\varphi'} - \frac{2}{p}d_\varphi + 1 - \frac{1}{q}\right)(1 + d_{\varphi'} - d_\varphi)z_1 \\ & + 2\left(1-\frac{1}{p}\right)d_\varphi\left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q}\right)(1 + d_{\varphi'} - d_\varphi) \\ = & -d_M\left(\frac{1}{q} - \frac{d_M}{p}\right)\left(2\left(1-\frac{1}{p}\right)d_\varphi + z_1\right) \\ & + (1 + d_{\varphi'} - d_\varphi)\left(2\left(1-\frac{1}{p}\right)d_\varphi + z_1\right)(S + d_M).\end{aligned}$$

Since $h' > 0$, $d_\varphi \leqslant 0$, we have

$$\begin{aligned}
\delta'(x) \leqslant h' \Bigg[& -1 + \frac{2\left(1-\frac{1}{p}\right)(1+d_{h'})}{2\left(1-\frac{1}{p}\right)d_\varphi + z_1} - \frac{2\left(1-\frac{1}{p}\right)d_M\left(1-\frac{1}{q}+d_{\varphi'}-d_\varphi+\frac{d_M}{p}\right)}{S\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_1\right)} \\
& - \frac{2\left(1-\frac{1}{p}\right)(1+d_{\varphi'}-d_\varphi)}{2\left(1-\frac{1}{p}\right)d_\varphi+z_1}\Bigg] \\
& = -h'\left[1-\frac{2\left(1-\frac{1}{p}\right)(d_M+d_\varphi)}{2\left(1-\frac{1}{p}\right)d_\varphi+z_1}+\frac{2\left(1-\frac{1}{p}\right)d_M\left(1-\frac{1}{q}+d_{\varphi'}-d_\varphi+\frac{d_M}{p}\right)}{S\left(2\left(1-\frac{1}{p}\right)d_\varphi+z_1\right)}\right] \\
& \sim \left(z_1-2\left(1-\frac{1}{p}\right)d_M\right)S+2\left(1-\frac{1}{p}\right)d_M\left(1-\frac{1}{q}+d_{\varphi'}-d_\varphi+\frac{d_M}{p}\right) \\
& = -\frac{1}{2}\left(1-\frac{1}{q}+d_{\varphi'}-2d_\varphi+\left(\frac{2}{p}-1\right)d_M-S\right)^2 \\
& \leqslant 0.
\end{aligned}$$

Hence $\delta(x) \geqslant \delta(x_0) = 0$ and $\Delta(x) \geqslant 0$ on (R_1^q, x_0).

This completes the proof of the theorem. □

THEOREM 2. *Suppose that $p > 1$, $q > 0$, $\varphi \in \Phi$ and $M \in \mathscr{M}$. Then $H(x^q)^{\frac{1}{p}}$ is convex for $x \in (R_1, R_2)$ if M and φ satisfy the following conditions:*

(i) $M' < 0$ and $M(x^q)^{\frac{1}{p}}$ is convex,

(ii) The inequality

$$\frac{1}{d_\varphi}\left[xD_{\varphi'}-\left(1-\frac{1}{q}+d_{\varphi'}\right)(1+d_{\varphi'}-d_\varphi)\right] \leqslant 0 \tag{22}$$

holds for $x \in (R_1^q, R_2^q)$.

Proof. To finish the proof of the theorem, we indicate how to adapt the proof of Theorem 1 to show that $\Delta(x) \geqslant 0$ on (x_0, R_2^q) and (R_1^q, x_0), respectively. So for the rest of this proof, we are going to use the notation from the proof of Theorem 1.

First, φ and h are defined by (8) and (9), respectively. It is easy to show that $d_h \leqslant d_\varphi$ for $x \in (R_1^q, x_0) \cup (x_0, R_2^q)$ since now M is decreasing on (R_1^q, R_2^q). Note that $d_M \leqslant 0$.

Second, S is well defined on (R_1^q, R_2^q). In fact, if $x < x_0$,

$$\left(d_{h'}-2d_\varphi+1-\frac{1}{q}\right)^2+4\left(1-\frac{1}{p}\right)d_\varphi d_M \geqslant 0;$$

If $x > x_0$, the assumption (ii) implies that $\left(1-\frac{1}{q}+d_{\varphi'}\right)/d_\varphi$ is a decreasing function of x, which leads to

$$1-\frac{1}{q}+d_{\varphi'}-2d_\varphi \leqslant 0,$$

$$1-\frac{1}{q}+d_{\varphi'}-\frac{p+1}{p}d_\varphi \leqslant 0,$$

and

$$\left(d_{h'}-\frac{2}{p}d_\varphi+1-\frac{1}{q}\right)^2-4\left(1-\frac{1}{p}\right)d_\varphi\left(d_{\varphi'}-\frac{p+1}{p}d_\varphi+1-\frac{1}{q}\right)\geqslant 0.$$

Thus z_1, z_2 are also well defined.

In the case $x \in (x_0, R_2^q)$, we have $z_1 \leqslant 0$, $z_2 \leqslant 0$, $2\left(1-\frac{1}{p}\right)d_\varphi + z_2 \geqslant 0$ and

$$z_1-2\left(1-\frac{1}{p}\right)(d_h-d_\varphi)\leqslant 2\left(1-\frac{1}{p}\right)d_\varphi+z_1\leqslant 0.$$

Therefore

$$\Delta(x)\sim h-\frac{2\left(1-\frac{1}{p}\right)xh'}{2\left(1-\frac{1}{p}\right)d_\varphi+z_2}=:\delta(x).$$

In the case $x \in (R_1^q, x_0)$, we have $z_1 \leqslant 0$, $z_2 \geqslant 0$, $2\left(1-\frac{1}{p}\right)d_\varphi + z_1 \leqslant 0$ and

$$z_2-2\left(1-\frac{1}{p}\right)(d_h-d_\varphi)\geqslant z_2\geqslant 0.$$

Therefore

$$\Delta(x)\sim -h+\frac{2\left(1-\frac{1}{p}\right)xh'}{2\left(1-\frac{1}{p}\right)d_\varphi+z_1}=:\delta(x).$$

The rest of the proof of Theorem 1 remains valid here. This completes the proof of Theorem 2. □

We remark that, when p or q is ∞, the proofs of Theorem 1 and 2 are also valid. When $p = 1$, we have the following two results.

THEOREM 3. *Suppose that $q > 0$, $\varphi \in \Phi$ and $M \in \mathscr{M}$. Then $H(x^q)$ is convex for $x \in (R_1, R_2)$ if M and φ satisfy the following conditions:*

(i) $M' > 0$ and $M(x^q)$ is convex,

(ii) The inequality (7) holds for $x \in (R_1^q, R_2^q)$.

Proof. According to the Lemma 5, we just need to prove that $\Delta(x) \geqslant 0$ for $x \in (R_1^q, R_2^q)$, where

$$\Delta(x) = d_h \left(d_{h'} - 2d_\varphi + 1 - \frac{1}{q} \right) - d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right).$$

By Lemma 6, $d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q}$ has at most one zero on (x_0, R_2^q), say x^*, so that $d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} < 0$ on (x_0, x^*) and $d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} > 0$ on $(R_1^q, x_0) \cup (x^*, R_2^q)$.

(a) Suppose that $x \in (x^*, R_2^q)$. Then $d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} > 0$. Noticing that $d_{h'} = d_M + d_{\varphi'}$, we have

$$\Delta(x) = d_h d_M + (d_h - d_\varphi) \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right).$$

Since $d_M \geqslant 0$, $d_h \geqslant 0$ and $d_h \geqslant d_\varphi$, we have $\Delta(x) \geqslant 0$ on (x^*, R_2^q).

(b) Suppose that $x \in (x_0, x^*)$. Then $d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} < 0$. Since $h > 0$, $d_\varphi > 0$, we have

$$\Delta(x) \sim h - xh' \left[\frac{d_M}{d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)} + \frac{1}{d_\varphi} \right] =: \delta(x).$$

Noticing that $h' = M\varphi' \geqslant 0$, it follows from direct computations that

$$\begin{aligned}
\delta'(x) =& h' - h'(1 + d_{h'}) \left[\frac{d_M}{d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)} + \frac{1}{d_\varphi} \right] \\
& - xh' \left[\frac{D_M}{d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)} - \frac{d_M D_\varphi}{d_\varphi^2 \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)} \right. \\
& \left. - \frac{d_M (D_{\varphi'} - 2D_\varphi)}{d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)^2} - \frac{D_\varphi}{d_\varphi^2} \right] \\
=& h' \left[-\frac{d_M}{d_\varphi} - \frac{d_M (1 + d_{M'} + d_\varphi)}{d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)} + \frac{x (D_{\varphi'} - 2D_\varphi) d_\varphi d_M}{d_\varphi^2 \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)^2} \right].
\end{aligned}$$

By Lemma 2,

$$1 - \frac{1}{q} + d_{M'} \geqslant 0.$$

Combining this with the assumption (ii) we have

$$\delta'(x) \geqslant \frac{h' d_M}{d_\varphi} \left[-1 - \frac{\frac{1}{q} + d_\varphi}{d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q}} + \frac{1 + d_{\varphi'} - d_\varphi}{d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q}} \right] = 0$$

on (x_0, x^*). Hence $\delta(x) \geqslant \delta(x_0) = 0$ and $\Delta(x) \geqslant 0$ on (x_0, x^*).

(c) Suppose that $x \in (R_1^q, x_0)$. Then $d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} > 0$. Since $h < 0$, $d_\varphi < 0$, we have

$$\Delta(x) \sim -h + xh' \left[\frac{d_M}{d_\varphi \left(d_{\varphi'} - 2d_\varphi + 1 - \frac{1}{q} \right)} + \frac{1}{d_\varphi} \right] = -\delta(x),$$

where $\delta(x)$ is defined in (b) above. Now from the proof in (b) we can get $\delta'(x) \leqslant 0$. Hence $\delta(x) \geqslant \delta(x_0) = 0$ and $\Delta(x) \geqslant 0$ on (R_1^q, x_0).

The proof of the theorem is completed. □

THEOREM 4. *Suppose that $q > 0$, $\varphi \in \Phi$ and $M \in \mathscr{M}$. Then $H(x^q)$ is convex for $x \in (R_1, R_2)$ if M and φ satisfy the following conditions:*

(i) $M' < 0$ and $M(x^q)$ is convex,

(ii) The inequality (22) holds for $x \in (R_1^q, R_2^q)$.

The proof is similar to that of Theorem 3. We omit the details here.

4. Convexities of area integral means

In this section we study convexities of area integral means of analytic functions on an annulus.

Recall that, for $f \in \mathscr{H}$, $0 < t < \infty$, $R_1 \leqslant c < R_2$, the area integral means of f are defined by

$$M_{t,\varphi,q,c}(f,r) = \frac{\displaystyle\int_{A(c,r)} |f(z)|^t \varphi'(|z|^q)|z|^{q-2}\, dA(z)}{\displaystyle\int_{A(c,r)} \varphi'(|z|^q)|z|^{q-2}\, dA(z)}, \quad R_1 \leqslant r < R_2,$$

where the functions f and φ' are such that the integrals exist.

Let $x_0 = c^q$, $x = r^q$ and $M(x) = M_t(f, x^{1/q})$. By polar coordinates, we have

$$M_{t,\varphi,q,c}(f,r) = \frac{h(r^q)}{\varphi(r^q)}, \quad R_1 \leqslant r < R_2,$$

where φ, h are defined by (8) and (9), respectively.

Now we turn to consider the convexity for area integral means of analytic functions f in $\mathbb{D}$. For the weighted area measure

$$dA_\varphi(z) = (1 - |z|^q)^\alpha |z|^{q-2}\, dA(z),$$

where dA is the Euclidean area measure on $\mathbb{D}$, the area integral means of f are denoted by $M_{t,\alpha,q,c}(f,r)$. Since now

$$\varphi'(x) = (1-x)^\alpha, \quad 0 \leqslant x < 1,$$

we have

$$\varphi(x)=\int_{x_0}^{x}(1-t)^{\alpha}\,dt,\quad 0\leqslant x<1, \tag{23}$$

where $x_0\in[0,1)$. One can easily verify that

$$d_{\varphi'}=d_{\varphi'}(x)=\frac{-\alpha x}{1-x}, \tag{24}$$

and

$$D_{\varphi'}=D_{\varphi'}(x)=\frac{-\alpha}{(1-x)^2}. \tag{25}$$

We have the following estimate for φ.

LEMMA 8. *Suppose that $q>0,\alpha\in\mathbb{R}$, φ is defined by (23). Then*

(i) (7) *holds for $x\in(0,1)$ if $q\geqslant 1$ and $-2\leqslant\alpha\leqslant 0$,*

(ii) (7) *holds for $x\in(0,x_0)$ if $q\geqslant 1$ and $\alpha<-2$,*

(iii) (22) *holds for $x\in(x_0,1)$ if $0<q\leqslant 1$ and $\alpha\geqslant 0$.*

Proof. Set

$$\Delta_1(x)=\left(\frac{xD_{\varphi'}}{(1-\frac{1}{q}+d_{\varphi'})^2}-\frac{1+d_{\varphi'}}{1-\frac{1}{q}+d_{\varphi'}}\right)\varphi+\frac{x\varphi'}{1-\frac{1}{q}+d_{\varphi'}},\quad x\in[0,1).$$

It follows from direct computations that

$$\begin{aligned}\Delta_1'(x)&=\left(\frac{D_{\varphi'}+xD_{\varphi'}'}{(1-\frac{1}{q}+d_{\varphi'})^2}-\frac{2xD_{\varphi'}^2}{(1-\frac{1}{q}+d_{\varphi'})^3}+\frac{\frac{1}{q}D_{\varphi'}}{(1-\frac{1}{q}+d_{\varphi'})^2}\right)\varphi\\&=\left[\left(1-\frac{1}{q}+d_{\varphi'}\right)\left(1+\frac{1}{q}+x\frac{D_{\varphi'}'}{D_{\varphi'}}\right)-2xD_{\varphi'}\right]\frac{\varphi D_{\varphi'}}{(1-\frac{1}{q}+d_{\varphi'})^3}\\&=\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}+\frac{(\alpha+2)x}{1-x}\right)\frac{\varphi D_{\varphi'}}{(1-\frac{1}{q}+d_{\varphi'})^3}.\end{aligned}$$

If $q\geqslant 1$ and $-2\leqslant\alpha\leqslant 0$, we have $D_{\varphi'}>0$ and $1-\frac{1}{q}+d_{\varphi'}>0$. So

$$\Delta_1'(x)\sim\varphi(x)=\int_{x_0}^{x}(1-t)^{\alpha}\,dt,$$

which implies that $\Delta_1'(x)<0$ on $(0,x_0)$ and $\Delta_1'(x)>0$ on $(x_0,1)$. Hence

$$\Delta_1(x)\geqslant\Delta_1(x_0)=\frac{x_0\varphi'(x_0)}{1-\frac{1}{q}+d_{\varphi'}(x_0)}\geqslant 0.$$

If $q \geqslant 1$ and $\alpha < -2$, we have $D_{\varphi'} > 0$ and $1 - \frac{1}{q} + d_{\varphi'} > 0$. So

$$\begin{aligned} &\frac{xD_{\varphi'}}{(1-\frac{1}{q}+d_{\varphi'})^2} - \frac{1+d_{\varphi'}}{1-\frac{1}{q}+d_{\varphi'}} \\ \leqslant &\frac{xD_{\varphi'}}{d_{\varphi'}(1-\frac{1}{q}+d_{\varphi'})} - \frac{1+d_{\varphi'}}{1-\frac{1}{q}+d_{\varphi'}} \\ = &\frac{(\alpha+1)x}{(1-x)(1-\frac{1}{q}+d_{\varphi'})} \\ \leqslant &0. \end{aligned}$$

Hence for any $x \in (0, x_0)$, we have

$$\Delta_1(x) \geqslant \Delta_1(x_0) = \frac{x_0 \varphi'(x_0)}{1 - \frac{1}{q} + d_{\varphi'}(x_0)} \geqslant 0.$$

If $0 < q \leqslant 1$ and $\alpha \geqslant 0$, we have $D_{\varphi'} < 0$ and $1 - \frac{1}{q} + d_{\varphi'} < 0$. So

$$\Delta_1'(x) \sim -\varphi(x) \leqslant 0$$

on $(x_0, 1)$ and hence $\Delta_1(x) \leqslant \Delta_1(x_0) \leqslant 0$ on $(x_0, 1)$.

The desired result follows. $\square$

Note that, Lemma 8 gives examples of a function φ satisfying (7) or (22).

As mentioned in the first section, if $f \in \mathscr{H}$, $\log M_t(f,r)$ is convex in $\log r$. Taking $p = q = \infty$ in Theorem 1, 2 and using Lemma 4, 8, we obtain the following result.

THEOREM 5. *Suppose that $q > 0$, $0 \leqslant c < 1$, and $f(z)$ is analytic in $\mathbb{D}$. Then*

- *(i) $\log M_{t,\alpha,q,c}(f,r)$ is convex in $\log r$ for $r \in (0,1)$ if $q \geqslant 1$, $-2 \leqslant \alpha \leqslant 0$, and $M_t(f,r)$ is increasing,*
- *(ii) $\log M_{t,\alpha,q,c}(f,r)$ is convex in $\log r$ for $r \in (0,c)$ if $q \geqslant 1$, $\alpha < -2$, and $M_t(f,r)$ is increasing,*
- *(iii) $\log M_{t,\alpha,q,c}(f,r)$ is convex in $\log r$ for $r \in (c,1)$ if $0 < q \leqslant 1$ and $\alpha \geqslant 0$, and $M_t(f,r)$ is decreasing,*

In [9] Shniad proved that, if f is analytic in $|z| < R$, $(M_4(f,r))^{\frac{1}{4}}$ is convex for $r \in [0, R)$. Taking $p = 4$, $q \geqslant 1$ and $c = 0$, we immediately obtain the following consequence of Theorem 1.

THEOREM 6. *Suppose $-2 \leqslant \alpha \leqslant 0$, $q \geqslant 1$, $f(z)$ is analytic in $\mathbb{D}$. Then the function $\left(M_{4,\alpha,q,0}(f,r)\right)^{\frac{1}{4}}$ is convex for $r \in (0,1)$.*

Every function $f \in \mathscr{H}$ has a Laurent expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \quad R_1 < |z| < R_2.$$

It is easy to check that

$$M_2(f,r) = \sum_{k=-\infty}^{\infty} |a_k|^2 r^{2k}, \quad R_1 \leqslant r < R_2.$$

Since for any integer k, $(r^{2k})^{\frac{1}{2}} = r^k$ is convex, by Lemma 7, $M_2(f,r)^{\frac{1}{2}}$ is convex. Taking $p=2$ in Theorem 1, we obtain the following result.

THEOREM 7. *Suppose that $-2 \leqslant \alpha \leqslant 0$, $q \geqslant 1$, $0 \leqslant c < 1$, and $f(z)$ is analytic in $\mathbb{D}$. Then $(M_{2,\alpha,q,c}(f,r))^{\frac{1}{2}}$ is convex for $r \in (0,1)$ if $M_2(f,r)$ is increasing for $r \in (0,1)$.*

Suppose that $f \in \mathscr{H}$ does not vanish. Then for any $t > 0$, $f^{\frac{t}{2}} \in \mathscr{H}$. It is easy to see that $M_t(f,r) = M_2(f^{\frac{t}{2}},r)$ is convex. Replacing f with $f^{\frac{t}{2}}$ in Theorem 7, we obtain the following result.

COROLLARY 1. *Suppose that $t > 0$, $-2 \leqslant \alpha \leqslant 0$, $q \geqslant 1$, $0 \leqslant c < 1$, and $f(z)$ is a nonvanishing analytic function in $\mathbb{D}$. Then $(M_{t,\alpha,q,c})(f,r)^{\frac{1}{2}}$ is convex for $r \in (0,1)$ if $M_t(f,r)$ is increasing for $r \in (0,1)$.*

For the weighted area measure

$$dA_\varphi(z) = e^{-\alpha|z|^q} |z|^{q-2} dA(z),$$

where dA is the Euclidean area measure on $\mathbb{C}$, the area integral means of f are denoted by $G_{t,\alpha,q,c}(f,r)$. Since now

$$\varphi'(x) = e^{-\alpha x}, \quad x \geqslant 0,$$

we have

$$\varphi(x) = \int_{x_0}^{x} e^{-\alpha t}\, dt, \quad x \geqslant 0, \tag{26}$$

where $x_0 \in [0,\infty)$. One can easily verify that

$$d_{\varphi'} = d_{\varphi'}(x) = -\alpha x, \tag{27}$$

and

$$D_{\varphi'} = D_{\varphi'}(x) = -\alpha. \tag{28}$$

Then we have the following estimate.

LEMMA 9. *Suppose that* $q>0$, $\alpha\in\mathbb{R}$, φ *is defined by* (26). *Then*

(i) (7) *holds for* $x\in(0,x_0)$ *if* $q\geqslant 1$ *and* $\alpha\leqslant 0$,

(ii) (22) *holds for* $x\in(x_0,\infty)$ *if* $0<q\leqslant 1$ *and* $\alpha\geqslant 0$.

Proof. Set

$$\Delta_1(x)=\frac{1}{d_\varphi}\left[xD_{\varphi'}-\left(1-\frac{1}{q}+d_{\varphi'}\right)\left(1+d_{\varphi'}-d_\varphi\right)\right].$$

Noticing that

$$\varphi(x)=\int_{x_0}^{x}e^{-\alpha t}dt=\frac{e^{-\alpha x}-e^{-\alpha x_0}}{-\alpha}=\frac{\varphi'(x)-\varphi'(x_0)}{-\alpha},\quad x\in[0,\infty),$$

we have

$$\begin{aligned}\Delta_1(x)&\sim\varphi\left[xD_{\varphi'}-\left(1-\frac{1}{q}+d_{\varphi'}\right)\left(1+d_{\varphi'}-d_\varphi\right)\right]\\&=\varphi\left[xD_{\varphi'}-\left(1-\frac{1}{q}+d_{\varphi'}\right)\left(1+d_{\varphi'}\right)\right]\\&\quad+\left(x\varphi'(x_0)-\alpha x\varphi\right)\left(1-\frac{1}{q}+d_{\varphi'}\right)\\&=-\left(1-\frac{1}{q}\right)\varphi+x\varphi'(x_0)\left(1-\frac{1}{q}+d_{\varphi'}\right)\\&=:\delta_1(x).\end{aligned}$$

If $q\geqslant 1$ and $\alpha\leqslant 0$, $1-\frac{1}{q}+d_{\varphi'}>0$. Therefore

$$\delta_1(x)\geqslant-\left(1-\frac{1}{q}\right)\varphi\geqslant 0$$

for any $x\in(0,x_0)$. Hence $\Delta_1(x)\geqslant 0$ on $(0,x_0)$.

If $0<q\leqslant 1$ and $\alpha\geqslant 0$, $\varphi'(x)\leqslant\varphi'(x_0)$ for any $x\in(x_0,\infty)$. Therefore

$$\begin{aligned}\delta_1'(x)&=-\left(1-\frac{1}{q}\right)\varphi'+\varphi'(x_0)\left(1-\frac{1}{q}-2\alpha x\right)\\&\leqslant-\left(1-\frac{1}{q}\right)\varphi'(x_0)+\varphi'(x_0)\left(1-\frac{1}{q}-2\alpha x\right)\\&=-2\alpha x\varphi'(x_0)\\&\leqslant 0\end{aligned}$$

on (x_0,∞). Hence $\delta_1(x)\leqslant\delta_1(x_0)\leqslant 0$ and $\Delta_1(x)\leqslant 0$ on (x_0,∞). $\square$

Taking $p=q=\infty$ in Theorem 1, 2 and using Lemma 4, 9, we obtain the following result.

THEOREM 8. *Suppose that* $q > 0$, $c \geqslant 0$, *and* $f(z)$ *is an entire function. Then*

(i) $\log G_{t,\alpha,q,c}(f,r)$ *is convex in* $\log r$ *for* $r \in (0,c)$ *if* $q \geqslant 1$, $\alpha \leqslant 0$, *and* $M_t(f,r)$ *is increasing,*

(ii) $\log G_{t,\alpha,q,c}(f,r)$ *is convex in* $\log r$ *for* $r \in (c,\infty)$ *if* $0 < q \leqslant 1$, $\alpha \geqslant 0$, *and* $M_t(f,r)$ *is decreasing,*

Taking $p = 4$ and $c \geqslant 0$ we obtain the following consequence of Theorem 1 and Lemma 9.

THEOREM 9. *Suppose that* $q \geqslant 1$, $c \geqslant 0$, $\alpha \leqslant 0$, $f(z)$ *is an entire function. Then the function* $\left(G_{4,\alpha,q,c}(f,r)\right)^{\frac{1}{4}}$ *is convex for* $r \in (0,c)$.

Suppose that f is analytic on $|z| > c$ and ∞ is a removable singularity of f. Then f has the Laurent expansion

$$f(z) = \sum_{k=0}^{\infty} a_{-k} z^{-k}, \quad |z| > c,$$

and

$$M_2(f,r) = \sum_{k=0}^{\infty} |a_{-k}|^2 r^{-2k}, \quad r \geqslant c.$$

By Lemma 7, $\log M_2(f,r)$ is convex in r. Taking $p = \infty$ and $q = 1$ in Theorem 2 and using Lemma 9(ii), we have the following result.

THEOREM 10. *Suppose* $\alpha \geqslant 0$, f *is analytic on* $|z| > c$ *and continuous on* $|z| \geqslant c$, ∞ *is a removable singularity of* f. *Then* $\log G_{2,\alpha,1,c}(f,r)$ *is convex for* $r \in (c,\infty)$. *Moreover, if* f *does not vanish, and* $t > 0$, *then* $\log G_{t,\alpha,1,c}(f,r)$ *is convex for* $r \in (c,\infty)$.

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(Received May 12, 2021)

Yucong Duan
Department of Mathematics
Hebei University of Technology
Tianjin 300401, China
e-mail: 18224562361@163.com

Chunjie Wang
Department of Mathematics
Hebei University of Technology
Tianjin 300401, China
e-mail: wcj498@126.com