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WEIGHTED SHIFTS ON DIRECTED TREES WITH ONE BRANCHING VERTEX: BETWEEN QUASINORMALITY AND PARANORMALITY

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Abstract. Let $\mathscr{T}_{\eta,\kappa}$ be a directed tree consisting of one branching vertex, η branches and a trunk of length κ and let $S_{\boldsymbol{\lambda}}$ be the associated weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weight sequence $\boldsymbol{\lambda}$. Introduced recently was a collection of classical weighted shifts, "the i-th branching weighted shifts" $W^{(i)}$ for $0 \leqslant i \leqslant \eta$, whose weights are derived from those of $S_{\boldsymbol{\lambda}}$ by slicing the branches of the tree $\mathscr{T}_{\eta,\kappa}$ ([9]). As a contrast contrasting to "slicing" we consider "collapsing the branches of a tree" and define "the k-step collapsed weighted shift" $S_{\widetilde{\boldsymbol{\lambda}}^{(k)}}$ on $\mathscr{T}_{\eta-k,\kappa}$ for $1 \leqslant k \leqslant \eta-1$ so that $S_{\widetilde{\boldsymbol{\lambda}}^{(\eta-1)}}$ may become the basic branching shift $W^{(0)}$. In this paper we discuss the relationships between operator properties of $S_{\boldsymbol{\lambda}}$ such as quasinormality, subnormality, ∞-hyponormality, p-hyponormality, and p-paranormality, and these properties for the $W^{(i)}$ and $S_{\widetilde{\boldsymbol{\lambda}}^{(k)}}$.

1. Introduction

Let $\mathscr{H}$ be an infinite dimensional complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. An operator T in $\mathscr{B}(\mathscr{H})$ is *normal* [resp., *quasinormal*, *hyponormal*] if $T^*T = TT^*$ [resp., $(T^*T)T = T(T^*T)$, $T^*T \geqslant TT^*$]. An operator T in $\mathscr{B}(\mathscr{H})$ is *subnormal* if T is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace. For a fixed $n \in \mathbb{N}$, an operator $T \in \mathscr{B}(\mathscr{H})$ is n-*contractive* [resp., n-*hypercontractive*] if $A_n(T) := \sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k \geqslant 0$ [resp., $A_k(T) \geqslant 0$ for all $1 \leqslant k \leqslant n$]. It is well-known that T is contractive subnormal if and only if T is n-contractive for all $n \in \mathbb{N}$ ([1]). For some $p > 0$, an operator T in $\mathscr{B}(\mathscr{H})$ is p-*hyponormal* if $(T^*T)^p \geqslant (TT^*)^p$ ([17], [24]). The Löwner-Heinz inequality implies that every p-hyponormal operator is q-hyponormal for $q \leqslant p$ ([17]). An operator T in $\mathscr{B}(\mathscr{H})$ is ∞-*hyponormal* if T is p-hyponormal for all $p > 0$. It is well-known that every quasinormal operator $T \in \mathscr{B}(\mathscr{H})$ is ∞-hyponormal. An operator $T \in \mathscr{B}(\mathscr{H})$ is *paranormal* if $\left\|T^2x\right\| \geqslant \|Tx\|^2$ for all unit vectors x in $\mathscr{H}$ ([16], [19]). Recall that every operator $T \in \mathscr{B}(\mathscr{H})$ has the

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(unique) polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is the partial isometry with $\ker U = \ker T$ and $\ker U^* = \ker T^*$. For each $p > 0$, an operator $T \in \mathscr{B}(\mathscr{H})$ is *p-paranormal* if $\||T|^pU|T|^px\| \geqslant \||T|^px\|^2$ for all unit vectors x in $\mathscr{H}$ ([14], [15]). Obviously, 1-paranormality and paranormality coincide. Every q-paranormal operator is p-paranormal for $q \leqslant p$. An operator $T \in \mathscr{B}(\mathscr{H})$ is *normaloid* if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T, which is equivalent to $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$. The following implications are well-known:

$$
\begin{array}{ccccccc}
\text{normal} & \Rightarrow & \text{quasinormal} & \Rightarrow & \text{subnormal} & \Rightarrow & \text{hyponormal} \\
 & & \Downarrow & & & & \Updownarrow \\
 & & \infty\text{-hyponormal} & \Rightarrow & \underset{(1<p<\infty)}{p\text{-hyponormal}} & \Rightarrow & 1\text{-hyponormal} \\
 & & & & & & \Downarrow \\
\text{normaloid} & \Leftarrow & \underset{(1<q<\infty)}{q\text{-paranormal}} & \Leftarrow & \underset{(0<q\leqslant 1)}{q\text{-paranormal}} & \Leftarrow & \underset{(0<p<1)}{p\text{-hyponormal}}
\end{array}
$$

and their converse implications do not hold in general ([4], [5], [6], [17]). There is no implication between p-hyponormality $(1 < p \leqslant \infty)$ and subnormality in general (see [21, Example 8.2.4]).

Let $\mathbb{Z}$ [resp., $\mathbb{Z}_+$, $\mathbb{N}$] be the set of integers [resp., nonnegative integers, positive integers]. We write $\mathbb{R}$ [resp., $\mathbb{R}_+$, $\mathbb{C}$] for the set of real [resp., nonnegative real, complex] numbers. And we set $\mathbb{N}_k = \{k, k+1, k+2, \ldots\}$ for $k \in \mathbb{N}$, and $J_\iota = \{k \in \mathbb{N} \colon k \leqslant \iota\}$, $\iota \in \mathbb{Z}_+$, with the convention that $J_0 = \varnothing$. For a subset J of $\mathbb{Z}$, we set $-J = \{-k : k \in J\}$.

As a generalization of the classical weighted shifts, Jabłoński-Jung-Stochel [21] introduced the weighted shift $S_{\boldsymbol{\lambda}}$ on a directed tree $\mathscr{T} = (V, E)$, where V and E are the sets of vertices and edges, respectively, whose definitions are given in Section 2. The weighted shifts $S_{\boldsymbol{\lambda}}$ on directed trees $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ with one branching vertex (see (1) and the Figure 1) have provided good information and several exotic examples to solve open problems in operator theory (see [2], [3], [12], [21], [22], [23]). In [12] and [13], the papers studied the subnormal completion problem for weighted shifts $S_{\boldsymbol{\lambda}}$ on the directed trees $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$. In [9] Exner-Jung-Lee studied the branching weighted shifts $W^{(i)}$ of $S_{\boldsymbol{\lambda}}$, $i \in J_\eta \cup \{0\}$, that are sliced from $\mathscr{T}_{\eta,\kappa}$ to analyze the structure of $S_{\boldsymbol{\lambda}}$ and proved the following statements:

1° $S_{\boldsymbol{\lambda}}$ is subnormal if and only if $W^{(i)}$ is subnormal for $i \in J_\eta \cup \{0\}$ (see a remark above Theorem 2.1 in [9] and also [21, Corollary 6.2.2]);

2° $S_{\boldsymbol{\lambda}}$ is n-contractive [resp., n-hypercontractive] if and only if $W^{(i)}$ is n-contractive [resp., n-hypercontractive] for $i \in J_\eta \cup \{0\}$;

3° if $S_{\boldsymbol{\lambda}}$ is hyponormal, then $W^{(i)}$ is hyponormal for $i \in J_\eta \cup \{0\}$. However the converse implication is not true.

We may apply this sort of study about hyponormality, subnormality, n-contractivity of $S_{\boldsymbol{\lambda}}$ and $W^{(i)}$ to other properties; we will use "property P" as placeholder for such properties, so, for example we say "property P is hyponormality", or "property P is p-paranormality", etc. Thus the following question arises:

Q1. Suppose $S_{\boldsymbol{\lambda}}$ is a weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$. Is it true that if $S_{\boldsymbol{\lambda}}$ has property P, then $W^{(i)}$ has property P for all $i \in J_\eta \cup \{0\}$?

As a concept complementary to that of "slicing tree", one may consider a "collapsing tree". In this paper, we give a weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ induced by such a directed tree $\mathscr{T}_{\eta-1,\kappa}$ – which we call "*the (first-step) collapsed weighted shift*" (see Definition 2). By repeating $\eta-1$ times the "collapsing" method from the given weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$, we may obtain lastly a classical weighted shift $\widetilde{W}$ which is called "*the last-step collapsed weighted shift*" of $S_{\boldsymbol{\lambda}}$ (see Definition 2). Hence the following parallel question arises from this notion:

Q2. Suppose $S_{\boldsymbol{\lambda}}$ is a weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$. Is it true that if $S_{\boldsymbol{\lambda}}$ has property P, then the collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ has property P?

In this paper we answer questions Q1 and Q2 for the properties of operators between quasinormality and paranormality such as quasinormality, subnormality, ∞-hyponormality, p-hyponormality, and p-paranormality.

The paper consists of five sections. In Section 2 we recall the notation and terminology for classical weighted shifts and for weighted shifts $S_{\boldsymbol{\lambda}}$ on directed trees $\mathscr{T}_{\eta,\kappa}$ and its sliced classical weighted shifts as in [9] and [21]. We introduce a new definition which we call "the collapsed weighted shift." In Section 3 we answer Q1 affirmatively when placeholder P is quasinormality, p-hyponormality $(0<p\leqslant\infty)$, and p-paranormality $(0<p\leqslant 1)$. When property P is p-paranormality $(1<p<\infty)$, we show that the answer to Q1 is negative. In addition, we discuss the question of the converse implication of the statement in Q1, namely "is it true that if $W^{(i)}$ has property P for all $i \in J_\eta \cup \{0\}$, then $S_{\boldsymbol{\lambda}}$ has property P?" We see that the converse implication is true when property P is quasinormality or p-paranormality $(1\leqslant p<\infty)$. In Section 4 we solve Q2 when the placeholder P in Q2 is some property between quasinormality and p-paranormality. We show that Q2 is affirmative when property P is quasinormality, subnormality, p-hyponormality $(0<p\leqslant\infty)$, and p-paranormality $(0<p\leqslant 1)$. Some counterexamples showing a negative answer to the question, "is it true that if $S_{\widetilde{\boldsymbol{\lambda}}}$ has property P, then $S_{\boldsymbol{\lambda}}$ has property P?" are given when property P is one of properties among quasinormality, subnormality, p-hyponormality $(0<p\leqslant\infty)$, and p-paranormality $(0<p<\infty)$. In Section 5, we see that if $S_{\boldsymbol{\lambda}}$ is 2-generation flat, then the answers to Q1 and Q2 are positive as are those for the converse implications of the statements in Q1 and Q2.

2. Preliminaries

2.1. Classical weighted shifts

We sketch here briefly some very standard notation and results for classical weighted shifts. Recall that given a weight sequence $\alpha=\{\alpha_n\}_{n=0}^{\infty}$ we define the weighted shift W_α on ℓ^2, equipped with the standard orthonormal basis $\{e_n\}_{n=0}^{\infty}$, by $W_\alpha(e_n)=\alpha_n e_{n+1}$ (and extend by linearity). For virtually all questions of interest it is sufficient to take the α_n to be strictly positive, and we do henceforth without further comment. The shift is bounded if the α_n are bounded above. The moment sequence $\gamma=\{\gamma_n\}_{n=0}^{\infty}$ of the shift is defined by $\gamma_0 = 1$ and $\gamma_n = \prod_{i=0}^{n-1} \alpha_i^2$ for $n \geqslant 1$. The subnormality and various related weak subnormalities of such shifts have been studied extensively (see, as a starting point, [6]); for example, the hyponormality of a classical weighted shift is easily seen to be equivalent to a non-decreasing weight sequence. Recall that a subnormal weighted shift W has representing (Berger) probability measure μ supported on $[0, \|W_\alpha\|]$ (see [18]) such that the moments of the measure are the moments of the shift:

$$\gamma_n = \int_{\mathbb{R}} t^n \, d\mu(t), \quad n = 0, 1, \ldots .$$

2.2. Weighted shifts on directed trees

In this section we recall briefly some basic terminology from [21] that will be required in this paper. Let $\mathscr{T} = (V, E)$ be a directed tree, where V and E are the sets of vertices and edges, respectively. A vertex $v \in V$ is the *parent* of u if $(v, u) \in E$, and denoted by $\mathsf{par}(u)$. A vertex of $\mathscr{T}$ which has no parent is called a *root* of $\mathscr{T}$. If $\mathscr{T}$ has a root, we denote it by root and write $V^\circ = V \setminus \{\mathsf{root}\}$. Set $\mathsf{Chi}(u) = \{v \in V \colon (u, v) \in E\}$ for $u \in V$. We call a member of $\mathsf{Chi}(u)$ a *child* of u. We write $V' = \{u \in V : \mathsf{Chi}(u) \neq \varnothing\}$. A vertex $u \in V \setminus V'$ is called a *leaf*. A vertex $v \in V$ is said to be a *descendant* of $u \in V$ if there exists a finite sequence $v_0, \ldots, v_n \in V$ with $n \in \mathbb{Z}_+$ such that $v_0 = v$, $v_n = u$ and $v_{j+1} = \mathsf{par}(v_j)$ for all $j = 0, \ldots, n-1$ (provided $n \geqslant 1$). We let $\mathsf{Des}(V)$ denote the set of all descendants of V.

For a directed tree $\mathscr{T} = (V, E)$, we let $\ell^2(V)$ be the usual Hilbert space of all square summable complex functions on V with the orthonormal basis $\{e_u\}_{u \in V}$ defined by

$$e_u(v) = \begin{cases} 1 & \text{if } v = u, \\ 0 & \text{otherwise,} \end{cases} \qquad v \in V.$$

For a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, the map $\Lambda_{\mathscr{T}}$ is defined on functions $f : V \to \mathbb{C}$ by

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases}$$

Then we can define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ with domain

$$\mathscr{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) : \Lambda_{\mathscr{T}} f \in \ell^2(V)\}$$

by

$$S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f, \qquad f \in \mathscr{D}(S_{\boldsymbol{\lambda}}).$$

The operator $S_{\boldsymbol{\lambda}}$ is called a *weighted shift* on the directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$ ([21]). In particular, if $S_{\boldsymbol{\lambda}} \in \mathscr{B}(\ell^2(V))$, then

$$S_{\boldsymbol{\lambda}} e_u = \sum_{v \in \mathsf{Chi}(u)} \lambda_v e_v, \quad u \in V, \quad \text{and} \quad \|S_{\boldsymbol{\lambda}}\| = \Bigl(\sup_{u \in V} \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \Bigr)^{1/2}.$$

(See [21] for more information concerning this notion.) Recall that a weighted shift $S_{\boldsymbol{\lambda}}$ has the unitary equivalence property ([21, Theorem 3.2.1]), and also that if $\lambda_u = 0$ for some $u \in V^\circ$, then $S_{\boldsymbol\lambda}$ can be decomposed into two nonzero weighted shifts on subtrees of $\mathscr{T}$ ([21, Theorem 3.1.6]). To study the structure of $S_{\boldsymbol\lambda}$, we therefore usually consider positive real values for the weights $\{\lambda_v\}_{v\in V^\circ}$ of $S_{\boldsymbol\lambda}$.

We now introduce a particular directed tree with one branching vertex which is the main model of this paper. Given $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$, we define the directed tree $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by (see Figure 1) [1]

$$\begin{aligned} V_{\eta,\kappa} &= \{-k\colon k \in J_\kappa\} \sqcup \{0\} \sqcup \{(i,j)\colon i \in J_\eta, j \in \mathbb{N}\},\\ E_\kappa &= \{(-k,-k+1)\colon k \in J_\kappa\},\\ E_{\eta,\kappa} &= E_\kappa \sqcup \{(0,(i,1))\colon i \in J_\eta\} \sqcup \{((i,j),(i,j+1))\colon i \in J_\eta, j \in \mathbb{N}\}. \end{aligned} \tag{1}$$

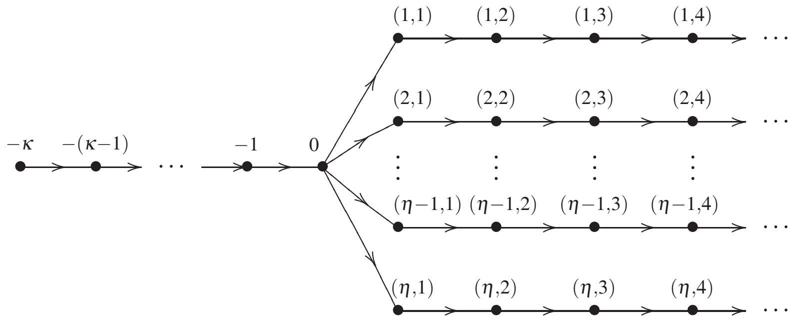

Figure 1: *Description of the directed tree* $\mathscr{T}_{\eta,\kappa}$.

Throughout this paper we only deal with bounded weighted shifts $S_{\boldsymbol\lambda}$ *on directed trees* $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ *with positive weights* $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$, *where* $\eta \in \mathbb{N}_2$ *and* $\kappa \in \mathbb{Z}_+$ *, unless we specify otherwise.*

2.3. Slicing trees and branching shifts

Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$. For $S_{\boldsymbol\lambda}$ we first recall the definition of branching shifts for the discussion of Q1.

DEFINITION 1. ([9]) Let $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ be the directed tree as in Figure 1 and let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$. In what follows we assume $\kappa \in \mathbb{Z}_+$ and $\eta \in \mathbb{N}_2$. We consider the i-th branching shifts $W^{(i)}$ which are sliced from the weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ as follows: let $W^{(i)}$ be the classical weighted shift with the weight sequence $\boldsymbol\alpha^{(i)}$ given by

$$\boldsymbol\alpha^{(i)}:\ \lambda_{i,2},\ \lambda_{i,3},\ \lambda_{i,4},\ \lambda_{i,5}, \ldots, \quad i \in J_\eta,$$

[1] The notation "$\sqcup$" denotes the pairwise disjoint union of sets.

under the order of branches as in Figure 2. As well, let $W^{(0)}$ be the classical weighted shift with the weight sequence $\boldsymbol{\alpha}^{(0)} = \{\alpha_j^{(0)}\}_{j=-\kappa+1}^{\infty}$ given by

$$\alpha_j^{(0)} = \lambda_j, \ \ j \in (-J_{\kappa-1}) \cup \{0\}, \text{ provided } \kappa \in \mathbb{N}, \tag{2}$$

$$\alpha_1^{(0)} := \Big(\sum_{i \in J_\eta} \lambda_{i,1}^2\Big)^{1/2}, \ \alpha_{j+1}^{(0)} := \left(\frac{\sum\limits_{i \in J_\eta} \prod\limits_{k \in J_{j+1}} \lambda_{i,k}^2}{\sum\limits_{i \in J_\eta} \prod\limits_{k \in J_j} \lambda_{i,k}^2} \right)^{1/2}, \quad j \in \mathbb{N}. \tag{3}$$

We say that $W^{(0)}$ is the *basic* (*sliced*) *branching shift* of $S_{\boldsymbol{\lambda}}$. For our convenience, we say that "$W^{(i)}$ is the *i-th* (*sliced*) *branching shift* of $S_{\boldsymbol{\lambda}}$ for $i \in J_\eta \cup \{0\}$".

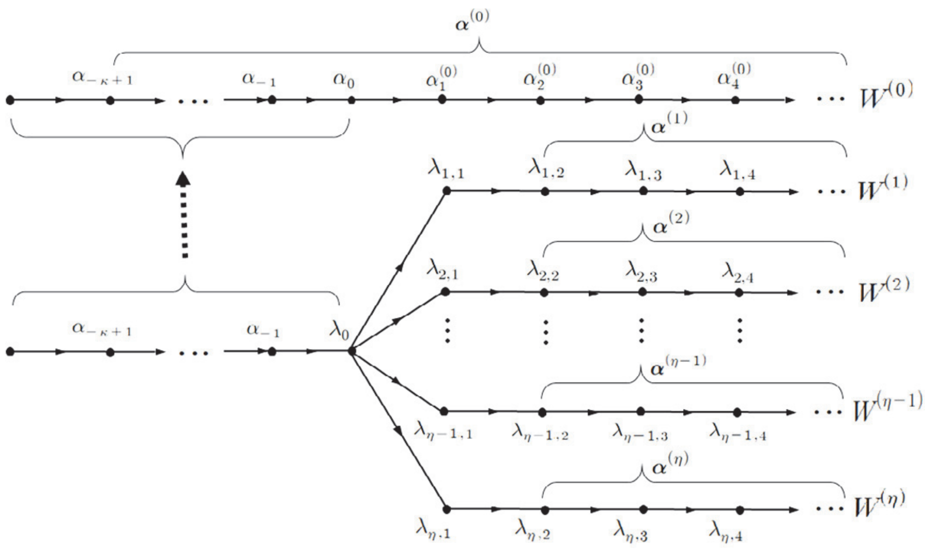

Figure 2: *The illustration of* $W^{(i)}$ *of* $S_{\boldsymbol{\lambda}}$ *for* $i \in J_\eta \cup \{0\}$.

2.4. Collapsing trees and collapsed shifts

Suppose $\eta \in \mathbb{N}_2$ and $\kappa \in \mathbb{Z}_+$. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^{\circ}}$. As a concept opposite to that of "slicing tree", we consider the collapsed tree of $\mathscr{T}_{\eta,\kappa}$ as in Figure 3, and introduce a new definition of the collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ with weights $\{\widetilde{\lambda}_v\}_{v \in V_{\eta-1,\kappa}^{\circ}}$ as in Definition 2. Consider a tree and operator $S_{\widetilde{\boldsymbol{\lambda}}}$ with weights as in Figure 3.

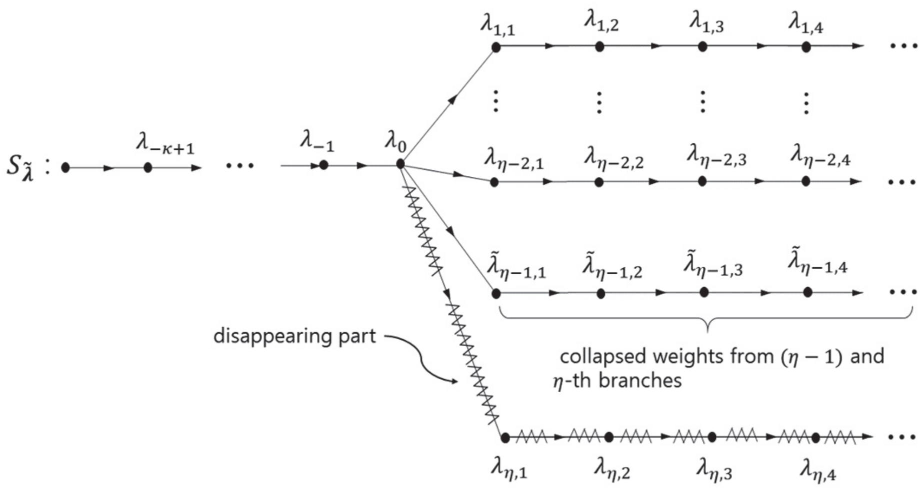

Figure 3: *The illustration of the collapsing tree* $\mathscr{T}_{\eta-1,\kappa}$ *with weights of* $\{\widetilde{\lambda}_v\}_{v\in V^\circ_{\eta-1,\kappa}}$.

DEFINITION 2. Let $S_{\widetilde{\boldsymbol{\lambda}}}$ be the weighted shift on the directed tree $\mathscr{T}_{\eta-1,\kappa}$ with weights $\boldsymbol{\lambda} = \{\widetilde{\lambda}_v\}_{v\in V^\circ_{\eta-1,\kappa}}$ which are given by

$$\widetilde{\lambda}_{\eta-1,1} := \left(\lambda^2_{\eta-1,1} + \lambda^2_{\eta,1}\right)^{1/2}, \tag{4}$$

$$\widetilde{\lambda}_{\eta-1,j} := \left(\frac{\sum\limits_{i=\eta-1}^{\eta} \prod\limits_{k\in J_j} \lambda^2_{i,k}}{\sum\limits_{i=\eta-1}^{\eta} \prod\limits_{k\in J_{j-1}} \lambda^2_{i,k}} \right)^{1/2}, \qquad j \in \mathbb{N}_2, \tag{5}$$

$$\widetilde{\lambda}_v := \lambda_v, \qquad \text{otherwise.} \tag{6}$$

We say that $S_{\widetilde{\boldsymbol{\lambda}}}$ is the *first-step collapsed weighted shift* of $S_{\boldsymbol{\lambda}}$. Collapsing the $(\eta-1)$-th branch with weights $\{\widetilde{\lambda}_{\eta-1,j}\}_{j\in\mathbb{N}}$ and the $(\eta-2)$-th branch with weights $\{\lambda_{\eta-2,j}\}_{j\in\mathbb{N}}$ again, we may obtain the *second-step collapsed weighted shift*, say $S_{\widetilde{\boldsymbol{\lambda}}^{(2)}}$, of $S_{\widetilde{\boldsymbol{\lambda}}}$ similarly. Repeating $(\eta-1)$-steps from $S_{\boldsymbol{\lambda}}$, we obtain a classical weighted shift $\widetilde{W} := S_{\widetilde{\boldsymbol{\lambda}}^{(\eta-1)}}$:

$$S_{\boldsymbol{\lambda}} \longrightarrow S_{\widetilde{\boldsymbol{\lambda}}} \longrightarrow S_{\widetilde{\boldsymbol{\lambda}}^{(2)}} \longrightarrow \cdots \longrightarrow S_{\widetilde{\boldsymbol{\lambda}}^{(\eta-1)}} = \widetilde{W}.$$

We say that $\widetilde{W}$ is the *last-step collapsed* (*classical*) *weighted shift* of $S_{\boldsymbol{\lambda}}$.

It is worth mentioning that the last-step collapsed weighted shift $\widetilde{W}$ and basic branching shift $W^{(0)}$ of $S_{\boldsymbol{\lambda}}$ coincide (see Corollary 20).

3. Slicing branching shifts and properties

It follows from [22, Proposition 3.1] that if $S_{\boldsymbol\lambda} \in \mathscr{B}(\ell^2(V))$ is a weighted shift on a directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v\in V^\circ}$, then

> *$S_{\boldsymbol\lambda} \in \mathscr{B}(\ell^2(V))$ is normal if and only if there exists a sequence $\{u_n\}_{n\in\mathbb{Z}} \subset V$ such that $u_{n-1} = \mathsf{par}(u_n)$ and $\left|\lambda_{u_{n-1}}\right| = |\lambda_{u_n}|$ for all $n \in \mathbb{Z}$, and $\lambda_v = 0$ for all $v \in V \setminus \{u_n : n \in \mathbb{Z}\}$.*

The above statement says that no nonzero weighted shift $S_{\boldsymbol\lambda}$ acting on $\mathscr{T}_{\eta,\kappa}$ with $\kappa < \infty$ can be normal, so we study only weak normalities of $S_{\boldsymbol\lambda} \in \mathscr{B}(\ell^2(V_{\eta,\kappa}))$ such as quasinormality, ∞-hyponormality, p-hyponormality $(p > 0)$ and p-paranormality $(p > 0)$.

3.1. Quasinormality

Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$. We first recall a condition equivalent to quasinormality of $S_{\boldsymbol\lambda}$ from [21, Proposition 8.1.7].

P1. *A weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ is quasinormal if and only if $\|S_{\boldsymbol\lambda} e_u\| = \|S_{\boldsymbol\lambda} e_v\|$ for all $u \in V$ and $v \in \mathsf{Chi}(u)$, which is equivalent to the following condition*:

$$\lambda_v^2 = \sum_{i\in J_\eta} \lambda_{i,1}^2, \quad v \in V^\circ_{\eta,\kappa} \setminus \{(i,1)\}_{i\in J_\eta}. \tag{7}$$

PROPOSITION 3. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$. Then $S_{\boldsymbol\lambda}$ is quasinormal if and only if every i-th branching shift $W^{(i)}$ is quasinormal for $i \in J_\eta \cup \{0\}$.*

Proof. Suppose that $S_{\boldsymbol\lambda}$ is quasinormal. By (7), all weights of $W^{(i)}$ are constant for $i \in J_\eta$. Thus $W^{(i)}$ is obviously quasinormal. And now we consider the basic branching shift $W^{(0)}$ with weight sequence $\boldsymbol\alpha^{(0)} = \{\alpha_j^{(0)}\}_{j=-\kappa+1}^{\infty}$ as in (2) and (3). According to P1, we obtain that $\alpha_1^{(0)} = \alpha_j^{(0)}$ for $j \in (-J_{\kappa-1}) \cup \{0\}$ (provided $\kappa \in \mathbb{N}$), and

$$\begin{aligned}\alpha_{j+1}^{(0)} &= \left(\frac{\sum\limits_{i\in J_\eta}\prod\limits_{k\in J_{j+1}} \lambda_{i,k}^2}{\sum\limits_{i\in J_\eta}\prod\limits_{k\in J_j} \lambda_{i,k}^2}\right)^{1/2} \overset{(7)}{=} \left(\frac{\sum\limits_{i\in J_\eta} \lambda_{i,1}^2 (\sum\limits_{i\in J_\eta} \lambda_{i,1}^2)^j}{\sum\limits_{i\in J_\eta} \lambda_{i,1}^2 (\sum\limits_{i\in J_\eta} \lambda_{i,1}^2)^{j-1}}\right)^{1/2} \\ &= \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)^{1/2} = \alpha_1^{(0)}, \quad j \in \mathbb{N},\end{aligned}$$

which shows that $W^{(0)}$ is quasinormal.

Conversely, we suppose that every i-th branching weighted shift $W^{(i)}$ is quasinormal for $i \in J_\eta \cup \{0\}$. Since the weights of $W^{(i)}$ are constant for each $i \in J_\eta \cup \{0\}$, their expressions are given by

$$\lambda_v = \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)^{1/2}, \quad v \in (-J_{\kappa-1}) \cup \{0\},$$

$$\alpha_{j+1}^{(0)} = \left(\frac{\sum\limits_{i\in J_\eta} \prod\limits_{k\in J_{j+1}} \lambda_{i,k}^2}{\sum\limits_{i\in J_\eta} \prod\limits_{k\in J_j} \lambda_{i,k}^2}\right)^{1/2} = \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)^{1/2}, \quad j \in \mathbb{N}, \tag{8}$$

$$\lambda_{i,2} = \lambda_{i,j}, \quad i \in J_\eta, \quad j \in \mathbb{N}_2. \tag{9}$$

By applying (8) with $j = 1,2$, and also (9) with $j = 3$, we have

$$\Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\lambda_{i,2}^2\Big)^2 = \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)\Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\lambda_{i,2}^4\Big), \tag{10}$$

which implies that

$$\begin{aligned}
&\sum_{1\leqslant k<l\leqslant\eta} \lambda_{k,1}^2\lambda_{l,1}^2(\lambda_{l,2}^2 - \lambda_{k,2}^2)^2 \\
&\quad = \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)\Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\lambda_{i,2}^4\Big) - \sum_{i\in J_\eta} \lambda_{i,1}^2\lambda_{i,2}^2\lambda_{i,1}^2\lambda_{i,2}^2 - \sum_{1\leqslant k<l\leqslant\eta} 2\lambda_{k,1}^2\lambda_{k,2}^2\lambda_{l,1}^2\lambda_{l,2}^2 \\
&\quad = \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)\Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\lambda_{i,2}^4\Big) - \Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\lambda_{i,2}^2\Big)^2 \overset{(10)}{=} 0.
\end{aligned}$$

Therefore $\lambda_{1,2} = \lambda_{k,2}$ for all $k \in J_\eta$. Applying these equalities to the right two terms of (8) with $j = 1$, we get $\Big(\sum_{i\in J_\eta} \lambda_{i,1}^2\Big)^{1/2} = \lambda_{k,2}$ for all $k \in J_\eta$. Hence $S_{\boldsymbol\lambda}$ satisfies (7), which completes the proof. $\square$

3.2. p-hyponormality

We now discuss the relationship for p-hyponormality of $S_{\boldsymbol\lambda}$ and its i-th branching shift $W^{(i)}$, $i \in J_\eta \cup \{0\}$. The following theorem answers question Q1 when property P is p-hyponormality $(p > 0)$.

THEOREM 4. *Suppose $p > 0$. If $S_{\boldsymbol\lambda}$ is a p-hyponormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$, then every i-th branching shift $W^{(i)}$ is p-hyponormal for $i \in J_\eta \cup \{0\}$.*

The following corollary comes immediately from applying Theorem 4 with all $p > 0$.

COROLLARY 5. *If $S_{\boldsymbol\lambda}$ is ∞-hyponormal, then every i-th branching shift $W^{(i)}$ is ∞-hyponormal for $i \in J_\eta \cup \{0\}$.*

To prove Theorem 4, we recall a condition equivalent to p-hyponormality of $S_{\boldsymbol{\lambda}}$ from [21, Corollary 8.2.3] as follows.

P2. *Suppose $p>0$. A weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$ is p-hyponormal if and only if the following four conditions hold:*

(i) $\lambda_{-(k+1)} \leqslant \lambda_{-k}$ *for* $k \in J_{\kappa-2}\cup\{0\}$, *if* $\kappa \in \mathbb{N}_2$,

(ii) $\lambda_0^2 \leqslant \sum_{i\in J_\eta} \lambda_{i,1}^2$, *if* $\kappa \in \mathbb{N}$,

(iii) $\left(\sum_{i\in J_\eta} \lambda_{i,1}^2\right)^{p-1}\left(\sum_{i\in J_\eta} \frac{\lambda_{i,1}^2}{\lambda_{i,2}^{2p}}\right) \leqslant 1$,

(iv) $\lambda_{i,j} \leqslant \lambda_{i,j+1}$ *for* $i\in J_\eta$ *and* $j \in \mathbb{N}_2$.

Note that a classical weighted shift is p-hyponormal ($p>0$) if and only if the weights are non-decreasing.

We introduce an elementary inequality for the proof of Theorem 4.

LEMMA 6. *Let a,b, and p be positive real numbers. Then it holds that*

$$(b^p x + a^p(1-x))^{1/p} \geqslant \frac{ab}{ax+b(1-x)}, \quad 0 \leqslant x \leqslant 1.$$

Proof. If $a=b$, then the result is obvious. Without loss of generality, we assume that $a>b$. Define a real function f on $[0,1]$ by

$$f(x) = (b^p x + a^p(1-x))(ax+b(1-x))^p - a^p b^p.$$

We will claim $f(x) \geqslant 0$ for x in $[0,1]$. Differentiating the function f, we can obtain that

$$F(x) := \frac{d}{dx} f(x) = (b+(a-b)x)^{p-1}\cdot(Ax+B)$$

with

$$A = -(a-b)(a^p-b^p)(1+p);\ B = (b^p-a^p)b + (a-b)pa^p.$$

Observe that $f(0)=f(1)=0$. For our purpose, we fix $b>0$ and $p>0$ and consider two real valued functions ϕ and ψ on $\mathbb{R}_+$ defined by $\phi(a)=B$ and $\psi(a)=A+B$, where we now view a as an independent variable. Some elementary computations show that

$$\phi(b)=\psi(b)=0,\ \frac{d}{da}\phi(a) = p(p+1)a^{p-1}(a-b),$$

$$\text{and } \frac{d}{da}\psi(a) = -(p+1)(a^p-b^p),$$

which implies that $\phi(a)>0$ and $\psi(a)<0$ for $a>b$, and so $F(0)>0$ and $F(1)<0$. Since f has the unique critical point on $(0,1)$ at $x=-\frac{B}{A}$, we can see that $f(x)\geqslant 0$ on $[0,1]$. The proof is complete. $\square$

Before proving Theorem 4, we consider first the case $\eta=2$.

PROPOSITION 7. *Suppose $p>0$. If $S_{\boldsymbol{\lambda}}$ is a p-hyponormal weighted shift on $\mathscr{T}_{2,\kappa}$ with weights $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{2,\kappa}^{\circ}}$, then every i-th branching shift $W^{(i)}$ is p-hyponormal, $i=0,1,2$.*

Proof. Since $S_{\boldsymbol{\lambda}}$ is p-hyponormal, the conditions (i)–(iv) of P2 hold. By P2(iv), it is obvious that $W^{(1)}$ and $W^{(2)}$ are p-hyponormal. Recall that $W^{(0)}$ is p-hyponormal if and only if

$$\lambda_{-\kappa+1}\leqslant\lambda_{-\kappa+2}\leqslant\cdots\leqslant\lambda_0, \tag{11}$$

$$\lambda_0^2\leqslant\lambda_{1,1}^2+\lambda_{2,1}^2, \tag{12}$$

$$(\lambda_{1,1}^2+\lambda_{2,1}^2)^2\leqslant\lambda_{1,1}^2\lambda_{1,2}^2+\lambda_{2,1}^2\lambda_{2,2}^2, \tag{13}$$

$$\Big(\sum_{i\in J_2}\prod_{k\in J_{j+1}}\lambda_{i,k}^2\Big)^2\leqslant\Big(\sum_{i\in J_2}\prod_{k\in J_j}\lambda_{i,k}^2\Big)\Big(\sum_{i\in J_2}\prod_{k\in J_{j+2}}\lambda_{i,k}^2\Big),\quad j\in\mathbb{N}. \tag{14}$$

Clearly, (11) [resp., (12)] is a condition equivalent to P2(i) [resp., P2(ii)] for $\eta=2$. Applying the Cauchy-Schwarz inequality (with $\prod_{k\in J_j}\lambda_{i,k}$ and $\lambda_{i,j+1}\prod_{k\in J_{j+1}}\lambda_{i,k}$) and using P2(iv), we see that the inequality (14) holds. The only question is whether we may obtain (13).

Observe first that for any $\theta\in\mathbb{C}\setminus\{0\}$, $T\in\mathscr{B}(\mathscr{H})$ is p-hyponormal if and only if θT is p-hyponormal, and check that conditions (i)–(iv) of P2 are unaffected by scaling; the only one not completely obvious is P2(iii). Obviously, $W^{(0)}$ is p-hyponormal if and only if $\theta W^{(0)}$ is p-hyponormal for $\theta>0$. So our first step is to scale the weights of $W^{(0)}$ so that

$$\alpha_1^{(0)}=(\lambda_{1,1}^2+\lambda_{2,1}^2)^{1/2}=1.$$

Then P2(iii) becomes

$$\frac{\lambda_{1,1}^2}{\lambda_{1,2}^{2p}}+\frac{\lambda_{2,1}^2}{\lambda_{2,2}^{2p}}\leqslant 1 \tag{15}$$

and (13) becomes

$$1\leqslant\lambda_{1,1}^2\lambda_{1,2}^2+\lambda_{2,1}^2\lambda_{2,2}^2. \tag{16}$$

Lemma 6, with $a=\lambda_{1,2}^2$, $b=\lambda_{2,2}^2$ and $x=\lambda_{1,1}^2$ (so $1-x=\lambda_{2,1}^2$), says that

$$\frac{1}{(\lambda_{1,1}^2\lambda_{1,2}^2+\lambda_{2,1}^2\lambda_{2,2}^2)^p}\leqslant\frac{\lambda_{1,1}^2}{\lambda_{1,2}^{2p}}+\frac{\lambda_{2,1}^2}{\lambda_{2,2}^{2p}}. \tag{17}$$

Using (15) and (17), we can obtain the inequality in (16), and thus (13) holds. Hence the proof is complete. $\square$

The proof of Theorem 4 which is generalized from Proposition 7 will appear in Subsection 4.3.

We now give a useful equivalent condition for ∞-hyponormality of $S_{\boldsymbol{\lambda}}$ which will be used later in the paper.

PROPOSITION 8. *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^{\circ}_{\eta,\kappa}}$. Then $S_{\boldsymbol{\lambda}}$ is ∞-hyponormal if and only if the conditions* (i), (ii) *and* (iv) *of P2 hold, and also the following inequality holds:*

$$c := \sum_{i \in J_\eta} \lambda_{i,1}^2 \leqslant \min_{i \in J_\eta} \{\lambda_{i,2}^2\}. \tag{18}$$

Proof. It is enough to show that condition P2(iii) for all $p > 0$ is equivalent to condition (18). Suppose P2(iii) holds for any $p > 0$. For the contrary, we suppose that $c > \lambda_{k,2}^2$ for some $k \in J_\eta$. Take $p > 0$ satisfying $\frac{\lambda_{k,1}^2}{c}\left(\frac{c}{\lambda_{k,2}^2}\right)^p > 1$. Then we can see that

$$\begin{aligned}\Big(\sum_{i \in J_\eta} \lambda_{i,1}^2\Big)^{p-1} \left(\sum_{i \in J_\eta} \frac{\lambda_{i,1}^2}{\lambda_{i,2}^{2p}}\right) &= \sum_{i \in J_\eta} \frac{\lambda_{i,1}^2}{c}\left(\frac{c}{\lambda_{i,2}^2}\right)^p \\ &= \sum_{i \in J_\eta \setminus \{k\}} \frac{\lambda_{i,1}^2}{c}\left(\frac{c}{\lambda_{i,2}^2}\right)^p + \frac{\lambda_{k,1}^2}{c}\left(\frac{c}{\lambda_{k,2}^2}\right)^p > 1,\end{aligned}$$

which contradicts P2(iii).

Conversely, suppose that $c \leqslant \lambda_{i,2}^2$ for all $i \in J_\eta$. Obviously $\left(\frac{c}{\lambda_{i,2}^2}\right)^p \leqslant 1$ for all $i \in J_\eta$ and $p > 0$. Then

$$\Big(\sum_{i \in J_\eta} \lambda_{i,1}^2\Big)^{p-1} \left(\sum_{i \in J_\eta} \frac{\lambda_{i,1}^2}{\lambda_{i,2}^{2p}}\right) = \sum_{i \in J_\eta} \frac{\lambda_{i,1}^2}{c}\left(\frac{c}{\lambda_{i,2}^2}\right)^p \leqslant \sum_{i \in J_\eta} \frac{\lambda_{i,1}^2}{c} = 1, \quad p > 0,$$

i.e., P2(iii) holds for all $p > 0$. Hence the proof is complete. □

3.3. p-paranormality

We discuss the relationship of p-paranormality between the weighted shift $S_{\boldsymbol{\lambda}}$ and its i-th branching shift $W^{(i)}$, $i \in J_\eta \cup \{0\}$, in this subsection. The following condition equivalent to p-paranormality of $S_{\boldsymbol{\lambda}}$ comes from [10, Theorem 6.5].

P3. *Suppose that $p > 0$. A weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^{\circ}_{\eta,\kappa}}$ is p-paranormal if and only if*

$$\sum_{v \in \mathsf{Chi}(u)} \lambda_v^2 \|S_{\boldsymbol{\lambda}} e_v\|^{2p} \geqslant \|S_{\boldsymbol{\lambda}} e_u\|^{2p+2}, \qquad u \in V_{\eta,\kappa},$$

which is equivalent to the three conditions (i), (ii), *and* (iv) *of P2, and with the further inequality:*

$$\Big(\sum_{i \in J_\eta} \lambda_{i,1}^2\Big)^{p+1} \leqslant \sum_{i \in J_\eta} \lambda_{i,1}^2 \lambda_{i,2}^{2p}. \tag{19}$$

Recall that p-paranormality for classical weighted shifts reduces to monotonicity of weights for $p > 0$.

We answer Q1 when property P is p-paranormality for $0 < p \leqslant 1$.

PROPOSITION 9. *Let $S_{\boldsymbol{\lambda}}$ and $W^{(i)}$ $(i \in J_\eta \cup \{0\})$ be as usual. Suppose $0 < p \leqslant 1$. If $S_{\boldsymbol{\lambda}}$ is p-paranormal, then the i-th branching shift $W^{(i)}$ is p-paranormal for $i \in J_\eta \cup \{0\}$.*

Proof. It holds obviously that every $W^{(i)}$ is p-paranormal ($p > 0$) for $i \in J_\eta \cup \{0\}$ if and only if three conditions (i), (ii) and (iv) of P2 hold as well as the following:

$$\Big(\sum_{i \in J_\eta} \lambda_{i,1}^2\Big)^2 \leqslant \sum_{i \in J_\eta} \lambda_{i,1}^2 \lambda_{i,2}^2. \tag{20}$$

This means that the above equivalent conditions for p-paranormality of $W^{(i)}$, $i \in J_\eta \cup \{0\}$, coincide with the equivalent conditions for 1-paranormality of $S_{\boldsymbol{\lambda}}$. Thus, if $S_{\boldsymbol{\lambda}}$ is p-paranormal for $0 < p \leqslant 1$ (therefore it is 1-paranormal), then it is obvious that every i-th branching shift $W^{(i)}$, $i \in J_\eta \cup \{0\}$, is p-paranormal. Hence the proof is complete. □

Note that if $p > 1$ in Proposition 9, the above statement is no longer true: see Subsection 3.4.

In the proof of Proposition 9, we can see that $S_{\boldsymbol{\lambda}}$ is 1-paranormal if and only if $W^{(i)}$ is p-paranormal, $i \in J_\eta \cup \{0\}$, for any [some] $p > 0$. Hence we obtain the following remark.

REMARK 10. Suppose $p \geqslant 1$. If every i-th branching shift $W^{(i)}$, $i \in J_\eta \cup \{0\}$, is p-paranormal, then $S_{\boldsymbol{\lambda}}$ is 1-paranormal, hence p-paranormal. However this assertion is not true in the case of $0 < p < 1$: see Subsection 3.4.

The following comes immediately from Proposition 9 and Remark 10.

COROLLARY 11. *Let $S_{\boldsymbol{\lambda}}$ and $W^{(i)}$ $(i \in J_\eta \cup \{0\})$ be as usual. Then $S_{\boldsymbol{\lambda}}$ is paranormal if and only if every i-th branching shift $W^{(i)}$ is paranormal for $i \in J_\eta \cup \{0\}$.*

3.4. Examples for relationships

In the previous subsections, we discussed some relationships between the two conditions below:

(C_1) $S_{\boldsymbol{\lambda}}$ has property P,

(C_2) $W^{(i)}$ has property P for all $i \in J_\eta \cup \{0\}$.

In this subsection we discuss the implications between (C_1) and (C_2) with some explicit examples.

Consider a weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{2,1}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{2,1}^\circ}$ such that $\lambda_0 = 1$, $\lambda_{1,1} = \sqrt{x}$, $\lambda_{2,1} = \sqrt{y}$ and $\lambda_{1,j} = \sqrt{u}$, $\lambda_{2,j} = \sqrt{v}$ for $j \in \mathbb{N}_2$, where x, y, u, and v are positive real variables. We denote this shift, here and subsequently, by $S_{\boldsymbol{\lambda}}(u,v,x,y)$; further, let $W^{(0)}(u,v,x,y)$ be the associated basic (sliced) branching shift of $S_{\boldsymbol{\lambda}}(u,v,x,y)$. According to Definition 1, we obtain the following sequences $\boldsymbol{\alpha}^{(i)}$, $i=0,1,2$:

$$\begin{aligned}
&\boldsymbol{\alpha}^{(0)}: 1, \sqrt{x+y}, \sqrt{\frac{ux+vy}{x+y}}, \sqrt{\frac{u^2x+v^2y}{ux+vy}}, \sqrt{\frac{u^3x+v^3y}{u^2x+v^2y}}, \cdots, \qquad (21)\\
&\boldsymbol{\alpha}^{(1)}: \sqrt{u}, \sqrt{u}, \sqrt{u}, \cdots,\\
&\boldsymbol{\alpha}^{(2)}: \sqrt{v}, \sqrt{v}, \sqrt{v}, \cdots.
\end{aligned}$$

Using the equivalent conditions in the previous subsections, we discuss operator properties of this weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{2,1}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{2,1}^{\circ}}$.

Quasinormality. By P1, we obtain easily that

(i) $S_{\boldsymbol{\lambda}}$ is quasinormal $\Leftrightarrow 1 = u = v = x+y$,

(ii) $W^{(0)}$ is quasinormal $\Leftrightarrow 1 = u = v = x+y$, i.e., $W^{(0)}$ is the unilateral shift of multiplicity one. Note that $W^{(1)}$ and $W^{(2)}$ are always quasinormal.

Subnormality. Consider $\mu_1 = \delta_u$ and $\mu_2 = \delta_v$, where $\delta_x := \delta_{\{x\}}$ denotes the usual Dirac measure. Obviously the measure μ_i above is the representing Berger measure for the branching shift $W^{(i)}$, $i = 1,2$, respectively. To find equivalent conditions for subnormality of the basic branching shift $W^{(0)}$, we first assume that $W^{(0)}$ is subnormal. Consider

$$\alpha' : \sqrt{\frac{ux+vy}{x+y}}, \sqrt{\frac{u^2x+v^2y}{ux+vy}}, \sqrt{\frac{u^3x+v^3y}{u^2x+v^2y}}, \cdots,$$

and let $W_{\alpha'}$ be the weighted shift corresponding to the weight sequence α'. Then $W_{\alpha'}$ is a bounded subnormal weighted shift with the corresponding Berger measure $\mu = \frac{x}{x+y}\delta_u + \frac{y}{x+y}\delta_v$. Since $W^{(0)}$ is a 2-step backward subnormal extension of $W_{\alpha'}$, it follows from [7, Theorem 3.5] (see also [8, Theorem 5.3] and [21, Corollary 6.2.2]) that

$$\int_{\mathbb{R}_+} \frac{1}{t}\, d\mu = 1 \quad \text{and} \quad \int_{\mathbb{R}_+} \frac{1}{t^2}\, d\mu \leqslant 1,$$

which implies that $\frac{x}{u^2} + \frac{y}{v^2} \leqslant 1$ and $\frac{x}{u} + \frac{y}{v} = 1$. Conversely, if the two conditions just before this sentence hold, the measure ν given by

$$\nu(\sigma) = \left(1 - \left(\frac{x}{u^2} + \frac{y}{v^2}\right)\right)\delta_0(\sigma) + \frac{x}{u^2}\delta_u(\sigma) + \frac{y}{v^2}\delta_v(\sigma), \qquad \sigma \in \mathscr{B}(\mathbb{R}_+),$$

where $\mathscr{B}(\mathbb{R}_+)$ is the family of Borel subsets of $\mathbb{R}_+$, is the Berger measure associated to $W^{(0)}$, which can be confirmed by computing the following moment equations:

$$\gamma_n = \int_{\mathbb{R}_+} t^n d\nu(t) = \begin{cases} 1, & n = 0,\\ 1, & n = 1,\\ u^{n-2}x + v^{n-2}y, & n \geqslant 2. \end{cases}$$

Therefore we can see that the following assertion holds.

(i) $W^{(0)}$ is subnormal if and only if $\frac{x}{u^2} + \frac{y}{v^2} \leqslant 1$ and $\frac{x}{u} + \frac{y}{v} = 1$.

Observe that the Borel probability measures μ_1, μ_2 and ν satisfy Corollary 6.2.2 (ii-b) in [21]. Thus we obtain the following assertion.

(ii) $S_{\boldsymbol{\lambda}}$ is subnormal if and only if $\frac{x}{u^2} + \frac{y}{v^2} \leqslant 1$ and $\frac{x}{u} + \frac{y}{v} = 1$.

p-hyponormality. According to P2 and Proposition 8, we get the following assertions.

(i) For $p > 0$, $S_{\boldsymbol{\lambda}}$ is p-hyponormal if and only if $1 \leqslant x + y$ and $\frac{x}{u^p} + \frac{y}{v^p} \leqslant (x+y)^{1-p}$.

(ii) $S_{\boldsymbol{\lambda}}$ is ∞-hyponormal if and only if $1 \leqslant x + y \leqslant \min\{u, v\}$.

(iii) For $p > 0$, $W^{(0)}$ is p-hyponormal if and only if $1 \leqslant x + y$ and $ux + vy \geqslant (x+y)^2$. Recall that every classical hyponormal weighted shift is p-hyponormal for any $p \in (0, \infty) \cup \{\infty\}$.

To show that the converse implication of the statement in Proposition 7 is not true, we consider $S_{\boldsymbol{\lambda}} = S_{\boldsymbol{\lambda}}(u, v, x, y)$ and $W^{(0)} = W^{(0)}(u, v, x, y)$ as at the start of Subsection 3.4 with $x = y = 1$. Then we obtain the following:

(i$'$) for $p > 0$, $S_{\boldsymbol{\lambda}}(u, v, 1, 1)$ is p-hyponormal if and only if

$$v \geqslant \frac{u}{\left(2(\frac{u}{2})^p - 1\right)^{1/p}},$$

(ii$'$) $S_{\boldsymbol{\lambda}}(u, v, 1, 1)$ is ∞-hyponormal if and only if $2 \leqslant \min\{u, v\}$,

(iii$'$) $W^{(0)}(u, v, 1, 1)$ is p-hyponormal if and only if $v \geqslant 4 - u$.

By (i$'$), (ii$'$) and (iii$'$), we obtain Figure 4 and may confirm that the converse implication of the statement of Proposition 7 is not true.

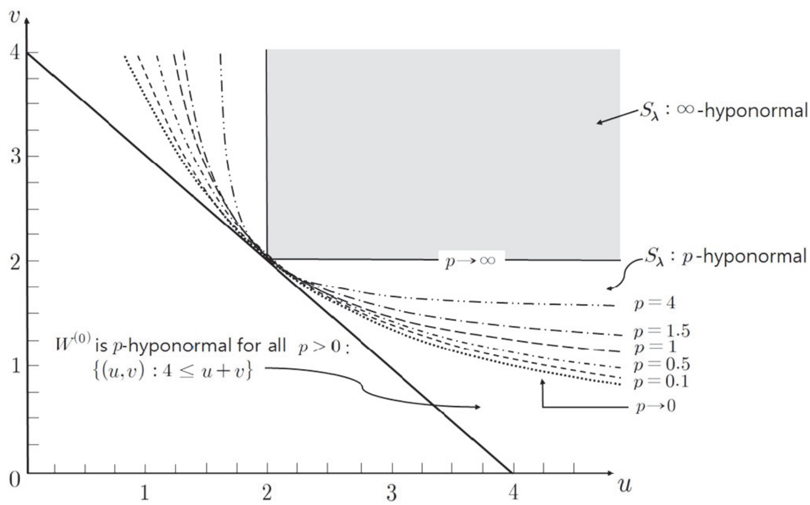

Figure 4: *Regions of p-hyponormality of $S_{\boldsymbol{\lambda}}$ and $W^{(0)}$ when $x = y = 1$.*

p-paranormality. Using P3, we can see that the following statements hold.

(i) For $p > 0$, $S_{\boldsymbol{\lambda}}$ is p-paranormal if and only if $1 \leqslant x + y$ and $u^p x + v^p y \geqslant (x+y)^{p+1}$.

(ii) For $p>0$, $W^{(0)}$ is p-paranormal if and only if $W^{(0)}$ is p-hyponormal, or equivalently $1 \leqslant x+y$ and $ux+vy \geqslant (x+y)^2$.

Again with $S_{\boldsymbol{\lambda}} = S_{\boldsymbol{\lambda}}(u,v,x,y)$ and $W^{(0)} = W^{(0)}(u,v,x,y)$, to show that

$$(C_2) \not\Rightarrow (C_1) \text{ for } 0<p<1 \text{ and } (C_1) \not\Rightarrow (C_2) \text{ for } 1<p<\infty \text{ when property } P \text{ is } p\text{-paranormality,} \tag{22}$$

we consider $S_{\boldsymbol{\lambda}}(u,v,1,1)$. Then we obtain

(i′) for $p>0$, $S_{\boldsymbol{\lambda}}(u,v,1,1)$ is p-paranormal if and only if $v \geqslant \left(2^{p+1}-u^p\right)^{1/p}$,

(ii′) for $p>0$, $W^{(0)}(u,v,1,1)$ is p-paranormal if and only if $W^{(0)}$ is p-hyponormal, or equivalently $v \geqslant 4-u$.

The regions of p-paranormality of $S_{\boldsymbol{\lambda}}(u,v,1,1)$ are described in Figure 5. One can find many counterexamples in Figure 5 to conclude as in (22).

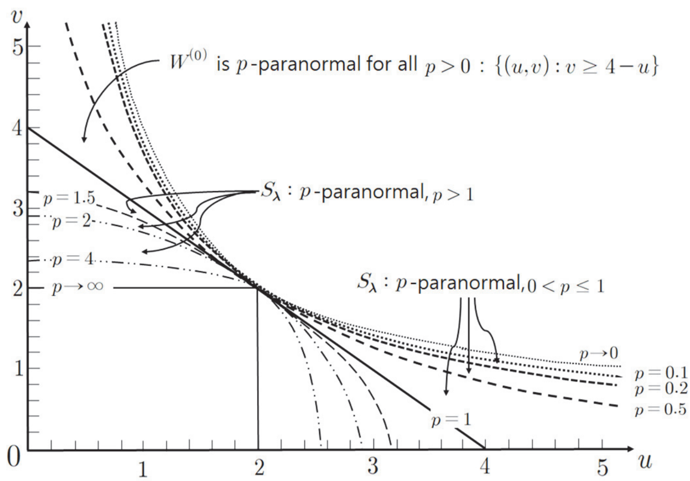

Figure 5: *Regions of p-paranormality of $S_{\boldsymbol{\lambda}}$ and $W^{(0)}$ with $x=y=1$.*

Summary. Summarizing results for solutions of Q1 in Section 3, we organize them in a table.

Property P	$(C_1) \Rightarrow (C_2)$	$(C_2) \Rightarrow (C_1)$
quasinormal	True	True
subnormal	True	True
∞-hyponormal	True	False
p-hyponormal $(p>0)$	True	False
p-paranormal $(0<p<1)$	True	False
1-paranormal	True	True
p-paranormal $(1<p<\infty)$	False	True

Table 3.1.

3.5. Remarks

There are various classes of weak hyponormal operators other than the operator classes that are considered above, such as absolutely p-paranormal, class $A(p)$, and normaloid operators in $\mathscr{B}(\mathscr{H})$. An operator $T \in \mathscr{B}(\mathscr{H})$ is a *class* $A(p)$ *operator* if $(T^*|T|^{2p}T)^{1/(p+1)} \geqslant |T|^2$ for $p > 0$, where $|T| = (T^*T)^{1/2}$. For $p > 0$, an operator T is *absolutely* p*-paranormal* if $\||T|^p Th\| \geqslant \|Th\|^{p+1}$ for all unit vectors $h \in \mathscr{H}$. It is well-known that the following implications hold for any $p > 0$ (see [17], [25]):

- p-hyponormal $\Rightarrow$ class $A(p)$ $\Rightarrow$ absolutely p-paranormal $\Rightarrow$ normaloid;
- p-paranormal $\Rightarrow$ absolutely p-paranormal (when $0 < p < 1$);
- class $A(p)$ $\Rightarrow$ absolutely p-paranormal $\Rightarrow$ p-paranormal (when $1 < p < \infty$);

the relationships among these classes have been studied by several operator theorists (see [4], [5], [11], [14], [15], [17], [25], etc.). The following remark provides information about these operator properties of $S_{\boldsymbol{\lambda}} \in \mathscr{B}(\ell^2(V_{\eta,\kappa}))$.

REMARK 12. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$. It follows from [10, Remark 6.6] that $S_{\boldsymbol{\lambda}}$ is p-paranormal if and only if $S_{\boldsymbol{\lambda}}$ is absolutely p-paranormal, or equivalently that $S_{\boldsymbol{\lambda}}$ is a class $A(p)$ operator for $p > 0$. Thus $S_{\boldsymbol{\lambda}}$ is an absolutely p-paranormal [or, a class $A(p)$] operator if and only if $\sum_{v\in\mathsf{Chi}(u)} \lambda_v^2 \|S_{\boldsymbol{\lambda}} e_v\|^{2p} \geqslant \|S_{\boldsymbol{\lambda}} e_u\|^{2p+2}$, $u \in V_{\eta,\kappa}$.

Recall that the largest class among classes of operators mentioned in the diagram in Section 1 is that of normaloid operators. It is natural to study whether $S_{\boldsymbol{\lambda}}$ is normaloid. The following remark provides some information to characterize $S_{\boldsymbol{\lambda}}$ normaloid.

REMARK 13. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T} = (V,E)$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$. To characterize $S_{\boldsymbol{\lambda}}$ normaloid we will compare $\|S_{\boldsymbol{\lambda}}^n\|$ and $\|S_{\boldsymbol{\lambda}}\|^n$ for $n \in \mathbb{N}$. It follows from [21, Lemma 6.1.1] that

$$S_{\boldsymbol{\lambda}}^n e_u = \sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} \lambda_{u|v} e_v, \qquad u \in V,\ n \in \mathbb{Z}_+, \tag{23}$$

where

$$\lambda_{u|v} = \begin{cases} 1, & \text{if } v = u, \\ \prod_{j=0}^{n-1} \lambda_{\mathsf{par}^j(v)}, & \text{if } v \in \mathsf{Chi}^{\langle n\rangle}(u),\quad n \geqslant 1. \end{cases}$$

Set $\widehat{C}_n := \sup_{u\in V} \sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} |\lambda_{u|v}|^2$ for $n \in \mathbb{N}$. To obtain a standard formula for $\|S_{\boldsymbol{\lambda}}^n\|$, take $f \in \ell^2(V)$. Since $f = \sum_{u\in V} f(u) e_u$, by (23) we get

$$\begin{aligned} \|S_{\boldsymbol{\lambda}}^n f\|^2 &= \sum_{u\in V} |f(u)|^2 \|S_{\boldsymbol{\lambda}}^n e_u\|^2 = \sum_{u\in V} \Big(\sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} |\lambda_{u|v}|^2 \Big) |f(u)|^2 \\ &\leqslant \widehat{C}_n \sum_{u\in V} |f(u)|^2 = \widehat{C}_n \|f\|_{\ell^2(V)}^2, \end{aligned}$$

which implies that $\left\|S_{\boldsymbol\lambda}^n\right\|^2 \leqslant \widehat{C}_n$ for all $n \in \mathbb{Z}_+$. By a method similar to that in the proof of [21, Lemma 3.18], we can see that $\left\|S_{\boldsymbol\lambda}^n\right\|^2 = \widehat{C}_n$, for all $n \in \mathbb{Z}_+$. Therefore $S_{\boldsymbol\lambda}$ is normaloid if and only if $\widehat{C}_n = \widehat{C}_1^n$ for all $n \in \mathbb{N}$, i.e.,

$$\sup_{u\in V} \sum_{v\in \mathsf{Chi}^{\langle n\rangle}(u)} |\lambda_{u|v}|^2 = \sup_{u\in V}\Big(\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2\Big)^n, \quad n \in \mathbb{Z}_+. \tag{24}$$

Applying (24) to the weighted shifts $S_{\boldsymbol\lambda}$ on directed trees $\mathscr{T}_{\eta,\kappa}$ and by direct computation, we obtain an equivalent condition for $S_{\boldsymbol\lambda}$ normaloid as follows.

PROPOSITION 14. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ as usual. For brevity, we set*

$$\widehat{a}_{n,\eta} = \sup_{\substack{i\in J_\eta\\ j\in\mathbb{N}}} \prod_{k\in J_n} \lambda_{i,j+k}^2, \quad n\in\mathbb{N}, \tag{25}$$

$$\widehat{b}_{n,\eta,\kappa} = \begin{cases} \max\limits_{0\leqslant k\leqslant \kappa-1} \lambda_{-k}^2, & \text{if } n=1,\\ \max\limits_{0\leqslant l\leqslant\kappa-1} \Omega_{n,l}, & \text{if } 1\leqslant\kappa<n,\\ \max\Big\{\max\limits_{0\leqslant l\leqslant \kappa-n}\prod\limits_{k=0}^{n-1}\lambda_{-k-l}^2,\ \max\limits_{0\leqslant l\leqslant n-2}\Omega_{n,l}\Big\}, & \text{if } 2\leqslant n\leqslant\kappa, \end{cases} \tag{26}$$

where $\Omega_{n,l} := \prod_{k=0}^{l}\lambda_{-k}^2\Big(\sum_{i\in J_\eta}\prod_{j\in J_{n-1-l}}\lambda_{i,j}^2\Big)$. *Define a sequence* $\{\widehat{C}_{n,\eta,\kappa}\}_{n\in\mathbb{N}}$ *by*

$$\widehat{C}_{n,\eta,\kappa} = \begin{cases} \max\{\widehat{a}_{n,\eta},\ \sum_{i\in J_\eta}\prod_{j\in J_n}\lambda_{i,j}^2\}, & \text{if } \kappa=0;\\ \max\{\widehat{a}_{n,\eta},\ \sum_{i\in J_\eta}\prod_{j\in J_n}\lambda_{i,j}^2, \widehat{b}_{n,\eta,\kappa}\}, & \text{if } \kappa\in\mathbb{N}; \end{cases} \quad n\in\mathbb{N}. \tag{27}$$

Then $S_{\boldsymbol\lambda}$ is normaloid if and only if $\widehat{C}_{n,\eta,\kappa} = \widehat{C}_{1,\eta,\kappa}^n$ for all $n\in\mathbb{N}$, which is equivalent to $\widehat{C}_{1,\eta,\kappa} = \lim_{n\to\infty}\widehat{C}_{n,\eta,\kappa}^{1/n}$.

COROLLARY 15. *Let $S_{\boldsymbol\lambda} := S_{\boldsymbol\lambda(u,v,x,y)}$ be a weighted shift as defined in Subsection 3.4. Then $S_{\boldsymbol\lambda}$ is normaloid if and only if $W^{(0)}$ is normaloid, or equivalently* $\max\{1,x+y\}\leqslant\max\{u,v\}$.

Proof. Firstly, we claim that $\|S_{\boldsymbol\lambda}^n\| = \|(W^{(0)})^n\|$. According to (25)–(27), we can see that $\widehat{C}_{1,2,1} = \max\{1, x+y, \max\{u,v\}\}$, and

$$\widehat{C}_{n,2,1} = \max\{u^{n-2}x+v^{n-2}y, u^{n-1}x+v^{n-1}y, \max\{u^n,v^n\}\}, \quad n\in\mathbb{N}_2. \tag{28}$$

Observe that the sequence $\left\{\frac{u^{k+1}x+v^{k+1}y}{u^kx+v^ky}\right\}_{k\in\mathbb{N}}$ is monotonically increasing. Using (21) and this observation, we can see that

$$\begin{aligned}\|W^{(0)}\|^2 &= \max\left\{1, x+y, \sup_{k\in\mathbb{Z}_+}\frac{u^{k+1}x+v^{k+1}y}{u^kx+v^ky}\right\}\\ &= \max\left\{1, x+y, \lim_{k\to\infty}\frac{u^{k+1}x+v^{k+1}y}{u^kx+v^ky}\right\} = \max\{1, x+y, u, v\}.\end{aligned}$$

Similarly, we may see that

$$\begin{aligned}\|(W^{(0)})^n\|^2 &= \max\left\{u^{n-2}x+v^{n-2}y, u^{n-1}x+v^{n-1}y, \sup_{k\in\mathbb{Z}_+}\frac{u^{k+n}x+v^{k+n}y}{u^kx+v^ky}\right\}\\ &= \max\{u^{n-2}x+v^{n-2}y, u^{n-1}x+v^{n-1}y, \max\{u^n, v^n\}\},\ \ n\in\mathbb{N}_2. \qquad (29)\end{aligned}$$

By (28) and (29), we have $\|(W^{(0)})^n\|^2 = \widehat{C}_{n,2,1}$ for all $n \geqslant 2$. Hence $W^{(0)}$ is normaloid if and only if $S_{\boldsymbol{\lambda}}$ is normaloid.

Furthermore, if the inequality $\max\{1, x+y\} \leqslant \max\{u, v\}$ holds, by a simple computation, we get $\widehat{C}_{n,2,1} = \widehat{C}^n_{1,2,1}$, $n \in \mathbb{N}_2$, and so $S_{\boldsymbol{\lambda}}$ is normaloid. Conversely, suppose that $S_{\boldsymbol{\lambda}}$ is normaloid, i.e., $\widehat{C}_{n,2,1} = \widehat{C}^n_{1,2,1}$, $n\in\mathbb{N}_2$. By using this equality and some elementary computations, we can see that $\max\{1, x+y, \max\{u, v\}\} = \max\{u, v\}$. Hence the proof is complete. □

It is well-known that there exists a 3×3 real matrix A such that A is normaloid but not p-paranormal for any $p > 0$ (see [17, Example 5, p. 179]). The following corollary provides such an example on an infinite dimensional Hilbert space.

COROLLARY 16. *There exists a normaloid weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{2,1}$ such that $S_{\boldsymbol{\lambda}}$ is not p-paranormal for any $p > 0$.*

Proof. Use Corollary 15 and equivalent conditions for p-paranormality of $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{2,1}$ as in Subsection 3.4. □

We now close this section with the following remark related to Q1 and Q2 for normaloidness of weighted shifts $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$.

REMARK 17. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ as usual. It seems difficult to solve Q1 or Q2 when P is normaloid, and we do not attempt it in this paper.

4. Collapsed branching shifts and properties

4.1. Basic properties

We consider question Q2 about quasinormality, subnormality, ∞-hyponormality, p-hyponormality ($p > 0$), and p-paranormality of the collapsed shift $S_{\widetilde{\boldsymbol{\lambda}}}$ of $S_{\boldsymbol{\lambda}} \in \mathscr{B}(\ell^2(V_{\eta,\kappa}))$. We start this section with basic lemmas about the collapsing method, which will be used frequently in subsequent parts of the paper.

LEMMA 18. *Suppose $\eta = 2$. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}_{2,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^{\circ}_{2,\kappa}}$. Then the last-step collapsed weighted shift $\widetilde{W}$ and basic branching shift $W^{(0)}$ of $S_{\boldsymbol{\lambda}}$ coincide.*

Proof. This is provided directly by Definitions 1 and 2. $\square$

LEMMA 19. *Suppose $\eta \geqslant 3$. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^{\circ}_{\eta,\kappa}}$ and let $S_{\widetilde{\boldsymbol{\lambda}}}$ be the first-step collapsed weighted shift of $S_{\boldsymbol{\lambda}}$ with weights $\widetilde{\boldsymbol{\lambda}} = \{\widetilde{\lambda}_v\}_{v \in V^{\circ}_{\eta-1,\kappa}}$. Then $\widetilde{W}^{(0)} = W^{(0)}$, where $W^{(0)}$ is the basic branching shift associated to $S_{\boldsymbol{\lambda}}$ and $\widetilde{W}^{(0)}$ is the basic branching shift associated to the first-step collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$. Moreover, $W^{(i)} = \widetilde{W}^{(i)}$ for $i \in J_{\eta-2}$, where $W^{(i)}$ $[$resp., $\widetilde{W}^{(i)}]$ is the i-th branching shift of $S_{\boldsymbol{\lambda}}$ $[$resp., $S_{\widetilde{\boldsymbol{\lambda}}}]$.*

Proof. We claim that the weights of $\widetilde{W}^{(0)}$ and $W^{(0)}$ coincide. The weight sequence $\{\widetilde{\alpha}_j^{(0)}\}_{j=-\kappa+1}^{\infty}$ of $\widetilde{W}^{(0)}$ is as follows:

$$\widetilde{\alpha}_j^{(0)} = \widetilde{\lambda}_j \text{ for } j \in (-J_{\kappa-1}) \cup \{0\},$$

$$\widetilde{\alpha}_1^{(0)} = \Big(\sum_{i \in J_{\eta-1}} \widetilde{\lambda}_{i,1}^2 \Big)^{1/2}, \qquad \widetilde{\alpha}_{j+1}^{(0)} = \left(\frac{\sum\limits_{i \in J_{\eta-1}} \prod\limits_{k \in J_{j+1}} \widetilde{\lambda}_{i,k}^2}{\sum\limits_{i \in J_{\eta-1}} \prod\limits_{k \in J_j} \widetilde{\lambda}_{i,k}^2} \right)^{1/2}, \quad j \in \mathbb{N}, \tag{30}$$

and the weight sequence $\{\alpha_j^{(0)}\}_{j=-\kappa+1}^{\infty}$ of $W^{(0)}$ is as in (2) and (3). By (4) and (5), it is easy to check that

$$\prod_{k \in J_j} \widetilde{\lambda}_{\eta-1,k}^2 = \sum_{i=\eta-1}^{\eta} \prod_{k \in J_j} \lambda_{i,k}^2, \quad j \in \mathbb{N}.$$

Using this equality and (6), we obtain

$$\begin{aligned}
\widetilde{\alpha}_{j+1}^{(0)} &= \left(\frac{\sum\limits_{i \in J_{\eta-2}} \prod\limits_{k \in J_{j+1}} \widetilde{\lambda}_{i,k}^2 + \prod\limits_{k \in J_{j+1}} \widetilde{\lambda}_{\eta-1,k}^2}{\sum\limits_{i \in J_{\eta-2}} \prod\limits_{k \in J_j} \widetilde{\lambda}_{i,k}^2 + \prod\limits_{k \in J_j} \widetilde{\lambda}_{\eta-1,k}^2} \right)^{1/2} \\
&= \left(\frac{\sum\limits_{i \in J_{\eta-2}} \prod\limits_{k \in J_{j+1}} \lambda_{i,k}^2 + \sum\limits_{i=\eta-1}^{\eta} \prod\limits_{k \in J_{j+1}} \lambda_{i,k}^2}{\sum\limits_{i \in J_{\eta-2}} \prod\limits_{k \in J_j} \lambda_{i,k}^2 + \sum\limits_{i=\eta-1}^{\eta} \prod\limits_{k \in J_j} \lambda_{i,k}^2} \right)^{1/2} \\
&= \left(\frac{\sum\limits_{i \in J_\eta} \prod\limits_{k \in J_{j+1}} \lambda_{i,k}^2}{\sum\limits_{i \in J_\eta} \prod\limits_{k \in J_j} \lambda_{i,k}^2} \right)^{1/2}, \qquad j \in \mathbb{N}.
\end{aligned} \tag{31}$$

Comparing (3) and (31), we have $\widetilde{\alpha}_{j+1}^{(0)} = \alpha_{j+1}^{(0)}$ for $j \in \mathbb{N}$. By (3), (4), (6), and (30), we have

$$\widetilde{\alpha}_1^{(0)} = \Big(\sum_{i \in J_{\eta-2}} \widetilde{\lambda}_{i,1}^2 + \widetilde{\lambda}_{\eta-1,1}^2 \Big)^{1/2} = \Big(\sum_{i \in J_\eta} \lambda_{i,1}^2 \Big)^{1/2} = \alpha_1^{(0)}. \tag{32}$$

Others are trivial. The "moreover" part of this proposition follows immediately from the definitions of $W^{(i)}$ and $\widetilde{W}^{(i)}$. Hence the proof is complete. □

For brevity, in the remaining part of this paper we will say simply "$S_{\widetilde{\boldsymbol{\lambda}}}$ is the collapsed weighted shift of $S_{\boldsymbol{\lambda}}$" instead of using "the first-step" when no confusion will arise.

Repeating the steps for collapsing branches in Lemma 19, and using Lemma 18, we may obtain the following corollary.

COROLLARY 20. *Suppose $\eta \geqslant 2$. Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. Then the last-step collapsed weighted shift $\widetilde{W}$ and the basic branching shift $W^{(0)}$ of $S_{\boldsymbol{\lambda}}$ coincide.*

4.2. Quasinormality and subnormality

First we answer question Q2 affirmatively when property P is quasinormality.

PROPOSITION 21. *If $S_{\boldsymbol{\lambda}}$ is a quasinormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$, then the collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ of $S_{\boldsymbol{\lambda}}$ is quasinormal.*

Proof. If $\eta = 2$, by Proposition 3 and Corollary 20, $S_{\widetilde{\boldsymbol{\lambda}}}$ is quasinormal. Thus we may assume $\eta \geqslant 3$. By (5) and (7), we may see that

$$\widetilde{\lambda}_{\eta-1,j} = \left(\sum_{i \in J_\eta} \lambda_{i,1}^2 \cdot \frac{\lambda_{\eta-1,1}^2 + \lambda_{\eta,1}^2}{\lambda_{\eta-1,1}^2 + \lambda_{\eta,1}^2} \right)^{1/2} = \Big(\sum_{i \in J_\eta} \lambda_{i,1}^2 \Big)^{1/2} = \lambda_{\eta-1,j}, \ j \in \mathbb{N}_2. \tag{33}$$

By (6) and (32), we get $\sum_{i \in J_{\eta-1}} \widetilde{\lambda}_{i,1}^2 = \widetilde{\lambda}_0^2 = \lambda_0^2 = \sum_{i \in J_\eta} \lambda_{i,1}^2$. According to P1, by (6) and (33), $S_{\widetilde{\boldsymbol{\lambda}}}$ is quasinormal. □

Next we answer question Q2 affirmatively when property P is subnormality.

THEOREM 22. *If $S_{\boldsymbol{\lambda}}$ is a subnormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$, then the collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ of $S_{\boldsymbol{\lambda}}$ is subnormal.*

Proof. The case $\eta = 2$ follows easily from Lemma 18 and 1° in Section 1. So we will consider $\eta \geqslant 3$. Recall that $S_{\boldsymbol{\lambda}}$ is subnormal if and only if every $W^{(i)}$ is subnormal for all $i \in J_\eta \cup \{0\}$. Similarly this fact holds for $S_{\widetilde{\boldsymbol{\lambda}}}$, and so it is enough to show that every i-th branching shift $\widetilde{W}^{(i)}$ is subnormal for $i \in J_{\eta-1} \cup \{0\}$. By Lemma 19, we have $\widetilde{W}^{(i)} = W^{(i)}$ for all $i \in J_{\eta-2} \cup \{0\}$. So to finish we need only that $\widetilde{W}^{(\eta-1)}$ is subnormal. We first scale the problem, namely, multiply all the weights of $S_{\boldsymbol{\lambda}}$ by a $c>0$ so small that $W^{(\eta-1)}$, $W^{(\eta)}$, and $\widetilde{W}^{(\eta-1)}$ are all contractions. This can surely be done for $W^{(\eta-1)}$, $W^{(\eta)}$, and an inspection of the resulting weights for $\widetilde{W}^{(\eta-1)}$ (see Figure 3) shows that these are also multiplied by c. Since all the shifts are bounded, this succeeds, and of course scaling the operators does not change subnormality.

But now we can detect subnormality by checking moment sequences of $W^{(\eta-1)}$, $W^{(\eta)}$ and $\widetilde{W}^{(\eta-1)}$. The moment sequence for $W^{(\eta-1)}$ is

$$1, \lambda_{\eta-1,2}^2, \lambda_{\eta-1,2}^2\lambda_{\eta-1,3}^2, \lambda_{\eta-1,2}^2\lambda_{\eta-1,3}^2\lambda_{\eta-1,4}^2, \cdots, \prod_{k=2}^{n}\lambda_{\eta-1,k}^2, \cdots,$$

and that for $W^{(\eta)}$ is

$$1, \lambda_{\eta,2}^2, \lambda_{\eta,2}^2\lambda_{\eta,3}^2, \lambda_{\eta,2}^2\lambda_{\eta,3}^2\lambda_{\eta,4}^2, \cdots, \prod_{k=2}^{n}\lambda_{\eta,k}^2, \cdots.$$

These sequences are completely monotone (see Section 3 of [20]), which is equivalent to the Agler condition

$$A(n,i) := \sum_{j=0}^{n}(-1)^j\binom{n}{j}\prod_{k=2}^{j+1}\lambda_{i,k}^2 \geqslant 0, \;\; n\in\mathbb{N},\; i\in\{\eta-1,\eta\}, \tag{34}$$

with the convention $\prod_{j=a}^{b}(\cdot)_j = 1$ for $a>b$. The moment sequence for $\widetilde{W}^{(\eta-1)}$ is

$$1, \frac{\lambda_{\eta-1,1}^2\lambda_{\eta-1,2}^2+\lambda_{\eta,1}^2\lambda_{\eta,2}^2}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}, \cdots, \frac{\sum\limits_{i=\eta-1}^{\eta}\prod\limits_{k\in J_n}\lambda_{i,k}^2}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}, \cdots.$$

Now we observe that for $n\in\mathbb{N}$,

$$\begin{aligned}
\sum_{i=\eta-1}^{\eta}\lambda_{i,1}^2 A(n,i) &= (\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2) + \left(\sum_{j\in J_n}(-1)^j\binom{n}{j}\sum_{i=\eta-1}^{\eta}\prod_{k\in J_{j+1}}\lambda_{i,k}^2\right)\\
&= (\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2)\left(1+\sum_{j\in J_n}(-1)^j\binom{n}{j}\frac{\sum\limits_{i=\eta-1}^{\eta}\prod\limits_{k\in J_{j+1}}\lambda_{i,k}^2}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}\right)\\
&= (\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2)\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\sum\limits_{i=\eta-1}^{\eta}\prod\limits_{k\in J_{j+1}}\lambda_{i,k}^2}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}.
\end{aligned}$$

By (34), it is obvious that

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\sum\limits_{i=\eta-1}^{\eta}\prod\limits_{k\in J_{j+1}}\lambda_{i,k}^2}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2} \geqslant 0,$$

(which means that the moment sequence for $\widetilde{W}^{(\eta-1)}$ is completely monotone). Thus $\widetilde{W}^{(\eta-1)}$ is subnormal, and so $S_{\widetilde{\boldsymbol{\lambda}}}$ is subnormal. Hence the proof is complete. □

4.3. p-hyponormality

In this subsection we solve question Q2 when property P of $S_{\boldsymbol{\lambda}}$ is p-hyponormality and prove Theorem 4 in the general case of η.

THEOREM 23. *Suppose $p>0$. If $S_{\boldsymbol{\lambda}}$ is a p-hyponormal weighted shift on a directed tree $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$, then the collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ of $S_{\boldsymbol{\lambda}}$ is p-hyponormal.*

Proof. Since the case $\eta=2$ follows from Lemma 18 and Proposition 7, we will consider only $\eta\geqslant 3$ as before. Recall that $S_{\boldsymbol{\lambda}}$ is p-hyponormal if and only if conditions (i)–(iv) of P2 hold. Let us write the associated conditions for p-hyponormality of $S_{\widetilde{\boldsymbol{\lambda}}}$ as P2$(\widetilde{\mathrm{i}})$–$(\widetilde{\mathrm{iv}})$ for the time being. According to (6), we can see that P2(i) and P2$(\widetilde{\mathrm{i}})$ coincide. Since

$$\frac{\widetilde{\lambda}_0^2}{\sum\limits_{i\in J_{\eta-1}}\widetilde{\lambda}_{i,1}^2}=\frac{\widetilde{\lambda}_0^2}{\sum\limits_{i\in J_{\eta-2}}\widetilde{\lambda}_{i,1}^2+\widetilde{\lambda}_{\eta-1,1}^2}\overset{(4)\&(6)}{=}\frac{\lambda_0^2}{\sum\limits_{i\in J_{\eta-2}}\lambda_{i,1}^2+\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}=\frac{\lambda_0^2}{\sum\limits_{i\in J_{\eta}}\lambda_{i,1}^2},$$

also conditions (ii) and $(\widetilde{\mathrm{ii}})$ of P2 coincide. By (6), each of conditions (iv) and $(\widetilde{\mathrm{iv}})$ of P2 coincide for $i\in J_{\eta-2}$. It follows from the Cauchy-Schwarz inequality that

$$\Big(\sum_{i=\eta-1}^{\eta}\prod_{k\in J_j}\lambda_{i,k}^2\Big)^2\leqslant\Big(\sum_{i=\eta-1}^{\eta}\prod_{k\in J_{j-1}}\lambda_{i,k}^2\Big)\Big(\sum_{i=\eta-1}^{\eta}\lambda_{i,j}^2\prod_{k\in J_j}\lambda_{i,k}^2\Big),\quad j\in\mathbb{N}_2,$$

and by using P2(iv), the inequality P2$(\widetilde{\mathrm{iv}})$ holds when $i=\eta-1$. So our concentration is on condition P2(iii). Observe that P2$(\widetilde{\mathrm{iii}})$ is equivalent to (cf. (32))

$$\Big(\sum_{i\in J_\eta}\lambda_{i,1}^2\Big)^{p-1}\left(\sum_{i\in J_{\eta-2}}\frac{\lambda_{i,1}^2}{\lambda_{i,2}^{2p}}+\frac{(\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2)^{p+1}}{(\lambda_{\eta-1,1}^2\lambda_{\eta-1,2}^2+\lambda_{\eta,1}^2\lambda_{\eta,2}^2)^p}\right)\leqslant 1. \tag{35}$$

It is clearly sufficient from P2(iii) and (35) to show

$$\frac{(\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2)^{p+1}}{(\lambda_{\eta-1,1}^2\lambda_{\eta-1,2}^2+\lambda_{\eta,1}^2\lambda_{\eta,2}^2)^p}\leqslant\frac{\lambda_{\eta-1,1}^2}{\lambda_{\eta-1,2}^{2p}}+\frac{\lambda_{\eta,1}^2}{\lambda_{\eta,2}^{2p}}. \tag{36}$$

Observe that we may scale the problem by multiplying each weight by $c>0$ such that

$$c^2\lambda_{\eta-1,1}^2+c^2\lambda_{\eta,1}^2=1.$$

This is because the total order of each side is c^{2-2p} in the scaling constant c. So we assume (without changing notation) that $\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2=1$, and then (36) becomes

$$\frac{1}{(\lambda_{\eta-1,1}^2\lambda_{\eta-1,2}^2+\lambda_{\eta,1}^2\lambda_{\eta,2}^2)^p}\leqslant\frac{\lambda_{\eta-1,1}^2}{\lambda_{\eta-1,2}^{2p}}+\frac{\lambda_{\eta,1}^2}{\lambda_{\eta,2}^{2p}}. \tag{37}$$

Let $x=\lambda_{\eta-1,1}^2$ (so $\lambda_{\eta,1}^2=1-x$), $\lambda_{\eta-1,2}^2=a$ and $\lambda_{\eta,2}^2=b$. Then (37) becomes

$$\frac{1}{(ax+b(1-x))^p}\leqslant\frac{x}{a^p}+\frac{1-x}{b^p},$$

which is equivalent to

$$(b^px+a^p(1-x))^{\frac{1}{p}}\geqslant\frac{ab}{ax+b(1-x)}. \tag{38}$$

By Lemma 6, (38) holds automatically, and so does (36). Hence $S_{\widetilde{\boldsymbol{\lambda}}}$ has condition P2$(\widetilde{\text{iii}})$. The proof is complete. □

PROPOSITION 24. *If $S_{\boldsymbol{\lambda}}$ is ∞-hyponormal, then $S_{\widetilde{\boldsymbol{\lambda}}}$ is ∞-hyponormal.*

Proof. The case $\eta=2$ follows from Lemma 18 and Corollary 5. Therefore we will consider $\eta\geqslant 3$, too. Recall that $S_{\boldsymbol{\lambda}}$ is ∞-hyponormal if and only if the conditions (i), (ii), (iv) of P2 and (18) hold (see Proposition 8). Let us write the corresponding conditions for ∞-hyponormality of $S_{\widetilde{\boldsymbol{\lambda}}}$ as $(\widetilde{\text{i}})$, $(\widetilde{\text{ii}})$, $(\widetilde{\text{iv}})$ of P2, and $(\widetilde{18})$. According to the proof of Theorem 23, conditions (i), (ii) and (iv) of P2 imply conditions $(\widetilde{\text{i}})$, $(\widetilde{\text{ii}})$, and $(\widetilde{\text{iv}})$ of P2. Since $c:=\sum_{i\in J_\eta}\lambda_{i,1}^2=\sum_{i\in J_{\eta-1}}\widetilde{\lambda}_{i,1}^2$, by (6) and (18), we see that $\sum_{i\in J_{\eta-1}}\widetilde{\lambda}_{i,1}^2\leqslant\min_{i\in J_{\eta-2}}\{\widetilde{\lambda}_{i,2}^2\}$. Using (5) and (18), we get that

$$\widetilde{\lambda}_{\eta-1,2}^2=\frac{\lambda_{\eta-1,1}^2\lambda_{\eta-1,2}^2+\lambda_{\eta,1}^2\lambda_{\eta,2}^2}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}\geqslant\frac{\lambda_{\eta-1,1}^2c+\lambda_{\eta,1}^2c}{\lambda_{\eta-1,1}^2+\lambda_{\eta,1}^2}=c.$$

Thus, the condition $(\widetilde{18})$ holds. This complete the proof. □

Now we will prove Theorem 4 by using Theorem 22.

Proof of Theorem 4. Suppose that $S_{\boldsymbol{\lambda}}$ is p-hyponormal on $\mathscr{T}_{\eta,\kappa}$. It follows from P2(iv) that $W^{(i)}$ is p-hyponormal for $i\in J_\eta$. So we will show by mathematical induction that

Claim: if $S_{\boldsymbol{\lambda}}$ is a p-hyponormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ $(p>0,\ \eta\in\mathbb{N}_2)$, then the basic branching shift $W^{(0)}$ is p-hyponormal.

The case $\eta=2$ follows from Proposition 7. We now assume that the statement holds when $\eta=m$. For the case of $\eta=m+1$, we suppose $S_{\boldsymbol{\lambda}}$ is a p-hyponormal weighted shift on $\mathscr{T}_{m+1,\kappa}$. By Theorem 23, the $(m+1)$-th collapsed shift $S_{\widetilde{\boldsymbol{\lambda}}}$ associated to $S_{\boldsymbol{\lambda}}$ is p-hyponormal on $\mathscr{T}_{m,\kappa}$. By the induction hypothesis in $\eta=m$, the basic branching shift $\widetilde{W}^{(0)}$ of $S_{\widetilde{\boldsymbol{\lambda}}}$ is p-hyponormal. Applying Lemma 19, we get $W^{(0)}=\widetilde{W}^{(0)}$, where $W^{(0)}$ is the basic branching shift of $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{m+1,\kappa}$, and so $W^{(0)}$ is p-hyponormal. Thus our statement holds. The proof is complete. □

4.4. p-paranormality

Now we solve question Q2 when property P is p-paranormality.

PROPOSITION 25. *Suppose* $0 < p \leqslant 1$. *If* $S_{\boldsymbol{\lambda}}$ *is a* p*-paranormal weighted shift on* $\mathscr{T}_{\eta,\kappa}$ *with weights* $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^{\circ}}$, *then the collapsed weighted shift* $S_{\widetilde{\boldsymbol{\lambda}}}$ *of* $S_{\boldsymbol{\lambda}}$ *is* p*-paranormal.*

Proof. Recall that $S_{\boldsymbol{\lambda}}$ is p-paranormal if and only if the conditions (i), (ii), (iv) of P2, and (19), hold (see (P3)). We write the corresponding conditions for p-paranormality of $S_{\widetilde{\boldsymbol{\lambda}}}$ as $\widetilde{\text{(i)}}$, $\widetilde{\text{(ii)}}$, $\widetilde{\text{(iv)}}$ of P2 and $(\widetilde{19})$. According to the proof of Theorem 23, conditions (i), (ii) and (iv) of P2 imply conditions $\widetilde{\text{(i)}}$, $\widetilde{\text{(ii)}}$, $\widetilde{\text{(iv)}}$, respectively. Since $\sum_{i \in J_\eta} \lambda_{i,1}^2 = \sum_{i \in J_{\eta-1}} \widetilde{\lambda}_{i,1}^2$, it is enough to show that

$$\lambda_{\eta-1,1}^2 \lambda_{\eta-1,2}^{2p} + \lambda_{\eta,1}^2 \lambda_{\eta,2}^{2p} \leqslant (\lambda_{\eta-1,1}^2 + \lambda_{\eta,1}^2) \left(\frac{\lambda_{\eta-1,1}^2 \lambda_{\eta-1,2}^2 + \lambda_{\eta,1}^2 \lambda_{\eta,2}^2}{\lambda_{\eta-1,1}^2 + \lambda_{\eta,1}^2} \right)^p. \tag{39}$$

As in the proof of Proposition 7, we scale the weights $\{\lambda_v\}_{v \in V_{\eta,\kappa}^{\circ}}$ of $S_{\boldsymbol{\lambda}}$ so that $\lambda_{\eta-1,1}^2 + \lambda_{\eta,1}^2 = 1$. Set $a = \lambda_{\eta-1,2}^2$, $b = \lambda_{\eta,2}^2$ and $x = \lambda_{\eta-1,1}^2$. Then condition (39) becomes

$$xa^p + (1-x)b^p \leqslant (xa + (1-x)b)^p. \tag{40}$$

Thus it is sufficient to show that (40) holds for $p \in (0,1]$. Since the function $f(t) := t^p$ is concave when $p \in (0,1)$, it is obvious that $xf(a) + (1-x)f(b) \leqslant f(xa + (1-x)b)$ for all $x \in (0,1)$ and $a, b > 0$. When $p = 1$, the equality in (40) holds. Hence the proof is complete. $\square$

REMARK 26. According to the "p-paranormality" part in Subsection 3.4, we see that there exists a p-paranormal weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{2,1}$ such that $W^{(0)}$ is not p-paranormal. Since in the case $\eta = 2$ the first collapsed weighted shift $S_{\widetilde{\boldsymbol{\lambda}}}$ becomes $W^{(0)}$, we can obtain the following, a counterpart of Proposition 25 for $p > 1$:

for $p > 1$, *it is not necessarily true that if* $S_{\boldsymbol{\lambda}}$ *is* p*-paranormal, then* $S_{\widetilde{\boldsymbol{\lambda}}}$ *is* p*-paranormal.* (41)

In Remark 26, we consider first the simplest case $\eta = 2$ to show (41). Furthermore, we will provide additional example to exhibit (41) for $\eta = 3$ in the next section (see Example 28).

4.5. Examples

We discussed relationships, for various P, between the following two conditions:

(C_1) $S_{\boldsymbol{\lambda}}$ has property P,

(C_3) $S_{\widetilde{\boldsymbol{\lambda}}}$ has property P.

In this section, we proved the implications $(\mathrm{C}_1) \Rightarrow (\mathrm{C}_3)$ when the "placeholder P" in Q2 is quasinormality, subnormality, p-hyponormality $(0 < p \leqslant \infty)$ or p-paranormality $(0 < p \leqslant 1)$. But the converse implications are not true. We provide counterexamples for these converse implications.

EXAMPLE 27. Consider a weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{3,1}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{3,1}^{\circ}}$ with $\lambda_0 \in (0,1]$ and consider weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{3,1}^{\circ}}$ of $\mathscr{T}_{3,1}$ given by

$$\begin{aligned} &\lambda_{1,1} = \sqrt{x},\ \lambda_{2,1} = \sqrt{uy},\ \lambda_{3,1} = \sqrt{vy},\ \lambda_{2,2} = \lambda_{3,3} = \sqrt{\frac{v}{u}},\ \lambda_{2,3} = \lambda_{3,2} = \sqrt{\frac{u}{v}}, \\ &\lambda_v = 1, \qquad \text{otherwise}, \end{aligned}$$

where $x, y, u, v > 0$ with $x + y = 1$ and $u + v = 1$. By Definition 2, the weights $\widetilde{\boldsymbol{\lambda}} = \{\widetilde{\lambda}_v\}_{v \in V_{2,1}^{\circ}}$ of $S_{\widetilde{\boldsymbol{\lambda}}}$ of $S_{\boldsymbol{\lambda}}$ are given by

$$\widetilde{\lambda}_0 = \lambda_0, \widetilde{\lambda}_{1,1} = \sqrt{x}, \widetilde{\lambda}_{2,1} = \sqrt{y}, \text{ and } \widetilde{\lambda}_v = 1 \text{ otherwise.}$$

By the corresponding equivalent conditions for each property, we obtain that

(i) $S_{\boldsymbol{\lambda}}$ is quasinormal if and only if $\lambda_0 = 1$ and $u = v$,

(ii) $S_{\widetilde{\boldsymbol{\lambda}}}$ is quasinormal if and only if $\lambda_0 = 1$,

(iii) $S_{\boldsymbol{\lambda}}$ is subnormal if and only if $u = v$,

(iv) $S_{\widetilde{\boldsymbol{\lambda}}}$ is always subnormal,

(v) $S_{\boldsymbol{\lambda}}$ is p-hyponormal if and only if $u = v$ $(0 < p \leqslant \infty)$,

(vi) $S_{\widetilde{\boldsymbol{\lambda}}}$ is always p-hyponormal $(0 < p \leqslant \infty)$,

(vii) $S_{\boldsymbol{\lambda}}$ is p-paranormal if and only if $u = v$ $(0 < p < \infty)$,

(viii) $S_{\widetilde{\boldsymbol{\lambda}}}$ is always p-paranormal $(0 < p < \infty)$.

According to (i)–(viii) above, we can find weighted shifts $S_{\boldsymbol{\lambda}}$ such that $S_{\widetilde{\boldsymbol{\lambda}}}$ has property P but $S_{\boldsymbol{\lambda}}$ does not when P is any of the operator properties quasi-, sub-, ∞-hypo-, p-hypo-, or p-paranormality. Moreover, we can confirm easily that this example shows $(\mathrm{C}_3) \not\Rightarrow (\mathrm{C}_1)$ when P is p-paranormality for $0 < p < \infty$.

EXAMPLE 28. Consider a weighted shift $S_{\boldsymbol{\lambda}} = S_{\boldsymbol{\lambda}}(x,y,z)$ on $\mathscr{T}_{3,1}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{3,1}^{\circ}}$ given by

$$\begin{aligned} &\lambda_0 \in (0,1],\ \lambda_{i,1} = \frac{1}{\sqrt{3}}, \qquad i \in J_3, \\ &\lambda_{1,j} = \sqrt{x},\ \lambda_{2,j} = \sqrt{y},\ \lambda_{3,j} = \sqrt{z}, \qquad j \in \mathbb{N}_2, \end{aligned}$$

where x, y and z are positive real numbers. Then weights $\{\widetilde{\lambda}_v\}_{v\in V_{2,1}^\circ}$ of the collapsed weighted shift $S_{\widetilde{\boldsymbol\lambda}}$ of $S_{\boldsymbol\lambda}$ are given by

$$\widetilde{\lambda}_0=\lambda_0,\ \widetilde{\lambda}_{1,1}=\frac{1}{\sqrt{3}},\ \widetilde{\lambda}_{2,1}=\sqrt{\frac{2}{3}},$$
$$\widetilde{\lambda}_{1,j}=\sqrt{x},\ \widetilde{\lambda}_{2,j}=\left(\frac{y^{j-1}+z^{j-1}}{y^{j-2}+z^{j-2}}\right)^{1/2},\qquad j\in\mathbb{N}_2.$$

By P2 and P3, we obtain without difficulty that

(i) $S_{\boldsymbol\lambda}$ is p-hyponormal if and only if $\frac{1}{x^p}+\frac{1}{y^p}+\frac{1}{z^p}\leqslant 3$,

(ii) $S_{\widetilde{\boldsymbol\lambda}}$ is p-hyponormal if and only if $\frac{1}{x^p}+2\left(\frac{2}{y+z}\right)^p\leqslant 3$,

(iii) $S_{\boldsymbol\lambda}$ is p-paranormal if and only if $3\leqslant x^p+y^p+z^p$,

(iv) $S_{\widetilde{\boldsymbol\lambda}}$ is p-paranormal if and only if $3\leqslant x^p+2(\frac{y+z}{2})^p$.

To show that

$$\begin{array}{l}(\mathrm{C}_3)\not\Rightarrow(\mathrm{C}_1)\text{ when }P\text{ is }p\text{-hyponormality for }0<p\leqslant\infty,\\ (\mathrm{C}_3)\not\Rightarrow(\mathrm{C}_1)\text{ when }P\text{ is }p\text{-paranormality for }0<p\leqslant 1,\text{ and}\\ (\mathrm{C}_1)\not\Rightarrow(\mathrm{C}_3)\text{ when }P\text{ is }p\text{-paranormality for }1<p<\infty,\end{array}\tag{42}$$

we consider $S_{\boldsymbol\lambda}(1,y,z)$. One can find many counterexamples satisfying (42) in Figures 6 and 7.

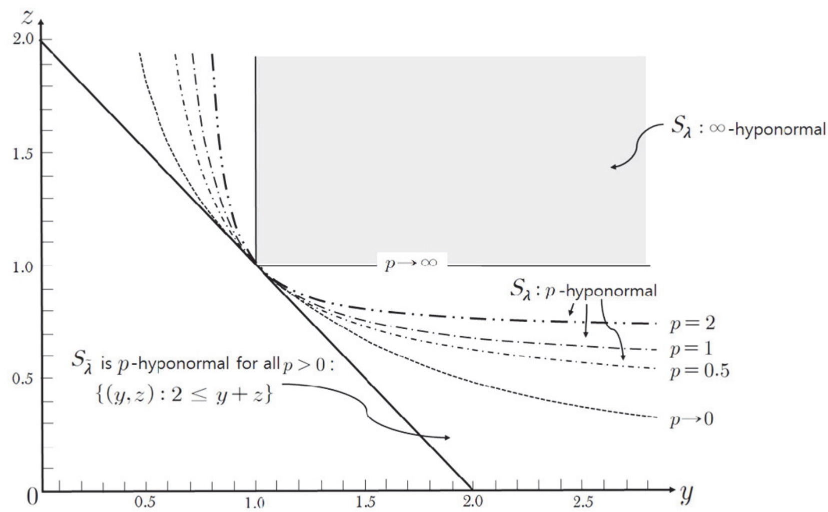

Figure 6: *Regions of p-hyponormality of $S_{\boldsymbol\lambda}$ and $S_{\widetilde{\boldsymbol\lambda}}$ with $x=1$.*

Summary. Combining Propositions 21, 24, and 25, Theorems 22 and 23, and Examples 27 and 28, we obtain Table 4.1.

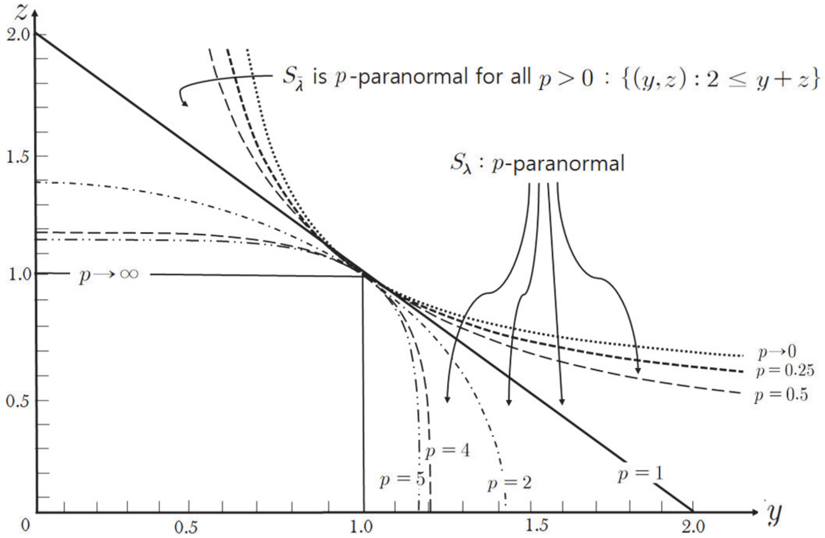

Figure 7: *Regions of p-paranormality of $S_{\boldsymbol{\lambda}}$ and $S_{\widetilde{\boldsymbol{\lambda}}}$ with $x=1$.*

Property P	$(C_1) \Rightarrow (C_3)$	$(C_3) \Rightarrow (C_1)$
quasi-, sub-, and p-hyponormal	True	False
p-paranormal $(0 < p \leqslant 1)$	True	False
p-paranormal $(1 < p < \infty)$	False	False

Table 4.1.

5. Generation flatness

In the previous sections, we discussed implications among conditions (C_1), (C_2) and (C_3) that are as in Sections 3 and 4. But, conditions (C_1) and (C_2) [(C_1) and (C_3)] are not equivalent in some of the standard operator properties. In this section we prove that if $S_{\boldsymbol{\lambda}}$ is generation flat (whose definition appears below), then (C_1), (C_2), and (C_3) are equivalent.

Recall from [12, Definition 6.1] that a weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$ is *r-generation flat* $(r \in \mathbb{N})$ if

$$\lambda_{i,j} = \lambda_{1,j}, \qquad i \in J_\eta,\, j \in \mathbb{N}_r. \tag{43}$$

The following properties immediately come from (43).

P4. Suppose a weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,\kappa}}$ is 2-generation flat. Then

(i) $\alpha^{(0)}_{j+1} = \lambda_{1,j+1}$ for $j \in \mathbb{N}$, where $\alpha^{(0)}_{j+1}$ is the weight of $W^{(0)}$ as in (3),

(ii) $\widetilde{\lambda}_{\eta-1,j} = \lambda_{\eta-1,j} = \lambda_{\eta,j}, \quad j \in \mathbb{N}_2.$

The following is the main result of this section.

THEOREM 29. *Suppose $p>0$. Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$. If $S_{\boldsymbol\lambda}$ is 2-generation flat, then the following statements are equivalent.*

(i) *$S_{\boldsymbol\lambda}$ is quasinormal [resp., subnormal, ∞-hyponormal, p-hyponormal, and p-paranormal],*

(ii) *$W^{(i)}$ of $S_{\boldsymbol\lambda}$ is quasinormal [resp., subnormal, ∞-hyponormal, p-hyponormal, and p-paranormal] for all $i\in J_\eta\cup\{0\}$,*

(iii) *$W^{(0)}$ of $S_{\boldsymbol\lambda}$ is quasinormal [resp., subnormal, ∞-hyponormal, p-hyponormal, and p-paranormal],*

(iv) *$S_{\widetilde{\boldsymbol\lambda}}$ is quasinormal [resp., subnormal, ∞-hyponormal, p-hyponormal, and p-paranormal].*

The proof of Theorem 29 will appear after the next proposition.

PROPOSITION 30. *Suppose $p>0$. Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$ and let $S_{\widetilde{\boldsymbol\lambda}}$ be the collapsed weighted shift of $S_{\boldsymbol\lambda}$. Assume that*

$$\lambda_{\eta-1,j}=\lambda_{\eta,j},\qquad j\in\mathbb{N}_2. \tag{44}$$

Then the following statements are equivalent.

(i) *$S_{\boldsymbol\lambda}$ is quasinormal [resp., subnormal, ∞-hyponormal, p-hyponormal, p-paranormal],*

(ii) *$S_{\widetilde{\boldsymbol\lambda}}$ is quasinormal [resp., subnormal, ∞-hyponormal, p-hyponormal, p-paranormal].*

Proof. Before proving this proposition, we observe that if (44) holds, then P4(ii) holds, which will be used in the proof.

(i)$\Rightarrow$(ii) In Section 4, we proved that this implication holds for any operator properties of $S_{\boldsymbol\lambda}$ except the case of p-paranormality for $p>1$. Hence it is sufficient to claim that if $S_{\boldsymbol\lambda}$ is p-paranormal, then $S_{\widetilde{\boldsymbol\lambda}}$ is p-paranormal $(p>1)$. Hence, using P4(ii) with $j=2$, we see that the equality in (39) holds. By a proof similar to that of Proposition 25, we get our claim.

(ii)$\Rightarrow$(i) We first prove the case of subnormality. Suppose $S_{\widetilde{\boldsymbol\lambda}}$ is subnormal. By P4(ii), since $W^{(\eta-1)}=W^{(\eta)}=\widetilde{W}^{(\eta-1)}$, it is obvious that every branching shift $W^{(i)}$ of $S_{\boldsymbol\lambda}$ is subnormal if and only if every branching shift $\widetilde{W}^{(i)}$ of $S_{\widetilde{\boldsymbol\lambda}}$ is subnormal. Hence $S_{\boldsymbol\lambda}$ is subnormal. Now we consider other properties. We obtained equivalent conditions (7), P2(i)–(iv), (18), (19) to characterize other properties, namely, quasinormality, ∞-hyponormality, p-hyponormality, p-paranormality of $S_{\boldsymbol\lambda}$. The conditions corresponding to all operator properties about $S_{\widetilde{\boldsymbol\lambda}}$ appearing in (ii) can be obtained naturally. We will write these conditions for $S_{\widetilde{\boldsymbol\lambda}}$ as $(\widetilde{7})$, P2$\,(\widetilde{\text{i}})$-$(\widetilde{\text{iv}})$, $(\widetilde{18})$, $(\widetilde{19})$. By using P4(ii) and

(32), we can see that condition (7) [resp., P2(i)–(iv), (18), (19)] for $S_{\boldsymbol{\lambda}}$ is equivalent to condition $(\widetilde{7})$ [resp., P2 $(\widetilde{\text{i}})$ - $(\widetilde{\text{iv}})$, $(\widetilde{18})$, $(\widetilde{19})$] for $S_{\widetilde{\boldsymbol{\lambda}}}$. Hence the proof is compete. □

We now prove Theorem 29.

Proof of Theorem 29. (i)$\Rightarrow$ (ii) Recall that this implication has been proved already except in the case of p-paranormality $(p > 1)$. If $S_{\boldsymbol{\lambda}}$ is p-paranormal $(p > 1)$, then P3 holds. Using (43) with $j = 2$, (19) is equivalent to (20) for any $p > 0$. By a proof similar to that of Proposition 9, $W^{(i)}$ of $S_{\boldsymbol{\lambda}}$ is p-paranormal for all $i \in J_\eta \cup \{0\}$.

(i)$\Leftrightarrow$(iv) See Proposition 30.

(ii)$\Rightarrow$(iii) Obvious.

(iii)$\Rightarrow$(ii) Let $\{e_i\}_{i=-\kappa}^{\infty}$ be an orthonormal basis for ℓ^2 such that

$$W^{(0)} e_i = \alpha_{i+1}^{(0)} e_i, \quad i \in (-J_\kappa) \cup \mathbb{Z}_+.$$

Then, by P4(i), we see that the restriction $W^{(0)}|_{\mathscr{M}}$ of $W^{(0)}$ is unitarily equivalent to $W^{(i)}$ for $i \in J_\eta$, where $\mathscr{M} := \bigvee_{k \in \mathbb{N}} \{e_k\}$ is the span of $\{e_k\}_{k \in \mathbb{N}}$. Since operator properties of $W^{(0)}$ appearing in (iii) are preserved for the restriction $W^{(0)}|_{\mathscr{M}}$, this implication is obvious.

(ii)$\Rightarrow$(i) When property P is quasinormality or subnormality, this implication was proved already. For the remaining parts, we suppose that $W^{(i)}$ is p-hyponormal for all $i \in J_\eta \cup \{0\}$ $(0 < p \leqslant \infty)$, i.e., $W^{(i)}$ is hyponormal for $i \in J_\eta \cup \{0\}$. By P4(i) and hyponormality of $W^{(0)}$, we get

$$\sum_{i \in J_\eta} \lambda_{i,1}^2 \leqslant \lambda_{1,2}^2. \tag{45}$$

Since (45) implies (18), by Proposition 8, $S_{\boldsymbol{\lambda}}$ is ∞-hyponormal. Then it is obvious that $S_{\boldsymbol{\lambda}}$ is p-hyponormal. Next, suppose $W^{(i)}$ is p-paranormal for all $i \in J_\eta \cup \{0\}$ (some $0 < p < \infty$). By p-paranormality of $W^{(0)}$ and P4(i), (45) holds, which is equivalent to (19) for any $p > 0$. Thus $S_{\boldsymbol{\lambda}}$ is p-paranormal. □

We give a natural question concerning the topics discussed in this paper.

QUESTION 31. Let $\mathscr{T} = (V, E)$ be a rooted directed tree with finitely many branching vertices and let $S_{\boldsymbol{\lambda}}$ be the associated weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_u\}_{u \in V^\circ}$. Is it possible to extend the notions about slicing and collapsing the branches of tree for the properties of $S_{\boldsymbol{\lambda}}$ between subnormality and normaloid of $S_{\boldsymbol{\lambda}}$ such as subnormality, p-hyponormality, p-paranormality, normaloidness, etc.?

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